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AUTOMORPHIC GRAPHS AND THE KREIN CONDITION

ABSTRACT. An automorphic graph is a distance-transitive graph, not a complete graph or a line graph, whose automorphism group acts primitively on the vertices. This paper shows that, for small values of the valency and diameter, such graphs are rare. The basic tool is the intersection array, for which there are several very restrictive feasibility conditions. In particular, a slight generalisation of the Krein condition of Scott and Higman is given, with a simplified proof.

1. INTRODUCTION

A graph Γ with distance function ∂ is said to be *distance-transitive* if, for any vertices u, v, x, y such that $\partial(u, v)=\partial(x, y)$, there is an automorphism of Γ taking u to x and v to y. Graphs with this property are rather scarce. For instance, there are just twelve trivalent distance-transitive graphs [4], and finitely many 4-valent ones [14, 15, 16]. Nevertheless, they are sufficiently numerous to make the problem of finding and classifying them highly non-trivial. In this paper we shall attack the problem by various theoretical and practical means.

The basic method is to seek graphs which have the combinatorial properties of a distance-transitive graph. A graph Γ is said to be *distance-regular* if it is a regular graph, of valency k and diameter d, and the following condition holds. There are natural numbers

$$b_0 = k, b_1, \ldots, b_{d-1}; \qquad c_1 = 1, c_2, \ldots, c_d,$$

such that for each pair (u, v) of vertices satisfying $\partial(u, v)=j$ we have

(i) the number of vertices w such that $\partial(u, w)=1$ and $\partial(v, w)=j-1$ is $c_j (1 \leqq j \leqq d)$;

(ii) the number of vertices w such that $\partial(u, w)=1$ and $\partial(v, w)=j+1$ is $b_j (0 \leqq j \leqq d-1)$.

The array $\iota(\Gamma)=\{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}$ is called the *intersection array* of Γ. It follows that if $a_j (0 \leqq j \leqq d)$ denotes the number of vertices w such that $\partial(u, w)=1$ and $\partial(v, w)=j$, then

$$a_0 = 0; \qquad a_j = k - b_j - c_j \,(1 \leqq j \leqq d-1);$$

$$a_d = k - c_d.$$

This definition and the consequent theory may be found in [2]. For completeness we shall summarise briefly the main results.

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Let n be the number of vertices of Γ. We define $d+1$ matrices $A_0, A_1, \ldots, A_d$, each having n rows and columns labelled by the vertices of Γ, as follows:

$$(A_h)_{uv} = \begin{cases} 1, & \text{if } \partial(u,v) = h, \\ 0, & \text{otherwise}. \end{cases}$$

The matrix $A_1 = A$ is the usual adjacency matrix of Γ. The adjacency algebra $\mathscr{A}(\Gamma)$ is the algebra of polynomials in A (over $\mathbb{C}$); in the case of a distance-regular graph this algebra has dimension $d+1$ and $\{A_0, A_1, \ldots, A_d\}$ is a basis for it. The multiplication of basis elements is given by

$$A_h A_i = \sum_{j=0}^{d} s_{hij} A_j \; (h, i \in \{0, 1, \ldots, d\}),$$

where the numbers s_{hij} are called the intersection numbers of Γ. They have the following combinatorial interpretation:

$$s_{hij} = |\{w \in V\Gamma \mid \partial(u,w) = h \quad \text{and} \quad \partial(v,w) = i\}|$$
$$\text{whenever} \quad \partial(u,v) = j.$$

Let $\mathbb{Q}[\lambda]$ denote the ring of polynomials in λ with rational coefficients, and let $v_0(\lambda), v_1(\lambda), \ldots, v_d(\lambda)$ be the elements of $\mathbb{Q}[\lambda]$ defined by the recursion

$$v_0(\lambda) = 1, \qquad v_1(\lambda) = \lambda,$$
$$c_{i+1} v_{i+1}(\lambda) + (a_i - \lambda) v_i(\lambda) + b_{i-1} v_{i-1}(\lambda) = 0$$
$$(i = 1, 2, \ldots, d-1).$$

The sequence $\{v_i(\lambda)\}$ is called the *eigenvector sequence* of Γ.

Since $\mathscr{A}(\Gamma)$ is the algebra of polynomials in A, each matrix $A_i (0 \le i \le d)$ is a polynomial in A; in fact $A_i = v_i(A)$.

