COLOURING SQUARE LATTICE GRAPHS

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1. *The colouring matrix*

We shall approximate to the infinite plane square lattice graph by means of the finite toroidal square lattice graphs. Although the methods used are well-known in mathematical physics (see for example Baxter [1] and Lieb [7]) they have yet to be exploited fully in graph theory.

Let k be a natural number and let $\mathbb{Z}_k$ denote the additive group of integers modulo k, $\mathbb{Z}_k = \{0, 1, \ldots, k-1\}$. For any natural numbers m, n, we define the toroidal square lattice graph $S_{m,n}$ as follows. The vertices are the elements of $\mathbb{Z}_m \times \mathbb{Z}_n$ and two vertices (i, j), (i', j') are adjacent if $i = i'$ and j, j' differ by one, or if $j = j'$ and i, i' differ by one. In particular, the graph $S_{1,n}$ is the n-circuit C_n $(n \geqslant 2)$. Let $c_n(z)$ denote the chromatic polynomial of this graph, that is [2, p. 59]

$$c_n(z) = C(C_n; z) = (z-1)^n + (-1)^n (z-1).$$

In addition to the assumption $n \geqslant 2$ we shall take z to be an integer not less than 3, so that $c_n(z)$ is a positive integer.

Let us say that two z-colourings γ, δ of C_n are *compatible* if $\gamma(v) \neq \delta(v)$ for each vertex v of C_n. Define a matrix $T_n(z)$, whose rows and columns correspond to the z-colourings of C_n, by the rule:

$$\big(T_n(z)\big)_{\gamma\delta} = \begin{cases} 1 & \text{if } \gamma \text{ and } \delta \text{ are compatible;} \\ 0 & \text{otherwise.} \end{cases}$$

We shall show that the chromatic polynomial of $S_{m,n}$ is given by

$$C(S_{m,n}; z) = \text{trace}\,[T_n(z)^m]. \tag{1}$$

For the proof let $T = T_n(z)$. For each fixed i in $\mathbb{Z}_m$ the vertices (i, j) of $S_{m,n}$ form an n-circuit $C_n^{(i)}$ and a z-colouring ζ of $S_{m,n}$ induces a z-colouring ζ_i of $C_n^{(i)}$. Conversely, if we are given such z-colourings of each $C_n^{(i)}$ then the expression

$$T_{\zeta_0\zeta_1} T_{\zeta_1\zeta_2} \cdots T_{\zeta_{m-1}\zeta_0}$$

is equal to one if $\zeta_0, \zeta_1, \ldots, \zeta_{m-1}$ unite to give a z-colouring of $S_{m,n}$ and is equal to zero otherwise. Thus the number of z-colourings of $S_{m,n}$ is

$$\sum_{\zeta_0, \ldots, \zeta_{m-1}} T_{\zeta_0\zeta_1} T_{\zeta_1\zeta_2} \cdots T_{\zeta_{m-1}\zeta_0} = \text{trace}\,(T^m).$$

2. *The eigenvalues of $T_n(z)$*

The expression (1) for the chromatic polynomial of $S_{m,n}$ may be written in terms of the eigenvalues of $T_n(z)$:

$$C(S_{m,n}; z) = \sum \lambda(z)^m, \tag{2}$$

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where $\lambda(z)$ runs through the eigenvalues of $T_n(z)$. As an example, the case $n = 2$ is amenable to direct calculation; $T_2(z)$ is a matrix with $z(z-1)$ rows and columns and its eigenvalues are z^2-3z+3 (once), $3-z$ ($z-1$ times), $1-z$ ($z-1$ times) and 1 (z^2-3z+1 times). This gives the result for $S_{m,2}$, the skeleton of an m-prism, in agreement with [4]:

$$C(S_{m,2};z) = (z^2-3z+3)^m+(z-1)\{(3-z)^m+(1-z)^m\}+z^2-3z+1.$$

We shall require this result presently.

Since $T_n(z)$ is a matrix with non-negative entries, the classical theory of Perron and Frobenius [6; p. 286] may be applied. It is convenient to regard $T_n(z)$ as the adjacency matrix of a graph whose vertices are the z-colourings of C_n. Then it is easy to show that (except when $z = 3$) the graph is connected and is not bipartite. This means that among the eigenvalues of $T_n(z)$ there is a unique one $\lambda_n(z)$ with greatest absolute value, and $\lambda_n(z)$

(i) is positive,

(ii) has multiplicity one,

(iii) is not greater than the maximum row sum $M_n(z)$ of $T_n(z)$,

(iv) is not less than the mean row sum $m_n(z)$ of $T_n(z)$.

