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Graphs with even girth and small excess

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1. *Introduction.* Let G be a regular graph with even girth $g = 2r \geqslant 4$ and valency $k \geqslant 3$. It is well known, and easy to prove, that G must have at least $n_0 = n_0(g, k)$ vertices, where

$$n_0(g, k) = 2\{1 + (k-1) + (k-1)^2 + \ldots + (k-1)^{r-1}\}.$$

It is also well known, but harder to prove, that, if there is a graph with the stated properties and exactly n_0 vertices, then g must be 4, 6, 8 or 12 (**3, 8, 10**). In the case $g = 4$, the minimal number of vertices $n_0 = 2k$ is attained for each value of k by the complete bipartite graph $K_{k,k}$. In the case $g = 6$, the existence of a graph with $n_0 = 2(k^2 - k + 1)$ vertices is equivalent to the existence of a projective plane of order $k-1$, that is, a symmetric $(v, k, 1)$-design with $v = k^2 - k + 1$. It is known that these designs exist for prime power values of $k-1$, but the existence question for many other values remains unsettled. In the cases $g = 8$ and $g = 12$ it is again possible to construct graphs with n_0 vertices when $k-1$ is a prime power (**2**), but here also the existence question is unresolved for most other values.

In view of the scarcity of graphs which attain the minimal number n_0, it is natural to investigate what happens when the number of additional vertices is small. Precisely, if G is a regular graph with girth g, valency k, and n vertices, then we define the *excess* e of G to be $n - n_0(g, k)$. This usage differs slightly from that employed in an earlier paper (**4**), but there should be no possibility of confusion.

For any given values of g and k it is possible to construct a regular graph with girth g and valency k: a recent survey of relevant methods is given by Bollobás (**5**). Thus the number

$$\min\{e \mid \exists G \text{ with girth } g, \text{ valency } k, \text{ excess } e\}$$

is defined for each g and k. It is possible that, for even values of $g \neq 4, 6, 8, 12$, the minimum tends to infinity with k. It is also possible that this remains true when

$g = 6$, 8, 12, if we exclude the graphs with $e = 0$ mentioned above. In this paper we begin the investigation of such questions, for the general case $g = 2r \geqslant 6$.

First, we show that if e is small ($e \leqslant k-2$) then G must be bipartite, and in particular e is even. Then we use algebraic methods to show that $e = 2$ is impossible for all $g = 2r \geqslant 8$. This rather unexpected result means that, for example, when $g = 8$ and $k-1$ is prime power there is a gap in the possible values of e: either $e = 0$ or $e \geqslant 4$. Finally, we investigate the case $g = 6$ in some detail. Here we obtain an interesting analogy between the case $e = 0$, when the graph corresponds to a symmetric $(v, k, 1)$-design, and some cases with $e \neq 0$, when the graphs correspond to symmetric (v, k, λ)-designs with $2(\lambda-1) = e$. We use this correspondence to show that $g = 6$ and $e = 2$ is impossible if $k \equiv 5$ or 7 (mod 8).

2. *The bipartition theorem.* We begin by introducing some notation, based on a standard decomposition for a graph G of girth $2r$. Choose an edge $\{\sigma, \tau\}$ of G and define, for $0 \leqslant i \leqslant r-1$,

$$S_i = \{\alpha \in VG \mid \partial(\sigma, \alpha) = i, \quad \partial(\tau, \alpha) = i+1\},$$
$$T_i = \{\alpha \in VG \mid \partial(\sigma, \alpha) = i+1, \quad \partial(\tau, \alpha) = i\}.$$

The fact that the girth of G is $2r$ implies that the sets S_i, T_i ($0 \leqslant i \leqslant r-1$) are mutually disjoint and, since G is regular of valency k, we have $|S_i| = |T_i| = (k-1)^i$. Let X denote the set of all remaining vertices of G, so that

$$X = \{\alpha \in VG \mid \partial(\sigma, \alpha) \geqslant r \quad \text{and} \quad \partial(\tau, \alpha) \geqslant r\}.$$

X is the *excess set* with respect to $\{\sigma, \tau\}$, and its cardinality is e, the excess of G. Occasionally we must specify $\{\sigma, \tau\}$, and write $X_{\sigma\tau}$ for X.