The multiplication formula in $\mathscr{A}(\Gamma)$ shows that the $(d+1) \times (d+1)$ matrices $B_0, B_1, \ldots, B_d$ defined by

$$(B_h)_{ij} = s_{hij}$$

generate an algebra isomorphic with $\mathscr{A}(\Gamma)$. We find that $B_0 = I$ and $B = B_1$ is the tridiagonal matrix

$$\begin{bmatrix} 0 & 1 & & & & 0 \\ k & a_1 & c_2 & & & \\ & b_1 & a_2 & \cdot & & \\ & & b_2 & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & c_d \\ 0 & & & & & \cdot & a_d \end{bmatrix}.$$

Further, the isomorphism of algebras implies that $B_i = v_i(B)$ $(2 \leqq i \leqq d)$, so that the matrices B_i can be calculated directly from the intersection array.

B has $d+1$ distinct eigenvalues $\lambda_0 = k, \lambda_1, \ldots, \lambda_d$, which are also the eigenvalues of A. Their multiplicities as eigenvalues of A can be calculated from $\iota(\Gamma)$ alone.

In the rest of the paper we shall take $d \geqq 2$; the only distance-transitive graphs with $d=1$ are the complete graphs. We also take $k \geqq 3$, thereby discarding only the graph K_2 $(k=1)$ and the polygons $(k=2)$.

2. Feasibility conditions

If there is a distance-regular graph Γ with intersection array $\{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}$ then many parameters associated with Γ can be calculated from the entries of this array. Conversely, if we wish to decide whether a given array arises from a graph then we can calculate the parameters of the putative graph and check that they satisfy certain 'feasibility' conditions. For example, the number

$$k_i = (kb_1 \cdots b_{i-1})/(c_2 c_3 \cdots c_i) \qquad (2 \leq i \leq d)$$

represents the number of vertices whose distance from a given vertex is i, and hence it must be a positive integer. (For uniformity, we set $k_0 = 1$, $k_1 = k$.)

Four such feasibility conditions are stated and proved in [2]. In the notation given above, they are:

I: The numbers $k_i = (kb_1 \cdots b_{i-1})/(c_2 c_3 \cdots c_i)$ are integers $(2 \leqq i \leqq d)$.

II: $k \geqq b_1 \geqq \cdots \geqq b_{d-1}$ and $1 \leqq c_2 \leqq \cdots \leqq c_d$.

III: $nk \equiv 0 \pmod 2$ and $k_i a_i \equiv 0 \pmod 2$.

IV: The number n/σ_i is a positive integer $(1 \leqq i \leqq d)$, where

$$\sigma_i = \sum_{j=0}^{d} k_j^{-1} v_j(\lambda_i)^2 .$$

These four conditions are not sufficient for the existence of a graph, but they are very restrictive. In particular, condition IV rules out a high proportion of arrays.

When the arrays which satisfy I, II, III and IV are listed for small values of k and d it appears that most of them do in fact correspond to a graph. But there are some awkward cases, and in order to deal with these it is helpful to have more conditions. There are three such conditions which have proved useful, and we shall present them here. The first two are elementary, and their justification is given in the next two propositions. The third is more subtle, and we shall devote the whole of Section 3 to it.

PROPOSITION. *Let k, a_1, a_2, c_2 denote the parameters associated with an intersection array, in the notation given above. Then if there is a distance-regular graph corresponding to this array we must have:*

V: (α) $a_1=0$ *and* $a_2\neq 0 \Rightarrow a_2 \geqq c_2$;

(β) $a_1=1 \Rightarrow a_2 \geqq c_2$;

(γ) $a_2=2$ *and* $3\nmid k \Rightarrow c_2 \geqq 2$.

Proof. (α) If $a_2 \neq 0$ then there are vertices v, y, u such that

$$\partial(v, u) = \partial(v, y) = 2 \quad \text{and} \quad \partial(u, y) = 1.$$

There is some vertex x adjacent to v and y. Since $a_1=0$ there are no triangles and x is not adjacent to u; thus $\partial(x, u)=2$. By the definition of c_2, there are c_2 paths of length 2 from x to u. Let xwu be such a path. Then w is not adjacent to v, since $a_1=0$, and so $\partial(v, w)=2$. That is, there are at least c_2 vertices w satisfying $\partial(u, w)=1$ and $\partial(v, w)=2$, while $\partial(u, v)=2$. By the definition of a_2, we have $a_2 \geqq c_2$.

