Girth, Valency, and Excess

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ABSTRACT

There is a simple lower bound for the number of vertices of a regular graph whose girth and valency are specified. If the graph is required to have certain additional properties, then the number of excess vertices needed may be unbounded.

1. INTRODUCTION

Let G be a regular graph with valency $k\geqslant 3$ and odd girth $g=2r+1$ $(r\geqslant 2)$. There is a simple lower bound

$$n_0(g,k)=1+\frac{k}{k-2}\{(k-1)^r-1\}$$

for the number of vertices of G, and it has been proved by Bannai and Ito [1], and by Damerell [4], that the bound can be attained only when $g=5$ and $k=3$, 7, or 57. On the other hand, attempts to find general constructions for graphs with given girth and valency seem to result, at best, in much larger graphs than the lower bound would predict. Bollobás [3] gives an account of the present state of knowledge, and asks some interesting questions. Many of these concern the behavior of the number of excess vertices, $n-n_0(g,k)$, where n is the smallest possible order of a graph G with the required properties.

Bollobás [3, p. 163] suggests that the excess cannot be arbitrarily large. In view of this, it may be of interest to show that for restricted classes of graphs the excess does tend to infinity as a function of g or as a function of k.

Let $\mathcal{P}$ be any property of graphs, and define $n(\mathcal{P};g,k)$ to be the minimum number of vertices of a regular graph with property $\mathcal{P}$ which has girth g and valency k, if such a graph exists. The existence of $n(\mathcal{P};g,k)$, and

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the behavior of the *excess*

$$e(\mathcal{P};g,k)=n(\mathcal{P};g,k)-n_0(g,k),$$

provide interesting unsolved problems for several properties. Although we shall not discuss them here, the Hamiltonian property and the property of vertex transitivity seem to be worth investigating. Our first result concerns the following property $\mathcal{D}$. The graph G has property $\mathcal{D}$ if there is a subset C of its vertices such that the sets

$$D(c)=\{x|x=c \text{ or } x \text{ is adjacent to } c\} \qquad (c\in C)$$

partition the vertex set of G. (Elsewhere, C has been called a perfect 1-code in G.) We shall prove that if $n(\mathcal{D};g,k)$ exists, then the excess $e(\mathcal{D};g,k)$ tends to infinity with g for fixed $k\geqslant 3$, and it tends to infinity with k for fixed $g\geqslant 5$.

Our second result concerns the property $\mathcal{C}_t$ of being t-colorable, in the usual graph-theoretic sense. In this case, we shall show not only that the excess $e(\mathcal{C}_t;5,k)$ tends to infinity with k, but that the "proportional excess" e/n_0 is bounded away from zero.

The proofs use fairly standard methods of algebraic graph theory, and these are explained more fully in [2]. Probably the results could be improved and extended by refining the basic approach.

2. A LOWER BOUND FOR EXCESS

Let G be a connected graph with n vertices and diameter d. For integers i in the range $0\leqslant i\leqslant d$ we define the $n\times n$ matrix $A_i=A_i(G)$ as follows. The rows and columns of A_i correspond to the vertices of G, and the entry in position (v,w) is 1 if the distance $\partial(v,w)$ between the vertices v and w is i, and zero otherwise. Clearly, $A_0=I$, $A_1=A$ (the usual adjacency matrix of G), and

$$\sum_{i=0}^{d} A_i=J,$$

where each entry of J is 1.

Now suppose that G is regular, with valency k, and has girth $g=2r+1\geqslant 5$. We claim that

$$\sum_{i=1}^{r} A_i=F_r(A),$$

where F_r is the polynomial expression defined by the recursion $F_0(x)=1$, $F_1(x)=x+1$, $F_s(x)=xF_{s-1}(x)-(k-1)F_{s-2}(x)$ $(s\geq 2)$. This follows from the matrix identities

$$A_2=A_1^2-kI, \qquad A_s=A_1A_{s-1}-(k-1)A_{s-2} \quad (3\leq s\leq r),$$

which in turn follow from the fact that if v and w are vertices with $\partial(v,w)=s\leq r$, then there is a unique path (without repeated vertices) joining them.

Since G is regular and connected, the matrix J is a polynomial function of A [2, p. 15]. Hence $F_r(A)+E=J$, where $E=A_{r+1}+\cdots+A_d$ is also a polynomial $E(A)$. Thus if λ is an eigenvalue of A, then $F_r(\lambda)+E(\lambda)$ is an eigenvalue of J. The eigenvalue $\lambda=k$ corresponds to the eigenvalue n of J, and we note that $F_r(k)=n_0(g,k)$, so that $E(k)=n-n_0=e$ is an eigenvalue of E. In fact, each row and column of E sums to e, and it follows that every eigenvalue μ of E satisfies $|\mu|\leq e$.