Since the girth of G is $2r$, there are no edges with both ends in S_i or both ends in T_i ($0 \leqslant i \leqslant r-1$), and no edges joining S_i to T_i ($0 \leqslant i \leqslant r-2$). However, there will be some edges joining S_{r-1} to T_{r-1}, and there may be edges joining X to S_{r-1}, X to T_{r-1}, and edges with both ends in X. For any vertex α and $W \subseteq VG$ we denote the set of vertices in W which are adjacent to α by $N(W, \alpha)$, and put $n(W, \alpha) = |N(W, \alpha)|$.

LEMMA 2·1. *Suppose that $r \geqslant 3$ and α, β are adjacent vertices of G, lying in $T_{r-1} \cup X$. Then*
$$n(S_{r-1}, \alpha) + n(S_{r-1}, \beta) \leqslant k-1.$$

The same result holds with S_{r-1} and T_{r-1} interchanged.

Proof. The $(k-1)^{r-1}$ vertices in S_{r-1} are partitioned into $k-1$ subsets of cardinality $(k-1)^{r-2}$, each subset consisting of those vertices which are at distance $r-2$ from a given vertex in S_1.

The sets $N(S_{r-1}, \alpha)$ and $N(S_{r-1}, \beta)$ are disjoint, otherwise G would contain a 3-cycle. Hence their union has cardinality $n(S_{r-1}, \alpha) + n(S_{r-1}, \beta)$. If this sum is greater than $k-1$ then, by the pigeon-hole principle, at least one of the $k-1$ subsets defined in the previous paragraph must contain two vertices θ, ϕ belonging to $N(S_{r-1}, \alpha) \cup N(S_{r-1}, \beta)$. Now θ and ϕ have a common vertex in S_1 at distance $r-2$ from both of them, and by construction there is a path of the form $\theta\alpha\beta\phi$, or $\theta\alpha\phi$, or $\theta\beta\phi$. Thus in any case we have a cycle of length $2r-1$ or less, and this contradiction gives the required result. ∎

LEMMA 2·2. *Suppose that $r \geqslant 3$ and α, β are adjacent vertices of G, with $\alpha \in X$, $\beta \in T_{r-1}$. Then*

$$n(X,\beta) \geqslant n(S_{r-1},\alpha),$$

and the same result holds with S_{r-1} and T_{r-1} interchanged.

Proof. The vertex β has k neighbours in all: one is in T_{r-2} and the others are in S_{r-1} or X. Hence

$$\begin{aligned} n(X,\beta) &= k-1-n(S_{r-1},\beta) \\ &\geqslant n(S_{r-1},\alpha) \quad \text{(by Lemma 2·1).} \end{aligned}$$ ∎

THEOREM A. *Let G be a regular graph with girth $2r \geqslant 6$, valency k, and excess e. If $e \leqslant k-2$ then G is bipartite and its diameter is $r+1$.*

Proof. We remark first that if a vertex in X has its k neighbours all in X then $e \geqslant k+1$. Thus, with the given hypothesis, every vertex in X must be adjacent to some vertex in $S_{r-1} \cup T_{r-1}$.

Suppose first that there is a vertex ξ in X adjacent to $\alpha \in S_{r-1}$ and $\beta \in T_{r-1}$. Then the sets $N(X,\xi)$, $N(X,\alpha)-\{\xi\}$, $N(X,\beta)-\{\xi\}$, $\{\xi\}$ are disjoint subsets of X, and so

$$\begin{aligned} e = |X| &\geqslant n(X,\xi)+n(X,\alpha)-1+n(X,\beta)-1+1 \\ &= n(X,\xi)+n(X,\alpha)+n(X,\beta)-1 \\ &\geqslant n(X,\xi)+n(T_{r-1},\xi)+n(S_{r-1},\xi)-1, \end{aligned}$$

by Lemma 2·2. Since the valency of ξ is k, and all the neighbours of ξ are in X, T_{r-1}, or S_{r-1}, it follows that $e \geqslant k-1$, contrary to hypothesis. Hence every vertex in X is joined to some vertices in one or other of S_{r-1}, T_{r-1}, but not both.