(β) The condition $a_1=1$ means that for each pair of adjacent vertices there is a unique vertex adjacent to both of them. Choose two vertices u and v so that $\partial(u, v)=2$; then there are c_2 vertices adjacent to both u and v. For each such vertex x there is a unique y adjacent to both x and u. If $\partial(v, y)=1$, then we should have both u and v adjacent to x and y, so $\partial(v, y)=2$. These y's are all different, otherwise we should have x_1 and x_2 adjacent to y and u. Thus there are at least c_2 vertices y satisfying

$$\partial(u, y) = 1 \quad \text{and} \quad \partial(v, y) = 2, \qquad \partial(u, v) = 2;$$

that is, $a_2 \geqq c_2$.

(γ) If $a_1=2$ then the vertex-subgraph induced by the vertices adjacent to u is a set of polygons. If $3\nmid k$ then these polygons are not all triangles. Choose v, x, y, so that $\partial(v, x)=\partial(x, y)=1$, $\partial(v, y)=2$, and y is not in a triangle adjacent to x. Then x and y have two common neighbours z and t. If $\partial(z, t)=1$ we should have a triangle yzt of vertices adjacent to x. So $\partial(z, t)=2$. Since both x and y are adjacent to z and t, $c_2 \geqq 2$.

PROPOSITION. *Let the array $\{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}$ be given. Define $B_0=I$ and let B_1 be the tridiagonal matrix defined in terms of the array as in Section 1. Define $B_2, B_3, \ldots, B_d$ in terms of the given array by the rule $B_h=v_h(B_1)$. Then, if there is a distance-regular graph with this intersection array,*

VI: the entries of B_h are non-negative integers $(0 \leqq h \leqq d)$.

Proof. As we saw in Section 1, $(B_h)_{ij}=s_{hij}$, and the combinatorial interpretation of the numbers s_{hij} shows that they must be non-negative integers.

3. The Krein condition

Let Γ be a distance-regular graph. If X is a matrix belonging to the adjacency algebra $\mathscr{A}(\Gamma)$ then $X=x(A)$ for some polynomial function x. We may define *characters* $\xi_0, \xi_1, \ldots, \xi_d$ on $\mathscr{A}(\Gamma)$ by setting

$$\xi_i(X) = x(\lambda_i) \qquad (0 \leqq i \leqq d),$$

where $\lambda_0, \lambda_1, \ldots, \lambda_d$ are the eigenvalues of A. So $\xi_i(X)$ is an eigenvalue of X. In particular, since $v_j(A)=A_j$, the eigenvalues of the basic matrices are given by

$$\xi_i(A_j) = v_j(\lambda_i).$$

Putting $i=0$ we have $\lambda_0=k$ and $\xi_0(A_j)=v_j(k)=k_j$.

We use the symbol $\circ$ to denote pointwise multiplication of matrices. The adjacency algebra is closed under this operation since $A_i \circ A_j=\delta_{ij}A_j$.

The following explicit formula gives a basis $\{J_0, J_1, \ldots, J_d\}$ of mutually orthogonal idempotents of $\mathscr{A}(\Gamma)$:

$$J_\alpha = \frac{1}{\sigma_\alpha} \sum_{r=0}^{d} \frac{v_r(\lambda_\alpha)}{k_r} A_r \qquad (\alpha = 0, 1, \ldots, d).$$

Consider the pointwise product $J_{\alpha_1} \circ J_{\alpha_2} \circ \cdots \circ J_{\alpha_t}$ of t such idempotents, not necessarily distinct. By a theorem of Schur this matrix has all its eigenvalues in the interval [0, 1], so for $0 \leqq \beta \leqq d$ we have

$$0 \leqq \xi_\beta\,(J_{\alpha_1} \circ J_{\alpha_2} \circ \cdots \circ J_{\alpha_t}) \leqq 1.$$

PROPOSITION. *For any choice of* $t+1$ *numbers* $\alpha_1, \alpha_2, \ldots, \alpha_t, \beta$, *(not necessarily distinct) from the set* $\{0, 1, \ldots, d\}$ *we have the 'generalised Krein condition':*

$$\text{VII:} \qquad 0 \leqq \sum_{j=0}^{d} k_j^{-t}\, v_j(\lambda_{\alpha_1}) \cdots v_j(\lambda_{\alpha_t})\, v_j(\lambda_\beta) \leqq \sigma_{\alpha_1}\sigma_{\alpha_2} \cdots \sigma_{\alpha_t}.$$

