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On the duality of interaction models

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Abstract. Two kinds of duality arising in studies of interaction models are discussed. The first kind, which has not previously been investigated, is related to algebraic properties of the coefficient ring. The second kind is the well-known geometric duality for planar graphs. The two dualities together lead to a perfectly symmetrical relationship for a general form of partition function.

1. *Introduction.* The prototype for combinatorial models of physical phenomena is the famous Lenz–Ising model of ferromagnetism, now over fifty years old. This stem has produced many offshoots, some of them straightforward generalisations such as the Potts (3) model, and others, such as the ice model (Pauling (2)), whose relationship with the prototype is less clear. However, it is apparent that the correct setting for all such models is the space of functions defined on the vertices and edges of a graph.

One concept which arises frequently in the discussion of the relationships between models is the idea of duality, as applied to planar graphs. In this context the idea dates back to a suggestion of Onsager (Wannier (7)) for obtaining the critical temperature of the Ising model on the plane square lattice. A recent survey of such methods is given by Syozi (4).

The purpose of this paper is to explain that there are really two kinds of duality involved in the study of interaction models. There is the 'geometric' duality, which applies only to planar graphs, and an 'algebraic' duality related to the choice of coefficient ring; the latter applies to any graph, and so it may prove to be more powerful in applications. Although several instances of the interplay between the two dualities have been noticed, most recently by Wu and Wang (9), the general nature of the relationship has been obscure.

An important feature of the theory is the connection between the generalized problem of 'colouring' the vertices of a graph and 'flows' on the edges of the graph. The formalization of this idea enables us to express models like the ice model directly in terms of interaction models, without the intrusion of geometric duality. The well-known equivalence between the ice model on a square lattice graph and the three-colouring problem is then seen to be a fortuitous accident, dependent upon the self-dual property of the graph. The true nature of the relationship may be more informative than the misleading special case.

2. *Graph functions with values in a ring.* We shall be concerned with functions defined on the vertices or edges of a graph and taking values in a ring. When the

function is defined on vertices, its values are often thought of as colours. There is no loss due to the introduction of ring structure, since any finite set of n elements is in 1–1 correspondence with some ring – the ring $\mathbb{Z}_n$ of integers modulo n, for example.

Let G be a graph, which we take to be simple; that is, it has no loops or multiple edges. The vertex-set and edge-set of G will be denoted by V and E respectively. We shall suppose that G is given an arbitrary *orientation*. In other words, for each edge e, one of the two incident vertices is chosen to be the positive end of e, and the other is chosen to be the negative end. The entries of the *incidence matrix* of G are defined as follows:

$$D_{ve} = \begin{cases} +1 & \text{if } v \text{ is the positive end of } e; \\ -1 & \text{if } v \text{ is the negative end of } e; \\ 0 & \text{if } v \text{ is not incident with } e. \end{cases}$$

The introduction of an orientation is necessary in order to yield satisfactory definitions, but the actual orientation chosen is immaterial.

Let A be a ring. (To avoid confusion, we shall postulate that A is finite, has a multiplicative identity, and that multiplication is commutative.) The set of functions $c: V \to A$ also has a ring structure, given by the rules

$$(c_1+c_2)(v) = c_1(v)+c_2(v),$$
$$(c_1 . c_2)(v) = c_1(v) . c_2(v).$$

This ring will be denoted by $C_0(G; A)$, and the analogous ring of functions $f: E \to A$ will be denoted by $C_1(G; A)$.

The *boundary* $\partial: C_1(G;A) \to C_0(G;A)$ and the *coboundary* $\delta: C_0(G;A) \to C_1(G;A)$ are defined as follows:

$$(\partial f)(v) = \sum_{e \in E} D_{ve} f(e) \quad (f: E \to A, \quad v \in V);$$
$$(\delta c)(e) = \sum_{v \in V} D_{ve} c(v) \quad (c: V \to A, \quad e \in E).$$

The right-hand side of $\delta c(e)$ contains only two non-zero summands, corresponding to the two ends of e, so that

$$(\delta c)(e) = c(x) - c(y),$$

where x and y are the positive and negative ends of e.