Define a partition $X = X_S \cup X_T$, such that X_S, X_T denote the subsets of X containing vertices joined to S_{r-1}, T_{r-1} respectively. Suppose that X_S contains two adjacent vertices ξ, η. By definition of X_S, there are vertices α, β in S_{r-1}, adjacent to ξ, η respectively. The sets $N(X,\xi)-\{\eta\}$, $N(X,\eta)-\{\xi\}$, $\{\xi\}$, $\{\eta\}$, are disjoint subsets of X, and so

$$\begin{aligned} e = |X| &\geqslant n(X,\xi)-1+n(X,\eta)-1+2 \\ &= n(X,\xi)+n(X,\eta). \end{aligned}$$

Thus we have

$$\begin{aligned} e &\geqslant (k-n(S_{r-1},\xi))+(k-n(S_{r-1},\eta)) \\ &= 2k-(n(S_{r-1},\xi)+n(S_{r-1},\eta)) \\ &\geqslant k+1, \end{aligned}$$

by Lemma 2·1. This contradicts the hypothesis $e \leqslant k-2$, and so we deduce that X_S (and similarly X_T) contains no pairs of adjacent vertices. Hence G is bipartite, the two parts consisting of alternate sets from the sequence

$$S_0, S_1, S_2, \ldots, S_{r-1}, X_S, X_T, T_{r-1}, \ldots, T_2, T_1, T_0.$$

Since all the excess vertices are in $X = X_S \cup X_T$, the diameter is $r+1$. ∎

The result stated in Theorem A is not the best possible. By using more carefu counting arguments we can show that $e \leqslant k-1$ is sufficient to give the same conclusions. For our present purposes we need this stronger result only for $k = 3$ and $e = 2$, when it can be established by fairly simple means.

3. *Algebraic treatment of the case $e = 2$, $g = 2r \geqslant 8$.* In this section we shall always suppose that G is a regular graph with girth $g = 2r \geqslant 8$, valency k, and excess 2. We set $n = |VG| = n_0 + 2$. By Theorem A and the remarks following it we know that G is bipartite and has diameter $r+1$.

Let A_i $(0 \leqslant i \leqslant r+1)$ denote the $n \times n$ matrix whose rows and columns correspond to the vertices of G, with

$$(A_i)_{\alpha\beta} = \begin{cases} 1 & \text{if } \partial(\alpha,\beta) = i; \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3·1. *The matrices A_i $(0 \leqslant i \leqslant r+1)$ satisfy the following identities (where $A = A_1$):*

(i) $AA_1 = kI + A_2$;

(ii) $AA_i = (k-1)A_{i-1} + A_{i+1}$ $\quad(2 \leqslant i \leqslant r-2)$;

(iii) $A(A_{r-1} + A_{r+1}) = (k-1)A_{r-2} + kA_r$.

Proof. We sketch the proof of (iii); (i) and (ii) are similar but simpler. The term in row α and column β of $A(A_{r-1}+A_{r+1})$ is equal to the number of vertices γ in G satisfying $\partial(\alpha,\gamma) = 1$ and $\partial(\beta,\gamma) = r \pm 1$. If $\partial(\alpha,\beta) \neq r-2$ or r, then there are no such vertices. If $\partial(\alpha,\beta) = r-2$, then there are $k-1$: all vertices adjacent to α except the unique one lying on a path of length $r-2$ from α to β. If $\partial(\alpha,\beta) = r$, then all k vertices adjacent to γ have the property, as a consequence of Theorem A. Thus

$$[A(A_{r-1}+A_{r+1})]_{\alpha\beta} = [(k-1)A_{r-2} + kA_r]_{\alpha\beta},$$

as required. ▮

Let J denote the $n \times n$ matrix whose entries are all 1. It is clear that

$$J = A_0 + A_1 + \ldots + A_{r+1}.$$

The identities given in (i) and (ii) of Lemma 3·1 enable us to express each of the matrices $A_0, A_1, \ldots, A_{r-1}$ in turn as a polynomial in A, and using (iii) to deal with A_r, we obtain

$$kJ = (A + kI)(E_r(A) + A_{r+1}),$$

where $\{E_i(x)\}$ is the sequence of polynomials defined by the recursion

$$E_0(x) = 0, \quad E_1(x) = 1,$$

$$E_i(x) = xE_{i-1}(x) - (k-1)E_{i-2}(x) \quad (i \geqslant 2).$$

Lemma 3·2. *If $\omega(\neq \pm k)$ is an eigenvalue of A, then*

$$E_r(\omega) - \epsilon = 0, \tag{$*$}$$

where $\epsilon = \pm 1$.

