Discrete Mathematics 30 (1980) 299–301

COMMUNICATION

A TRIVALENT GRAPH WITH 58 VERTICES AND GIRTH 9

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Communicated by the Managing Editor
Received 23 January 1980

A regular graph with valency k and girth g will be referred to as a (k, g)-graph. Petersen's graph is a (3, 5)-graph; indeed, it is the (unique) smallest (3, 5)-graph. In general, the problem of finding a smallest (k, g)-graph is hard, and the answer is known only for a few values of k and g.

The particular case $k = 3$, $g = 9$ has been the subject of much endeavour. A lower bound of 46 for the number of vertices can be obtained by simple arguments, and it is easy to show that, in fact, this bound cannot be attained. With progressively more effort, it can be shown that 48, 50, and 52 vertices are likewise impossible. The alternative approach is to establish an upper bound by actually constructing a (3, 9)-graph. As long ago as 1952, R.M. Foster knew of a (3, 9)-graph with 60 vertices: this graph was mentioned by Frucht [3] in 1955, and included in a list of symmetric trivalent graphs distributed by Foster at a conference held in 1966 at Waterloo, Canada. From about 1968 onwards many attempts have been made to improve on Foster's result. Balaban [1], Coxeter, Evans, Harries, Wynn, and Foster himself, have all made contributions. The sum total of these efforts, up to November 1979, was the construction of no fewer than 19 mutually non-isomorphic (3, 9)-graphs with 60 vertices—but no smaller ones. Little of this work has been published, since each attempt to prepare a paper has been overtaken by the discovery of a new graph or graphs. In fact, three more (3, 9)-graphs with 60 vertices were found in the preliminary stages of the present investigation.

The main purpose of this note is to announce the existence of a (3, 9)-graph with 58 vertices. This graph is a significant, but not necessarily final, contribution to the problem, since it is possible that smaller (3,9)-graphs exist.

The graph is displayed in Figs. 1 and 2, demonstrating that it has at least two different Hamiltonian cycles. There are 80 9-cycles. The eigenvalues of the adjacency matrix are all distinct, and this means that any non-identity automorphism must be an involution [2, p. 103]. The automorphism group is a non-cyclic

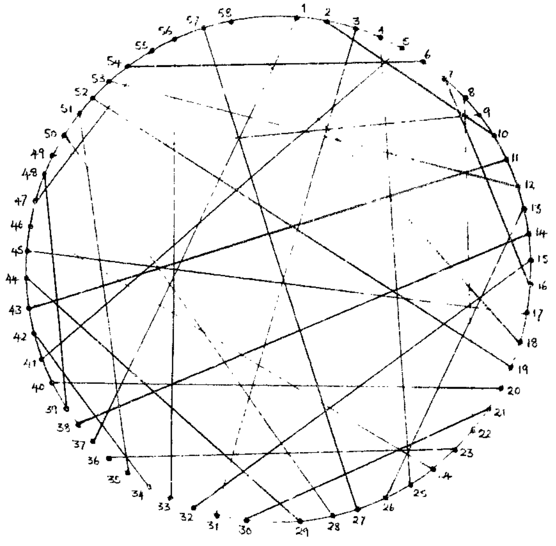
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58

Fig. 1

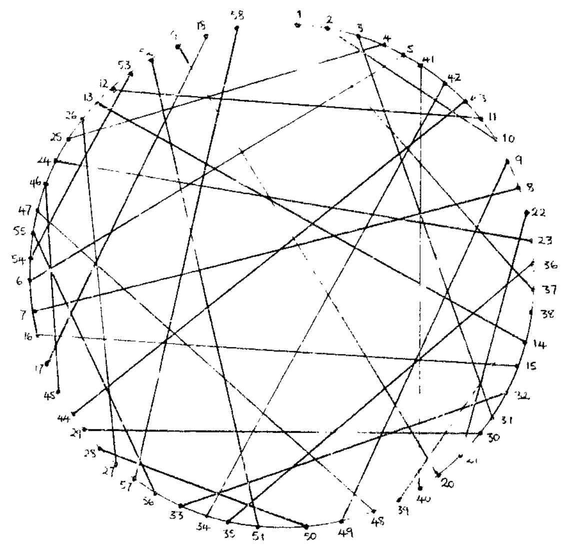
1 2 3 4 5 41 42 43 11 10 9 8 22 23 36 37 38 14 15 32 31 30 20 40 39 48 49 50 51 35 34 33 36 57 27 28 29 44 45 17 16 7 6 54 55 47 46 24 25 26 13 12 53 58

Fig. 2

group of order 4, generated by α and β, where

$$\alpha = (\ 2\ 58)(\ 3\ 57)(\ 4\ 56)(\ 5\ 55)(\ 6\ 54)(\ 7\ 53)(\ 8\ 52)$$
$$(\ 9\ 19)(10\ 18)(11\ 17)(12\ 16)(13\ 15)(20\ 49)(21\ 50)$$
$$(22\ 51)(23\ 35)(24\ 34)(25\ 33)(26\ 32)(27\ 31)(28\ 30)$$
$$(40\ 48)(41\ 47)(42\ 46)(43\ 45),$$

$$\beta = (\ 1\ 36)(\ 2\ 23)(\ 3\ 22)(\ 4\ \ 8)(\ 5\ \ 7)(\ 9\ 25)(10\ 24)$$
$$(11\ 46)(12\ 47)(13\ 48)(14\ 39)(15\ 40)(16\ 41)(17\ 42)$$
$$(18\ 34)(19\ 33)(20\ 32)(21\ 31)(26\ 49)(27\ 50)(35\ 58)$$
$$(43\ 45)(51\ 57)(52\ 56)(53\ 55).$$

The strategy employed in the search for this graph was based upon the use of a computer, mainly for the hard work involved in testing various suggested methods of construction. This is a field where it is easy to conceive possible means of attack, but very tedious to carry them out in a systematic way. In fact, the successful approach which resulted in the graph depicted here involved using the computer rather less than might have been expected.

References

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