LEMMA 1. *Let c in $C_0(G;A)$ and f in $C_1(G;A)$ be given. Then*

$$\sum_{v \in V} (\partial f . c)(v) = \sum_{e \in E} (f . \delta c)(e). \tag{1}$$

Proof.

$$\begin{aligned} \sum_v (\partial f . c)(v) &= \sum_v \partial f(v) . c(v) \\ &= \sum_v \sum_e D_{ve} f(e) c(v) \\ &= \sum_e \sum_v D_{ve} c(v) f(e) \\ &= \sum_e \delta c(e) f(e) \\ &= \sum_e (f . \delta c)(e). \blacksquare \end{aligned}$$

3. *Characters and transforms.* We shall denote the multiplicative group of complex numbers of modulus 1 by S. A *character* of an abelian group A is a group homomorphism $k: A \to S$; recalling that the group operation in A is written additively, this means that

$$k(a+b) = k(a)\,k(b) \quad (a, b \in A).$$

For any character we must have $k(0) = 1$, and the character k is said to be non-trivial if $k(a) \neq 1$ for some $a \in A$. If b is an element of order m, then $k(b)$ is a complex mth root of unity.

Now suppose that A is also a ring, and consider the sum

$$\sum_{a \in A} k(ab).$$

If $b = 0$, then the sum is equal to $|A|$. We shall say that k is *ring-like* if the sum is zero for all $b \neq 0$. In many cases this property is actually a consequence of prior properties of k, but it is convenient to postulate it explicitly.

Given any ring-like character k, and any function x from A to the complex numbers $\mathbb{C}$, we define the *k-transform* of x to be the function $\hat{x}: A \to \mathbb{C}$ given by

$$\hat{x}(a) = |A|^{-\frac{1}{2}} \sum_{b \in A} x(b)\,k(-ab). \tag{2}$$

It is a simple consequence of the ring-like property that this transform may be inverted by the formula

$$x(a) = |A|^{-\frac{1}{2}} \sum_{b \in A} \hat{x}(b)\,k(ab). \tag{3}$$

4. *The interaction model.* We now turn to matters more directly related to models of physical phenomena. We may think of the vertices of a graph as a set of 'sites', where edges link those sites which are 'nearest neighbours'. Each site may have one of several 'configurations', and an assignment of a configuration to each site defines a 'state' of the model. Examples will be given after the general framework has been established.

We consider a graph G and a function $x: A \to \mathbb{C}$, where A is a ring. For each function c in $C_0(G; A)$ and each edge e in E we have an 'interaction' $x[\delta c(e)]$, which depends only on x and the values of c at the ends of e. The expression

$$Z(G; x) = \sum_{c \in C_0} \prod_{e \in E} x[\delta c(e)] \tag{4}$$

will be called the *Z-function* related to the *interaction model* formed by G, x and A. Several instances of the interaction model are quite well-known. In theoretical physics, the most famous is the Ising model (with zero external field), which is obtained by taking A to be the ring $\mathbb{Z}_2$ and x to be the function given by

$$x(0) = e^{\beta J}, \quad x(1) = e^{-\beta J}.$$

The Potts model is obtained by replacing $\mathbb{Z}_2$ by $\mathbb{Z}_q$, and defining x to be $e^{-\beta J}$ for all non-zero elements of the ring.

In graph theory the most familiar case is the chromatic function, which gives the number of proper colourings of the graph; the adjective 'proper' signifies that adjacent vertices must have different colours. This is obtained from the general case by taking A to be the ring of integers modulo n (the number of colours available) and x to be the function $x(0) = 0$, $x(a) = 1$ $(a \neq 0)$.

The foregoing examples have a 2-*valued* property; there are numbers α and β such that
$$x(0) = \alpha, \quad x(a) = \beta \quad (a \neq 0).$$
It is clear that in such cases Z depends essentially only on the ratio α/β, so that there is no loss in taking $\beta = 1$. For a given complex number t, we shall use the same symbol to denote the typical 2-valued function:
$$t(0) = t, \quad t(a) = 1 \quad (a \neq 0). \tag{5}$$
With this convention the chromatic function is $Z(G; 0)$.