Proof. Since the excess is 2, every vertex of G has just one vertex at distance $r+1$ from it. Thus A_{r+1} is a permutation matrix satisfying $A_{r+1}^2 = I$, and its eigenvalues are ± 1. (The trace of A_{r+1} is zero, so each value occurs $\frac{1}{2}n$ times.)

Suppose that ω is an eigenvalue of A. Since G is regular, a result of Hoffman (**3**, p. 15) implies that J is a polynomial in A, and so any eigenvector of A is an eigenvector of J. The equation $kJ = (A+kI)(E_r(A)+A_{r+1})$ shows that such an eigenvector is also an eigenvector of A_{r+1}, whence $(\omega+k)(E_r(\omega)\pm 1)$ is an eigenvalue of kJ. But the eigenvalues of kJ are kn (once) and 0 ($n-1$ times). The eigenvalue kn corresponds to putting $\omega = k$, and so all remaining eigenvalues except $\omega = -k$ satisfy $(*)$. ∎

LEMMA 3·3. *For either value of ϵ, the equation $(*)$ has $r-1$ distinct roots*

$$\omega_1 < \omega_2 < \dots < \omega_{r-1}.$$

If we set $s = \sqrt{(k-1)}$, then $\omega_i = -2s\cos\phi_i$ $(0 < \phi_i < \pi)$ and

$$i\pi/r^+ \leqslant \phi_i \leqslant i\pi/r \quad \text{if} \quad \eta_i = 1,$$

$$i\pi/r \leqslant \phi_i \leqslant i\pi/r^- \quad \text{if} \quad \eta_i = -1,$$

where

$$\eta_i = (-1)^{r+i}\epsilon, \quad r^+ = r+s^{1-r}, \quad r^- = r-s^{1-r}.$$

Proof. Putting $\omega = -2s\cos\phi$, we find

$$E_r(\omega) = (-s)^{r-1}\frac{\sin r\phi,}{\sin\phi},$$

and $E_r(\omega)$ has zeros when $\phi = i\pi/r$ $(1 \leqslant i \leqslant r-1)$. Hence we put $\phi = (i\pi-\delta)/r$, and $(*)$ becomes

$$\sin\delta - \eta_i s^{1-r}\sin(i\pi-\delta)/r = 0,$$

where $\eta_i = (-1)^{r+i}\epsilon$.

Suppose $\eta_i = 1$. The left-hand side of the equation is negative when $\delta = 0$, and positive when $\delta = \min\{\phi s^{1-r}, (\pi-\phi)s^{1-r}\}$, since the sine function is convex upwards. Hence there is a root $\phi_i = (i\pi-\delta_i)/r$, with δ_i between these bounds. This gives the required bounds for ϕ_i.

The case $\eta_i = -1$ is similar. ∎

LEMMA 3·4. *The multiplicity $m(\omega_i)$ of $\omega_i = -2s\cos\phi_i$ as an eigenvalue of A is*

$$m(\omega_i) = \frac{nk}{4}(1-s^{2-2r})\,p(\cos\phi_i)/q(-\eta_i\cos\phi_i),$$

where

$$p(t) = (1-t^2)/[(k^2/4s^2)-t^2],$$

$$q(t) = \{r(1-s^{2-2r})+(r-1)s^{1-r}t[s^{1-r}t+\sqrt{\{1-s^{2-2r}(1-t^2)\}}]\}.$$

Proof. A standard method of calculating multiplicities, as in (**1**), leads to the formula

$$m(\omega) = \frac{nk(k-1)}{2}\cdot\frac{E_{r-1}(\omega)}{(k^2-\omega^2)E_r'(\omega)},$$

where E_r' is the derivative of E_r. Substituting our expressions for E_{r-1} and E_r', we obtain the stated formula. ∎

LEMMA 3·5. *Let* $\lambda_1 < \lambda_2 < \ldots < \lambda_{r-1}$ *be the roots of* $E_r(x) - 1 = 0$ *and*

$$\mu_1 < \mu_2 < \ldots < \mu_{r-1}$$

the roots of $E_r(x) + 1 = 0$. *Then we have:*

(i) *If r is odd,* $m(\lambda_i) = m(\lambda_{r-i})$, $m(\mu_i) = m(\mu_{r-i})$, *while if r is even,* $m(\lambda_i) = m(\mu_{r-i})$ $(1 \leqslant i \leqslant r-1)$.