Proof. From the explicit formula for J_α we find

$$\begin{aligned}
&\xi_\beta\,(J_{\alpha_1} \circ J_{\alpha_2} \circ \cdots \circ J_{\alpha_t})\\
&= \xi_\beta \left[(\sigma_{\alpha_1} \cdots \sigma_{\alpha_t})^{-1} \left\{ \sum_r \frac{v_r(\lambda_{\alpha_1})}{k_r} A_r \right\} \circ \cdots \circ \left\{ \sum_r \frac{v_r(\lambda_{\alpha_t})}{k_r} A_r \right\} \right]\\
&= \xi_\beta \left[(\sigma_{\alpha_1} \cdots \sigma_{\alpha_t})^{-1} \sum_j k_j^{-t}\, v_j(\lambda_{\alpha_1}) \cdots v_j(\lambda_{\alpha_t})\, A_j \right]
\end{aligned}$$

(since $A_r \circ \cdots \circ A_s \neq 0$ only if $r = \cdots = s$)

$$= (\sigma_{\alpha_1} \cdots \sigma_{\alpha_t})^{-1} \sum_j k_j^{-t}\, v_j(\lambda_{\alpha_1}) \cdots v_j(\lambda_{\alpha_t})\, v_j(\lambda_\beta).$$

The result now follows from the remark preceding the statement of the theorem.

We note that in the case $t=1$ the bounds are necessarily attained: when $\alpha_1 \neq \beta$ the lower bound applies, while for $\alpha_1 = \beta$ the upper bound holds, by the definition of σ_β.

Condition VII is a generalisation of the Krein condition of L. L. Scott [12] and D. G. Higman [10], who state it for the case $t=2$ only. In fact, the generalised condition is a consequence of the condition for $t=2$; this was pointed out by P.J. Cameron and P. Delsarte.

4. Method of computation

The seven conditions now available make it possible to list by computer all the feasible arrays for small values of k and d. This project has been carried out intermittently during the years 1970–4, with the assistance of D. H. Smith, C. Penman and G. H. J. Meredith. The results are now complete for all pairs (d, k) in the range $d \leqq 5$, $k \leqq 13$. For $d=2, 3$ (corresponding to rank 3 and rank 4 in permutation group terminology) the lists can be extended to much greater values of k.

The method adopted is to generate the arrays which satisfy conditions I, II and III in sequence for fixed k and d. The conditions IV, V, VI and VII are then applied, and the arrays which pass all these tests are listed. The procedure is justified by the fact that a high proportion of the listed arrays can be shown to correspond to graphs.

Condition IV is the most restrictive and also the most time-consuming. Originally it was applied in the form given in Section 2. That method involves the estimation of the eigenvalues λ_i of B, calculation of values $v_j(\lambda_i)$ from the recursion formula, calculation of σ_i, and testing n/σ_i for integrality. Small errors in the estimation of the eigenvalues may accumulate in the subsequent calculations, and this leads to difficulties with the integrality test. An improved technique involves the direct estimation of the numbers σ_i, and this can be done by means of the following result.

PROPOSITION. *The numbers* σ_i $(0 \leqq i \leqq d)$ *are the eigenvalues of the matrix*

$$S = \sum_{j=0}^{d} \left(\frac{\operatorname{tr} B_j}{k_j} \right) B_j.$$

Proof. The expression defining σ_i and the fact that λ_i is an eigenvalue of B_1

shows that σ_i is an eigenvalue of

$$S = \sum_{j=0}^{d} k_j^{-1}\, v_j(B_1)^2 = \sum_{j=0}^{d} k_j^{-1} B_j^2 .$$

But $B_h B_i = \sum_j s_{hij} B_j$ and so

$$S = \sum_j k_j^{-1} \sum_h s_{jjh} B_h = \sum_h B_h \sum_j s_{jjh}/k_j .$$

Now $s_{jjh}k_h = s_{hjj}k_j$. To see this, fix a vertex u in $V\Gamma$ and count the pairs (v, w) such that

$$\partial\,(u, v) = j, \qquad \partial\,(u, w) = h \qquad \text{and} \qquad \partial\,(v, w) = j .$$

Counting v first, we get $k_j s_{hjj}$ pairs, and counting w first we get $k_h s_{jjh}$ pairs. So we have

$$S = \sum_h B_h \sum_j \frac{s_{hjj}}{k_h} = \sum_h \left(\frac{\operatorname{tr} B_h}{k_h} \right) B_h, \quad \text{as required}.$$

5. Reduction to automorphic cases

At this point we are faced with the problem of analysing and interpreting the computed lists of arrays. In this task we are aided by several theorems of the following general kind. If an array of a certain type arises from a graph, then there is a corresponding 'smaller' array arising from a 'smaller' graph.