Our main result is a relationship between the Z-function and another function, defined in terms of the elements of $C_1(G; A)$. Let us denote the element of $C_0(G; A)$ which is zero at every vertex by 0. Then an element of $C_1(G; A)$, for which $\partial f = 0$, will be called a *flow* on G, and the set of all such flows will be denoted by $K_1 = K_1(G; A)$. K_1 is the kernel of the group homomorphism ∂. If we think of $f(e)$ as a quantity flowing along e from its negative to its positive end, then the condition $\partial f = 0$ means that, at each vertex, there is no net accumulation of flow. Taking G, x, and A as before, we define
$$Y(G; x) = \sum_{f \in K_1} \prod_{e \in E} x[f(e)]. \tag{6}$$
When x is the 2-valued function t, with $t = 0$, the product is zero unless $f(e) \neq 0$ for all edges e; thus $Y(G; 0)$ is the number of flows on G which vanish on no edge of G. This function has been known for nearly thirty years (Tutte (5)), but its relationship with physical models has not hitherto been noticed.

Let G be a connected regular graph of valency 4. Since each vertex has even valency, G has an Eulerian path, that is, a path which uses each edge just once. Let us give G an orientation by following the direction in which a fixed Eulerian path is traversed, so that each vertex is the positive end of two edges and the negative end of two edges. Suppose that f is an element of $K_1(G; \mathbb{Z}_3)$ with the property that $f(e)$ is never zero. The possible values of f are 1 and 2, and we may define an assignment of 'arrows' to the edges of G as follows: the arrow on e points towards its positive end if $f(e) = 1$, but towards its negative end if $f(e) = 2$. The condition $\partial f = 0$ ensures that, at each vertex, two arrows point inwards and two arrows point outwards. In other words, the so-called 'ice condition' is satisfied, and $Y(G; 0)$ is the number of ice states on G.

Similarly, if G is a regular graph with valency 3, and A is the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$, it turns out that $Y(G; 0)$ is the number of 'Tait colourings' of G. It may also be of interest to recall a strangely intractable problem of Tutte (6). Let A be the ring $\mathbb{Z}_5$; Tutte's conjecture is simply that $Y(G; 0)$, for this ring and any graph G, is not zero. In other words, every graph has a flow over $\mathbb{Z}_5$ which vanishes nowhere.

5. *The main theorem*. We need an elementary lemma, which will facilitate the manipulation of cumbersome expressions.

LEMMA 2. *Let X and Y be finite sets, and ϕ a complex-valued function defined on $X \times Y$. Then*
$$\prod_{x \in X} \sum_{y \in Y} \phi(x, y) = \sum_{F: X \to Y} \prod_{x \in X} \phi(x, F(x)). \tag{7}$$

Proof. The identity is really just a general form of the distributive law. For example, when $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$, it reduces to

$$\{\phi(x_1, y_1) + \phi(x_1, y_2)\}\{\phi(x_2, y_1) + \phi(x_2, y_2)\} = \sum_F \phi(x_1, F(x_1))\,\phi(x_2, F(x_2)).$$

This may be verified explicitly, and the general result proved by induction on $|X|$ and $|Y|$. ∎

We now begin the proof of our main theorem, which establishes a relationship between the functions Z and Y for the same graph. To begin, we need the result of operating on (1) with a character.

LEMMA 3. *Let k be a character on the ring A, and suppose that c and f belong to $C_0(G; A)$ and $C_1(G; A)$ respectively. Then*

$$\prod_{v \in V} k[\partial f(v)\, c(v)] = \prod_{e \in E} k[f(e)\, \delta c(e)]. \tag{8}$$

Proof. Apply k to the identity (1) and use the fact that k is a homomorphism from the abelian group A into the multiplicative group S. ∎

LEMMA 4. *Suppose that k and f are as in the preceding lemma, and that in addition k is ring-like. Let $C_0 = C_0(G; A)$; then*

$$\sum_{c \in C_0} \prod_{e \in E} k[\delta c(e) f(e)] = \begin{cases} |A|^{|V|} & \text{if} \quad \partial f = 0; \\ 0 & \text{if} \quad \partial f \neq 0. \end{cases} \tag{9}$$

Proof.