(ii) *If r is odd, then* $m(\mu_1) < m(\mu_i)$ *for* $i = 2, \ldots, r-2$, *while if r is even,* $m(\lambda_1) < m(\lambda_i)$ *for* $i = 2, \ldots, r-2$.

(iii) *If* $(r, k) \neq (4, 3), (4, 4), (5, 3), (6, 3)$, *then* $m(\lambda_{r-1}) < m(\lambda_i)$ *for* $i = 2, \ldots, r-2$.

Proof. (i) If r is odd, $E_r(-x) = E_r(x)$ and so $\lambda_i + \lambda_{r-i} = \mu_i + \mu_{r-i} = 0$, and the formula for the multiplicity gives $m(\lambda_i) = m(\lambda_{r-i})$, $m(\mu_i) = m(\mu_{r-i})$ $(1 \leqslant i \leqslant r-1)$. If r is even, $E_r(-x) = -E_r(x)$ and so $\lambda_i + \mu_{r-i} = 0$, $m(\lambda_i) = m(\mu_{r-i})$.

(ii) We remark that, in the notation of Lemma 3·4, $p(t)$ is an even function, and is convex upwards, while $q(t)$ is monotonic increasing. Let $\mu_i = -2st_i$. Then it follows that

$$p(t_1) < p(t_i) \quad \text{and} \quad q(t_1) > q(\pm t_i),$$

for $2 \leqslant i \leqslant r-2$. Since $\eta_1 = -1$ when r is odd and $\epsilon = -1$, we get $m(\mu_1) < m(\mu_i)$ in this case. The case when r is even is similar.

(iii) This is proved by direct calculation, using the inequality

$$(1 - s^{1-r})\, r^+ < q(t) < (1 + s^{1-r})\, r^- \quad (|t| < 1).$$

THEOREM B. *There is no regular graph G with girth* $2r \geqslant 8$ *and excess* 2.

Proof. For the major part of the proof we shall suppose that (r, k) is not one of the exceptions listed in Lemma 3·5 (iii). The exceptional cases will be dealt with separately at the end.

Suppose that r is even and (r, k) is not one of the exceptions. Then Lemma 3·5 implies that λ_1 and λ_{r-1} have multiplicities different from (in fact, strictly less than) the other eigenvalues. Now the eigenvalues are all algebraic integers, and if one of them is of degree l then its $l-1$ algebraic conjugates will be eigenvalues with the same multiplicity. Hence λ_1 and λ_{r-1} are either rational integers or they are conjugate quadratic irrationals. In either case $\lambda_1 + \lambda_{r-1}$ is an integer.

But now Lemma 3·3 tells us that $\lambda_1 + \lambda_{r-1}$ is positive and

$$\begin{aligned} \lambda_1 + \lambda_{r-1} &< 2s\{-\cos(\pi/r^-) + \cos(\pi/r^+)\} \\ &< s\{(\pi/r^-)^2 - (\pi/r^+)^2\} \\ &= 4\pi^2 r / s^{r-2}(r^2 - s^{2-2r})^2. \end{aligned}$$

The final expression is strictly less than 1 when $r \geqslant 4$ and $k \geqslant 3$, contradicting the fact that $\lambda_1 + \lambda_{r-1}$ is an integer. Hence the result is proved in this case.

Suppose that r is odd and not one of the exceptions. By a similar argument using Lemma 3·5 we find that $\lambda_1^2 - \mu_1^2$ must be an integer, and using the bounds established in Lemma 3·3 we obtain the contradiction $0 < \lambda_1^2 - \mu_1^2 < 1$.

For the exceptional cases $(4, 3)$, $(4, 4)$, $(6, 3)$, we note that $E_r(x) - 1$ is irreducible and the formula for $m(\lambda)$ in Lemma 3·4 leads to irrational, and therefore impossible, values. Finally, the case $(5, 3)$ can be excluded by direct elementary arguments. ∎

4. *Graphs with girth* 6. The algebraic methods used in the previous section do not lead to any conclusion for graphs with girth 6. However, we can obtain some very interesting results for this case by exploiting a relationship with symmetric designs.