The largest class of arrays of this kind are those corresponding to *bipartite* graphs. The intersection arrays of bipartite graphs are just those for which

$$a_1 = a_2 = \cdots = a_d = 0,$$

and so they can be instantly recognised. Further, associated with a bipartite distance-transitive graph Γ is a 'halved' graph $\bar{\Gamma}$, also distance-transitive but not in general bipartite. The vertices of $\bar{\Gamma}$ are those in one part of the bipartition of Γ, and two such are adjacent in $\bar{\Gamma}$ when their distance in Γ is 2. The intersection array of $\bar{\Gamma}$ is determined by that of Γ.

Another class of arrays with such a property are those corresponding to *antipodal* graphs [13]. An antipodal distance-transitive graph Γ has a 'derived' graph Γ' which is also distance-transitive, and the intersection array of Γ' is determined by that of Γ [6].

Thirdly, there are the *line* graphs. It has been proved [3] that the line graph $L(\Gamma)$ is distance-transitive only if Γ itself is distance-transitive and belongs to a very small class of graphs. The intersection arrays of distance-transitive line graphs can thus be readily identified.

So if we set aside the feasible arrays corresponding to bipartite, antipodal, or line graphs, those that remain must be of three types:

(i) those for which no distance-regular graph exists;

(ii) those for which a distance-regular graph exists, but not a distance-transitive graph;

(iii) those corresponding to an *automorphic* graph [2, p. 152].

The automorphic graphs are important from the group-theoretical viewpoint. They are the distance-transitive graphs for which the automorphism group acts primitively on the vertices [13], with the exclusion of the complete graphs ($d=1$) and the line graphs.

6. Survey of automorphic graphs

In the range $3 \leqq k \leqq 13$, $2 \leqq d \leqq 5$ there are more than a million arrays of natural numbers $\{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}$ satisfying the elementary feasibility conditions I, II and III. Less than 60 of these satisfy the conditions IV, V, VI, VII and are of the type which might arise from an automorphic graph. This fact and the survey which follows are intended to justify the claim that automorphic graphs are especially interesting objects, worthy of further study.

The case $d=2$ has already been the subject of several investigations and will not be discussed here. An account may be found in [1], and in the works of D. G. Higman and his associates [8], [9]. In addition, we shall not investigate questions of uniqueness, being satisfied with finding one graph corresponding to each given array.

To begin, we survey those arrays which belong to uniform families. These make up more than one-third of the total number under discussion.

For any natural numbers $m \geqq 2$, $q \geqq 2$ the array

$$\{m(q-1), (m-1)(q-1), \ldots, (q-1); 1, 2, \ldots, m\}$$

passes all our tests. It is the intersection array of the graph $\Gamma(m, q)$ defined as follows. Let $Q=\{1, 2, \ldots, q\}$ and take the vertex-set to be $Q \times Q \times \cdots \times Q = Q^m$, with two vertices being joined by an edge if and only if they differ in just one coordinate. $\Gamma(m, q)$ is distance-transitive, and it is automorphic if $m>2$ and $q>2$. (The graph $\Gamma(k, 2)$ is antipodal and bipartite; it is the cube Q_k [2, p. 138]. The graph $\Gamma(2, r)$ is the line graph of the complete bipartite graph $K_{r,r}$.) So there are just six automorphic graphs $\Gamma(m, q)$ in the range being considered, corresponding to $(m, q)=(3, 3)$, $(4, 3)$, $(5, 3)$, $(3, 4)$, $(4, 4)$, $(3, 5)$.

The derived graph of the antipodal graph Q_k is automorphic if k is odd, and it is denoted by $\square_k$. The intersection array of $\square_{2l+1}$ is

$$\{2l + 1, 2l, \ldots, l + 1; 1, 2, \ldots, l\}.$$

The graphs $\square_7$, $\square_9$, $\square_{11}$ are relevant for us.