Now if $\lambda\neq k$ is an eigenvalue of A, then $F_r(\lambda)+E(\lambda)$ is the zero eigenvalue of J, and $|E(\lambda)|\leq e$. Thus $|F_r(\lambda)|\leq e$. This is our lower bound for the excess. To summarize: if G is a graph with girth g, valency k, and order $n_0(g,k)+e$, then every eigenvalue $\lambda\neq k$ of G satisfies

$$e\geq |F_r(\lambda)|, \qquad (*)$$

where $\{F_s\}$ is the sequence of polynomials defined above.

3. PROPERTIES WITH UNBOUNDED EXCESS

It may be possible to derive general results from the inequality (*), by means of a careful study of the polynomials $\{F_s\}$. Here we shall simply assume that G has certain properties which are known to have spectral consequences.

Let $\mathfrak{P}$ be the property of having a perfect 1-code C, as defined in the introduction. We define a column vector z, whose n rows correspond to the vertices of G, as follows:

$$(z)_v=\begin{cases} -k & \text{if} \quad v\in C, \\ 1 & \text{if} \quad v\notin C. \end{cases}$$

It is easy to check that $Az=-z$, so that -1 is an eigenvalue of G. Thus (*) implies that

$$e(\mathfrak{P};g,k)\geq |f_r| \qquad (g=2r+1),$$

where $f_r = F_r(-1)$ is determined by the recursion $f_0 = 1$, $f_1 = 0$, $f_s = -f_{s-1} - (k-1)f_{s-2}$ $(s \geqslant 2)$. It is easy to see that f_r is a polynomial of degree $[\frac{1}{2}r]$ in $k-1$, so that for fixed $r \geqslant 2$, $|f_r|$ tends to infinity with k. Also, for fixed $k \geqslant 3$, $|f_r|$ tends to infinity with r, although this is rather more difficult to prove. (See, for example, [6].)

In the preceding example, the excess e is of order $(k-1)^{\frac{1}{2}r}$, while n_0 is of order $(k-1)^r$. Hence the proportional excess e/n_0 tends to zero with increasing k, for fixed $g \geqslant 5$. We now give an example where the proportional excess is bounded away from zero.

Let G be a graph with girth 5 and valency k which has the property $\mathcal{C}_t$ of being t-colorable. A result of Hoffman [5] (see also [2, p. 54]) tells us that the smallest eigenvalue $\lambda_{\min}$ of G satisfies

$$\lambda_{\min} \leqslant -\frac{k}{t-1}.$$

Our basic inequality (*) yields $e(\mathcal{C}_t; 5, k) \geqslant |F_2(\lambda_{\min})|$. Now the quadratic expression $F_2(x) = x^2 + x - (k-1)$ is positive and decreasing for $x \leqslant X = \frac{1}{2}(-1 - \sqrt{4k-3})$. Thus if $F_2(-k/(t-1)) \geqslant 0$, we must have $\lambda_{\min} \leqslant -k/(t-1) \leqslant X$, and $F_2(\lambda_{\min}) \geqslant F_2(-k/(t-1))$. Consequently,

$$e(\mathcal{C}_t; 5, k) \geqslant F_2(-k/(t-1)).$$

In fact, this result is always true, since it is trivially satisfied if the right-hand side is negative.

Explicitly, we have shown that

$$e(\mathcal{C}_t; 5, k) \geqslant (t-1)^{-2}k^2 - (t-1)^{-1}k - (k-1).$$

Since $n_0(5, k) = k^2 + 1$, we see that e/n_0 has order $(t-1)^{-2}$ as k increases. For example, a graph of girth 5 and large valency must, if it is to have a 3-coloring, possess about 25% more vertices than predicted by the simple lower bound n_0.

4. DISCUSSION

It will be noted that we have not discussed the existence problem for graphs with girth g, valency k, and a specific property $\mathcal{P}$. It seems that such problems are best attacked by probabilistic methods.

Finally, it is worth remarking that, for trivalent graphs, and quite small values of g, there are still many open questions. Particularly remarkable is the case $g=9$ and $k=3$, for which $n_0=46$. In spite of many attempts, no construction has produced a graph with fewer than 60 vertices. Some years ago I suggested that one might be able to set up a construction on the basis that a "perfect 1-code" existed, and H. S. M. Coxeter succeeded in doing this. His graph was the fifth (9,3) graph with 60 vertices to be discovered. I understand that there are now at least twenty different graphs of this kind known.

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