As before, we assume $e \leqslant k-2$, so that G is bipartite, its diameter is 4 and e is even. In the minimal case $e = 0$ it is known (10) that the two parts of G may be regarded as the sets of points and lines of a projective plane, adjacent vertices corresponding to an incident pair.

Suppose that $e \neq 0$, and let $\langle X \rangle$ denote the vertex-subgraph induced on X. $\langle X \rangle$ is bipartite, and the valency of each of its vertices is at least 1, since any vertex in X is joined to at most $k-1$ vertices in S_2 or T_2 (Lemma 2·1). Thus $\langle X \rangle$ has at least $\frac{1}{2}e$ edges. The case when $\langle X \rangle$ has just $\frac{1}{2}e$ edges, so that the e vertices are joined in pairs and there are no other joins, is of particular interest, as the next theorem shows. This case certainly must happen when $e = 2$ (even when $k = 3$, in fact), and it may be that it necessarily occurs if e is small compared with k. However, we have not succeeded in proving a result of this kind.

In order to state the next theorem we shall need some definitions.

A *symmetric* (v, k, λ)-*design* is a set P of v *points* and a set B of v *blocks*, such that each block is a k-subset of P, and any two distinct points belong to exactly λ blocks. It follows that any two distinct blocks intersect in just λ common points, and

$$v = (k^2 - k + \lambda)/\lambda.$$

Associated with a symmetric (v, k, λ)-design there is a bipartite graph whose vertex-set is $P \cup B$ and whose edge-set consists of the pairs $\{p, b\}$ with $p \in b$ ($p \in P$, $b \in B$). We shall denote a graph which arises in this way by $D(k, \lambda)$, noting that the parameters k and λ do not necessarily determine a unique graph.

A graph $D(k, \lambda)$ has the following four properties:

(1) it is regular, with valency k;

(2) it is bipartite;

(3) it has diameter 3;

(4) given any two vertices at distance 2 there are just λ vertices adjacent to both of them.

This is equivalent to saying that $D(k, \lambda)$ is a distance-regular graph with intersection array

$$\{k, k-1, k-\lambda; 1, \lambda, k\}.$$

Conversely, if we are given a graph with properties (1)–(4), then it gives rise to a symmetric design in the obvious way, and so it is a $D(k, \lambda)$ graph.

A *local isomorphism* of two graphs G and H is a mapping f from VG onto VH such that the neighbours of v in G are mapped in a one-to-one fashion onto the neighbours of $f(v)$ in H. We shall say that G is an *s-fold cover* of H if there is a local isomorphism

$f: VG \to VH$ such that $|f^{-1}(w)| = s$ for each $w \in VH$. It follows that if $\{w_1, w_2\}$ is an edge of H then there are just s edges $\{v_1, v_2\}$ such that $f(v_1) = w_1, f(v_2) = w_2$.

THEOREM C. *Let G be a regular graph with valency k, girth 6, and excess $e \leqslant k-2$. Suppose that for each edge $\{\sigma, \tau\}$ of G the excess set $X_{\sigma\tau}$ induces a subgraph with just $\frac{1}{2}e$ edges; then G is a λ-fold cover of a graph $D(k, \lambda)$, with $\lambda = \frac{1}{2}e+1$.*

Proof. (I) We begin by constructing a graph G' such that G is a λ-fold cover of G'.

Choose an edge $\{\sigma, \tau\}$ of G and let α, β be distinct vertices in $X = X_{\sigma\tau}$, both at distance 4 from σ. We claim that $\partial(\alpha, \beta) = 4$. If not, then since G is bipartite we should have $\partial(\alpha, \beta) = 2$, and there is some vertex γ adjacent to both α and β. Now γ cannot be in X, since $\langle X \rangle$ is 1-valent, so γ must be in T_2. There are at least two edges from γ to X, one edge from γ to T_1, and so at most $k-3$ edges from γ to S_2. Hence not all vertices in S_1 are at distance 2 from γ, and there is a vertex $\delta \in S_1$ such that $\partial(\gamma, \delta) = 4$. But now the excess set $X_{\sigma\delta}$ contains the vertices α, β, γ, and γ has valency 2 in $\langle X_{\sigma\delta} \rangle$, contrary to our assumption. Hence $\partial(\alpha, \beta) = 4$.