$$\begin{aligned} \sum_{c \in C_0} \prod_{e \in E} k[\delta c(e) f(e)] &= \sum_{c \in C_0} \prod_{v \in V} k[c(v)\, \partial f(v)] \quad \text{by} \quad (8) \\ &= \prod_{v \in V} \sum_{a \in A} k[a \,.\, \partial f(v)] \quad \text{by} \quad (7). \end{aligned}$$

Since k is ring-like, each sum is zero unless $\partial f(v) = 0$, when it takes the value $|A|$. Hence the product is zero unless $\partial f = 0$, when it takes the value $|A|^{|V|}$. ∎

THEOREM. *The functions Z and Y are related by the identity*

$$Z(G; x) = |A|^{|V| - \frac{1}{2}|E|}\, Y(G; \hat{x}), \tag{10}$$

where $\hat{x}$ is the k-transform of x and k is any ring-like character on A.

Proof. From the definition of Z (4) and the inversion formula for k-transforms (3), we have

$$\begin{aligned} Z(G; x) &= \sum_c \prod_e x[\delta c(e)] \\ &= \sum_c \prod_e |A|^{-\frac{1}{2}} \sum_b \hat{x}(b)\, k[\delta c(e)\, b] \\ &= |A|^{-\frac{1}{2}|E|} \sum_c \prod_e \sum_b \hat{x}(b)\, k[\delta c(e)\, b]. \end{aligned}$$

Applying (7) to the $\prod \sum$ term, we obtain

$$\begin{aligned} |A|^{\frac{1}{2}|E|} Z(G; x) &= \sum_{c \in C_0} \sum_{f \in C_1} \prod_e \hat{x}[f(e)]\, k[\delta c(e) f(e)] \\ &= \sum_{f \in C_1} \sum_{c \in C_0} \prod_e \hat{x}[f(e)]\, k[\delta c(e) f(e)]. \end{aligned}$$

Now the $\Pi\hat{x}$ product does not depend on c, and so it may be moved to the left of the sum over C_0, giving

$$|A|^{\frac{1}{2}|E|}Z(G;x) = \sum_{f\in C_1} \{\prod_e \hat{x}[f(e)]\} \sum_{c\in C_0} \prod_e k[\delta c(e) f(e)].$$

By (9), the final $\Sigma\Pi$ term is zero unless $\partial f = 0$, when it takes the value $|A|^{|V|}$. Hence

$$|A|^{\frac{1}{2}|E|}Z(G;x) = |A|^{|V|} \sum_{f\in K_1} \prod_e \hat{x}[f(e)]$$
$$= |A|^{|V|} Y(G;\hat{x}). \blacksquare$$

The k-transform of a 2-valued function is also 2-valued. In particular, we have the following

COROLLARY. *The number $Y(G;0)$ of nowhere zero flows on G is expressible in terms of Z-function of a 2-valued interaction model on G:*

$$Y(G;0) = (-1)^{|E|}|A|^{-|V|}Z(G;1-|A|). \blacksquare$$

For example, we may obtain expressions for the number of ice states on a 4-valent graph, and the number of Tait colourings of a 3-valent graph, in terms of suitable interaction models.

6. *Dual graphs.* The result given in section 5 is a consequence of the duality between x and its k-transform $\hat{x}$. We shall now consider a different duality – the geometric notion for planar graphs.

In this section, G will always denote a connected planar graph. It is well-known (see Wilson (8) for example) that such a graph G has a *dual* graph G^*; the vertices of G^* are in 1–1 correspondence with the regions into which the plane is divided by a planar representation of G, and the edges of G^* join vertices corresponding to adjacent regions. It follows that there is a 1–1 correspondence between the edge-set E^* of G^* and the edge-set E of G. In a pictorial representation, we emphasize this correspondence by drawing corresponding edges so that they intersect at right angles.