The exceptional case $z = 3$ is not a serious difficulty, since in that case we have the exact calculation of Lieb [7].

We wish to investigate the limit of the sequence $\{C(S_{n,n};z)^{1/n^2}\}$ for fixed z. This "chromatic limit" is the analogue, for the infinite square lattice graph, of the chromatic polynomial for a finite graph. It is known [3] that

$$L(z) = \lim_{n\to\infty} C(S_{n,n};z)^{1/n^2}$$

exists for each $z \geqslant 4$, and that the same limit is obtained when $S_{n,n}$ is replaced by an $n\times n$ subgraph of the plane square lattice. The corresponding results for $z = 3$ may be deduced from a paper of Brascamp, Kunz and Wu [5].

The properties of $\lambda_n(z)$ and the formula (2) show that

$$L(z) = \lim_{n\to\infty} \lambda_n(z)^{1/n}.$$

We shall obtain upper and lower bounds for this limit.

3. *Bounds for* $\lambda_n(z)$

The mean row sum of $T_n(z)$ is easily found. The sum of all the entries of $T_n(z)$ is just the number of compatible pairs of z-colourings of C_n, and this is the same as the number of z-colourings of the n-prism $S_{2,n}$. Thus

$$m_n(z) = \frac{(z^2-3z+3)^n+(z-1)\{(3-z)^n+(1-z)^n\}+z^2-3z+1}{(z-1)^n+(-1)^n(z-1)}.$$

When $n = 2$ or $n = 3$ all colourings of C_n are alike and $M_n(z) = \lambda_n(z) = m_n(z)$. In fact $\lambda_2(z) = z^2-3z+3$, $\lambda_3(z) = z^3-6z^2+14z-13$. When $n > 3$ the bounds are not equal and the upper bound may be estimated as follows.

We wish to find an upper bound for the number of z-colourings of C_n which are compatible with a given such colouring γ.

Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of C_n and let $u_0, u_1, \ldots, u_{n-1}$ be the vertices of a path of length n. (The reason for choosing a path rather than a circuit is simply convenience; the order of magnitude of the result is not affected.) Let X_i $(0 \leqslant i \leqslant n-1)$ denote the number of z-colourings ε of the path from u_0 to u_i such that

$$\varepsilon(u_j) \neq \gamma(v_j) \quad (0 \leqslant j \leqslant i).$$

Then it is clear that

$$X_0 = z-1, \quad X_1 = z^2-3z+3, \quad X_{n-1} \geqslant M_n(z).$$

Each of the X_{i-1} colourings of the path from u_0 to u_{i-1} may be extended in (at least) $z-2$ ways to u_i, since we have only to ensure that $\varepsilon(u_i)$ is not either $\varepsilon(u_{i-1})$ or $\gamma(v_i)$. Let Y_{i-1} be the cardinality of the subset of the X_{i-1} colourings which have the property $\varepsilon(u_{i-1}) = \gamma(v_i)$. Each of these colourings may be extended in $z-1$ ways to u_i, and so $X_i = (z-2)X_{i-1} + Y_{i-1}$. But every one of the Y_{i-1} colourings induces, by restriction, a colouring of the path from u_0 to u_{i-2}, and these induced colourings are all different. Consequently, $Y_{i-1} \leqslant X_{i-2}$ and we have

$$X_i \leqslant (z-2)X_{i-1} + X_{i-2} \quad (2 \leqslant i \leqslant n-1).$$

From this we deduce that

$$X_{n-1} \leqslant p_{n-2} X_1 + p_{n-3} X_0,$$

where the coefficients p_k are given by the recursion $p_0 = 1$, $p_1 = z-2$,

$$p_k = (z-2)p_{k-1} + p_{k-2} \ (k \geqslant 2).$$

Explicitly

$$p_k = (\tfrac{1}{2})^k (z-2+\surd(z^2-4z+8))^k + (\tfrac{1}{2})^k (z-2-\surd(z^2-4z+8))^k.$$

Since $X_{n-1} \geqslant M_n(z)$ we have an explicit upper bound for $\lambda_n(z)$.

Taking the limit as $n \to \infty$.

$$\tfrac{1}{2}(z-2+\surd(z^2-4z+8)) \geqslant L(z) \geqslant \frac{z^2-3z+3}{z-1}.$$

When $z = 3$ the bounds are $\frac{1}{2}(1+\surd 5) = 1{\cdot}618\ldots$ and $3/2 = 1{\cdot}500\ldots$. For this case the calculations of Lieb [7] give the exact result $L(3) = (4/3)^{3/2}$, which is numerically $1{\cdot}540\ldots$. For larger values of z the bounds are better, since they differ by a term of order z^{-2}.

References

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