We have shown that G has the antipodal property: if $\partial(\sigma, \alpha) = \partial(\sigma, \beta) = 4$ (the diameter of G), then $\partial(\alpha, \beta) = 4$ also. Hence we may define an equivalence relation $\sim$ on VG by the rule

$$\mu \sim \nu \Leftrightarrow \partial(\mu, \nu) = 0 \text{ or } 4.$$

Let μ' denote the equivalence class of $\mu \in VG$, and let V' denote the set of equivalence classes. Define E' by the rule that $\{\mu', \nu'\} \in E'$ if and only if there are vertices $\xi \in \mu'$, $\eta \in \nu'$ such that $\{\xi, \eta\}$ is an edge of G, and let G' denote the graph with vertex-set V' and edge-set E'. The mapping f taking μ to μ' is a local isomorphism of G onto G', and $f^{-1}(\mu')$ consists of μ and the $\frac{1}{2}e$ vertices at distance 4 from it, so G is a λ-fold cover of G', where $\lambda = \frac{1}{2}e+1$.

(II) We now check that G' is a graph satisfying the conditions (1)–(4) which characterize a $D(k, \lambda)$ graph.

(1) Since f is a local isomorphism, the valency of G' is k.

(2) If $VG = V_1 \cup V_2$ is the bipartition of G, then $V' = V_1' \cup V_2'$ is a bipartition of G' where

$$V_i' = \{\mu' \in V' | \mu \in V'\} \quad (i = 1, 2).$$

(3) Since G has diameter 4, the diameter d' of G' is at most 4. Vertices at distance 4 in G are identified in G', and so $d' \leqslant 3$. It is easy to check that some pairs μ, ν of vertices of G with $\partial(\mu, \nu) = 3$ give $\partial(\mu', \nu') = 3$ in G'. (The only other possibility is that $\partial(\mu', \nu') = 1$, and for a given μ there are too many vertices ν for $\partial(\mu', \nu') = 1$ to hold always.) Hence $d' = 3$.

(4) Suppose that $\partial(\phi', \psi') = 2$ in G', and choose ϕ_0, ψ_0 in G covering ϕ', ψ' respectively. Let χ_0 be the unique vertex of G adjacent to ϕ_0 and ψ_0. The excess set with respect to $\{\phi_0, \chi_0\}$ consists of $e = 2(\lambda-1)$ vertices $\chi_1, \ldots, \chi_{\lambda-1}$, $\phi_1, \ldots, \phi_{\lambda-1}$, where ϕ_i, χ_i are at distance 4 from ϕ_0, χ_0 respectively ($1 \leqslant i \leqslant \lambda-1$). Also, there are $\lambda-1$ vertices at distance 4 from ψ_0. If we use the standard notation of Section 2 with σ, τ replaced by ϕ_0, χ_0, then $\psi_0 \in T_1$ and its $\lambda-1$ antipodes are in S_2; we may suppose they are labelled $\psi_1, \ldots, \psi_{\lambda-1}$ so that ψ_i is adjacent to χ_i. Each ψ_i is adjacent to a unique

vertex α_i in S_1, and since $\partial(\psi_i, \psi_j) = 4$, α_i and α_j are different when $i \neq j$. Now $\partial(\alpha_i, \alpha_j) = 2$ and so $\alpha'_i \neq \alpha'_j$. Thus we have, as required, λ vertices χ', $\alpha'_1, \ldots, \alpha_{\lambda-1}$ adjacent to both ϕ' and ψ' in G', and these are the only such vertices. Hence G' is a $D(k, \lambda)$ graph. ∎

It is possible that for a given value of $\lambda > 1$ there are only finitely many values of k for which a symmetric (v, k, λ)-design exists. If this is so, then Theorem C provides some evidence in favour of the conjecture that the number

$$\min\{e \mid \exists G \text{ with girth } 6, \text{ valency } k, \text{ and excess } e \neq 0\}$$

tends to infinity with k. However, the conjectured result on symmetric designs is probably quite deep, and it may be easier to attack our graph-theoretic problem more directly. This is certainly true when $e = 2$, as the next theorem will show.