The *odd graphs* O_k have been investigated in various connections [7], [11]. They are automorphic graphs with valency k and diameter $k-1$. The intersection arrays of those in our range are

$$\begin{aligned} \iota(O_4) &= \{4, 3, 3; 1, 1, 2\} \\ \iota(O_5) &= \{5, 4, 4, 3; 1, 1, 2, 2\} \\ \iota(O_6) &= \{6, 5, 5, 4, 4; 1, 1, 2, 2, 3\}, \end{aligned}$$

from which the general pattern may be inferred.

The intersection array

$$\{xy, (x-1)(y-1), (x-2)(y-2), \ldots, (y-x+1); \\ 1, 4, 9, \ldots, x^2\}$$

for $y > x \geqq 1$, is feasible. It is realised by the graph $J(x+y, x)$ whose vertices are the subsets of cardinality x chosen from a set of cardinality $x+y$, and whose edges join subsets which intersect in a set of cardinality $x-1$. For our purposes, only $J(7, 3)$ is relevant.

In the range $3 \leqslant k \leqslant 13$, $3 \leqslant d \leqslant 5$ there remain just 24 feasible arrays which might correspond to an automorphic graph. These can be divided as follows. There are six arrays for which it has been shown, by arguments special to each case, that no automorphic graph exists. They are:

$d = 3$	$\{5, 4, 3; 1, 1, 2\}$	$(n = 56)$
	$\{7, 6, 6; 1, 1, 2\}$	$(n = 176)$
	$\{8, 7, 5; 1, 1, 4\}$	$(n = 135)$
	$\{13, 10, 7; 1, 2, 7\}$	$(n = 144)$
$d = 4$	$\{5, 4, 3, 3; 1, 1, 1, 2\}$	$(n = 176)$
	$\{10, 5, 4, 2; 1, 2, 2, 10\}$	$(n = 96)$.

Then there are eleven arrays for which the existence problem is, as yet, unresolved. They are:

$d = 3$	$\{10, 8, 7; 1, 1, 4\}$	$(n = 231)$
	$\{10, 6, 4; 1, 2, 5\}$	$(n = 65)$
	$\{11, 10, 4; 1, 1, 5\}$	$(n = 210)$
	$\{12, 10, 5; 1, 1, 8\}$	$(n = 208)$
	$\{12, 10, 2; 1, 2, 8\}$	$(n = 88)$
	$\{12, 10, 3; 1, 3, 8\}$	$(n = 68)$
	$\{12, 9, 9; 1, 1, 4\}$	$(n = 364)$

$d = 4$	$\{10, 8, 8, 8; 1, 1, 1, 5\}$	$(n = 1755)$
	$\{10, 8, 8, 2; 1, 1, 4, 5\}$	$(n = 315)$
	$\{12, 8, 8, 8; 1, 1, 1, 3\}$	$(n = 2925)$
	$\{12, 8, 6, 4; 1, 1, 2, 9\}$	$(n = 525)$.

Finally, there are seven arrays for which an automorphic graph is known to exist.

$d = 3$	$\{5, 4, 2; 1, 1, 4\}$	$(n = 36)$
	$\{6, 5, 2; 1, 1, 3\}$	$(n = 57)$
	$\{6, 4, 4; 1, 1, 3\}$	$(n = 63)$
$d = 4$	$\{3, 2, 2, 1; 1, 1, 1, 2\}$	$(n = 28)$
	$\{7, 6, 4, 4; 1, 1, 1, 6\}$	$(n = 330)$
	$\{9, 8, 6, 4; 1, 1, 3, 8\}$	$(n = 280)$
	$\{11, 10, 6, 1; 1, 1, 5, 11\}$	$(n = 266)$.

The graphs corresponding to these seven arrays are of great significance, from the viewpoint of both combinatorics and group theory. For instance, the 36 vertex graph [2] is related to the existence of an outer automorphism of the symmetric group of degree six (the only symmetric group admitting such an automorphism). The 330 vertex graph is constructed using the Steiner system $S(5, 8, 24)$, and the 266 vertex graph is related to Janko's smallest simple group [5, p.223]. The resemblance between the sporadic nature of feasible arrays and the isolated occurrences of combinatorial and group-theoretical phenomena is remarkable, and it should provide impetus for the further study of automorphic graphs.

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