We shall continue to suppose that G has been given some arbitrary orientation. The planar representation then yields a compatible orientation of G^*, by the rule that the positive directions along corresponding edges may be brought into coincidence by a clockwise rotation through $\frac{1}{2}\pi$. Consequently, the boundary ∂^* and the coboundary δ^* in G^* may be defined. Suppose that e^* is the edge of G^* corresponding to the edge e of G; then each f in $C_1(G;A)$ has a corresponding f^* in $C_1(G^*;A)$, defined by

$$f^*(e^*) = f(e).$$

Furthermore, the rules for orientation imply that $\partial f = 0$ if and only if $f^* = \delta^* c^*$, for some c^* in $C_0(G^*;A)$. If we write I_1^* for the image of $\delta^*: C_0(G^*;A) \to C_1(G^*;A)$, then we have

$$f \in K_1 \Leftrightarrow f^* \in I_1^*. \tag{11}$$

If we are given that f^* is in I_1^*, so that there is some c^* for which $\delta^* c^* = f^*$, then each of the $|A|$ functions c_a^* $(a\in A)$ defined by

$$c_a^*(v^*) = c^*(v^*) + a,$$

satisfies $\delta^* c_a^* = f^*$. These are the only members of $C_0(G^*; A)$ with this property; the proof of this is just a modification of the usual argument for the rank of the incidence matrix, as given, for example, in Biggs (1).

We have exhibited a $|A|$-to-1 correspondence between the elements of $C_0^* = C_0(G^*; A)$ and those of I_1^*. By (11), the latter are in 1–1 correspondence with the flows on G, that is, the members of $K_1(G; A)$. Consequently,

$$\begin{aligned} Z(G^*; x) &= \sum_{c \in C_0^*} \prod_{e^*} x[\delta^* c^*(e^*)] \\ &= |A| \sum_{f^* \in I_1^*} \prod_{e^*} x[f^*(e^*)] \\ &= |A| \sum_{f \in K_1} \prod_{e} x[f(e)]. \end{aligned}$$

In other words,

$$Z(G^*; x) = |A| \, Y(G; x). \tag{12}$$

When $x = 0$ this reduces to an old result of Tutte (5).

Combining the two dualities, (10) and (12), we obtain

$$Z(G^*; x) = |A|^{N^*} Z(G; \hat{x}), \tag{13}$$

where $N^* = |V^*| - \frac{1}{2}|E^*| - 1$. It is worth remarking that this result is perfectly symmetric, since $|E^*| = |E|$, and by Euler's formula

$$|V| - |E| + |V^*| = 2;$$

consequently, writing N for $|V| - \frac{1}{2}|E| - 1$, we have

$$N + N^* = 0.$$

7. *Conclusion.* The double duality expressed in equation (13) underlies most applications of duality to physical models. However, it is clear that (13) is the consequence of two more fundamental results, (10) and (12). In particular, equation (10), which holds without any restrictions on the graph G, may well find applications in theoretical physics.

Equation (12) yields an expression for $Y(G; 0)$ in terms of the chromatic function $Z(G^*; 0)$ of the dual graph G^*, when G is planar. If G is 4-valent and A is the ring $\mathbb{Z}_3$, this means that the number of ice states on G is three times the number of proper 3-colourings of G^*. But, whether or not G is planar, the Corollary in section 5 tells us that the number of ice states on G is the Z-function $Z(G; -2)$ of a suitable interaction model. Only in the self-dual case $G = G^*$ do we get the well-known equivalence with a colouring problem.

REFERENCES

(1) Biggs, N. L. *Algebraic graph theory* (Cambridge University Press, 1974).

(2) Pauling, L. The structure and entropy of ice and of other crystals with some randomness of atomic arrangement. *J. Amer. Chem. Soc.* **57** (1935), 2680–2684.

(3) Potts, R. B. Some generalized order–disorder transformations. *Proc. Cambridge Philos. Soc.* **48** (1952), 106–109.

(4) Syozi, I. Transformation of Ising models. *Phase transitions and critical phenomena*, vol. 1. (London; Academic Press, 1972).

(5) TUTTE, W. T. A ring in graph theory. *Proc. Cambridge Philos. Soc.* **43** (1947), 26–40.
(6) TUTTE, W. T. On the imbedding of linear graphs in surfaces. *Proc. London Math. Soc.* **51** (1949), 474–483.
(7) WANNIER, G. H. The statistical problem in cooperative phenomena. *Rev. Mod. Phys.* **17** (1945), 50–60.
(8) WILSON, R. J. *Introduction to graph theory* (Edinburgh; Oliver and Boyd, 1972).
(9) WU, F. Y. and WANG, Y. K. Duality transformation in a many-component spin model. *J. Math. Phys.* **17** (1976), 439–440.