If we are given that $e = 2$, then the condition that $\langle X \rangle$ has just $\frac{1}{2}e$ edges is necessarily satisfied: the two excess vertices must be joined by an edge. Theorem C now tells us that a graph with $e = 2$ and $g = 6$ is a 2-fold covering of a $D(k, 2)$ graph, which corresponds to a symmetric $(v, k, 2)$-design, or *biplane*. Biplanes are known to exist for $k = 3, 4, 5, 6, 9, 11, 13$ (6). The following theorem shows that, in general, the existence of a biplane is not sufficient for the existence of a 2-fold covering of the associated graph. (Results of this kind have been obtained independently by J. Kahn.)

THEOREM D. *A graph G with girth 6, valency k and excess 2 cannot exist if $k \equiv 5$ or 7 (mod 8).*

Proof. We already know, by Theorem C, that G is a 2-fold covering of a graph $D(k, 2)$. A typical vertex π of $D(k, 2)$ is covered by two vertices of G, which we shall denote by π^+ and π^- in some arbitrary fashion. A typical edge $\{\pi, \beta\}$ of $D(k, 2)$ is covered by two edges of G, and there are just two possibilities:

(i) the covering edges are $\{\pi^+, \beta^+\}$ and $\{\pi^-, \beta^-\}$;

(ii) the covering edges are $\{\pi^+, \beta^-\}$ and $\{\pi^-, \beta^+\}$.

In case (ii) we shall say that $\{\pi, \beta\}$ belongs to the subset E^- of the edge-set E of $D(k, 2)$.

Consider a typical 4-cycle $(\pi, \beta, \omega, \gamma)$ of $D(k, 2)$. If it contains an even number of edges in E^-, then the edges of G covering it will comprise two 4-cycles. Since G has girth 6, this is impossible, and we conclude that every 4-cycle must contain just 1 or 3 edges in E^-.

Let C_1, C_3 denote the number of 4-cycles in $D(k, 2)$ which contain 1, 3 edges in E^- respectively. Since any two of the v points of the biplane determine a unique 4-cycle, there are $\frac{1}{2}v(v-1)$ 4-cycles in all, and

$$C_1 + C_3 = \tfrac{1}{2}v(v-1).$$

Each edge of $D(k, 2)$ corresponds to a point π and a block β of the biplane and so it belongs to $k-1$ 4-cycles $(\pi, \beta, \omega, \gamma)$, where ω runs through the $k-1$ points of β different from π and γ is the unique block containing π and ω. Thus, counting the edges in E^-, we obtain

$$C_1 + 3C_3 = (k-1)|E^-|.$$

Eliminating C_1, we have

$$2C_3 = (k-1)|E^-| - \tfrac{1}{2}v(v-1).$$

Now $v = \frac{1}{2}(k^2 - k + 2)$, and if $k \equiv 5$ or 7 (mod 8), we find that $\frac{1}{2}v(v-1)$ is odd and $k-1$ is even, so that the equation for C_3 has no integral solution. ∎

Theorem D shows that even if a biplane exists, its graph need not have a double covering G of the kind we require. This is certainly the case when $k = 5$ or 13, for example. For the other residue classes (mod 8) of k we can, for the moment, say no more than that a biplane must exist. Necessary conditions for this are provided by the Bruck–Ryser–Chowla theorem, and they may be summarized as follows (7, p. 104).

Let $X(n)$ denote the square-free part of n. If there is a biplane with k points in a block, then

(i) $k \equiv 2, 3, 6 \pmod 8 \Rightarrow X(k-2) = 0$;

(ii) $k \equiv 0, 1 \pmod 8 \Rightarrow$ any odd prime dividing $X(k-2)$ is congruent to 1 or 7 (mod 8);

(iii) $k \equiv 4 \pmod 8 \Rightarrow$ any odd prime dividing $X(k-2)$ is congruent to 1 or 3 (mod 8).

For example, there are no biplanes for $k = 7, 8, 10, 12$. However, the conditions allow biplanes with $k = 3, 4, 6, 9, 11$, and examples are known in each of these cases. The first two values give unique biplanes, and there is a unique covering graph in both cases. When $k = 3$ the graph $D(3, 2)$ is just the ordinary cube, and it has a unique 2-fold covering with girth 6: this graph was first discussed by R. M. Foster (9, p. 315).

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