EXCESS IN VERTEX-TRANSITIVE GRAPHS

N. L. BIGGS

Let G be a regular graph with valency k and girth $g = 2r+1$ $(r = 1, 2, 3, \ldots)$. Then for each vertex v of G, and each integer $i = 1, 2, \ldots, r$, the set $\Delta_i(v)$ of vertices whose distance from v is i has cardinality $k(k-1)^{i-1}$. It follows that the number n of vertices of G satisfies

$$n \geqslant n_0(k, g) = 1 + \sum_{i=1}^{r} k(k-1)^{i-1}$$

$$= 1 + k(k-2)^{-1}\{(k-1)^{\frac{1}{2}(g-1)} - 1\}.$$

It is known [1], [3] that the lower bound $n_0(k, g)$ is attained only in a few cases. On the other hand, general properties of the *excess*, $e = n - n_0(k, g)$, are very hard to establish (see, for example [2, Chapter 3]). The subject of this paper is the behaviour of the excess in the case when G is vertex-transitive: that is, when G has a group of automorphisms acting transitively on its vertices, so that each vertex has the same properties relative to G. It will be shown that, for each odd value of k, the minimum excess $e_{T,k}(g)$ of a vertex-transitive graph with valency k and girth g is unbounded as a function of g. In other words,

$$\limsup_{g \to \infty} e_{T,k}(g) = \infty .$$

In the remainder of the paper G will always denote a vertex-transitive graph with odd valency k and odd girth $g = 2r+1$. For each vertex v of G the number of g-cycles containing v is equal to the number of edges of G which join two vertices in $\Delta_r(v)$, and this number is a constant X, independent of v.

LEMMA. *If e is the excess of G and $Y = \frac{1}{2}k(k-1)^r$, then*

$$Y - \tfrac{1}{2}ke \leqslant X \leqslant Y.$$

Proof. Let $E(v)$ denote the set of vertices of G whose distance from v is strictly greater than r, and let J denote the number of edges of G which join a vertex in $\Delta_r(v)$ to one in $E(v)$. Then each vertex in $\Delta_r(v)$ is adjacent to one vertex in $\Delta_{r-1}(v)$ and $k-1$ other vertices, so that

$$2X + J = (k-1)|\Delta_r(v)| = 2Y.$$

Since $|E(v)| = e$, and each vertex has valency k, we have $0 \leqslant J \leqslant ke$. Putting $J = 2(Y - X)$ gives the required result.

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THEOREM. *For each odd integer $k \geqslant 3$ there is an infinite sequence of values of g such that the excess e of any vertex-transitive graph with valency k and girth g satisfies $e > g/k$.*

Proof. Let S_k denote the set of primes $p > k$ such that $k-1$ is a square modulo p. It follows from the laws of quadratic reciprocity and Dirichlet's theorem on primes in arithmetic progressions that S_k is an infinite set.

If there are N g-cycles in G, then each of the n vertices is contained in X of them, so that $gN = nX$. Now if we take g to be in S_k, so that g is a prime, g must divide at least one of n and X.

Suppose g divides X. Since $k-1$ is a square modulo g we have

$$(k-1)^{\frac{1}{2}(g-1)} \equiv 1 \pmod g.$$

Since $g > k$ and k is odd, the least positive residue of $Y = \frac{1}{2}k(k-1)^{\frac{1}{2}(g-1)}$ modulo g is $\frac{1}{2}(g+k)$. But if g divides X, and X satisfies the inequality proved in the lemma, this means that

$$\tfrac{1}{2}ke \geqslant \tfrac{1}{2}(g+k).$$

That is, $e \geqslant (g+k)/k$, or $e > g/k$.

Suppose that g does not divide X, so that g must divide n. Then we have

$$\begin{aligned} n &= n_0(k,g)+e \\ &= 1+k(k-2)^{-1}\{(k-1)^{\frac{1}{2}(g-1)}-1\}+e \\ &\equiv 1+e \pmod g. \end{aligned}$$

Thus $e \geqslant g-1 > g/k$. Hence $e > g/k$ for each g in the infinite set S_k.

The simplest specific case of the theorem occurs when $k = 3$ and $g = 7$. Here the lower bound $n_0(3,7) = 22$ is not attained, but there is a unique smallest graph with the required properties [5]. It has $e = 2$, $n = 24$, but it is not vertex-transitive. In fact, our theorem applies, since $k-1 = 2$ is a square modulo 7, and we deduce that for a vertex-transitive graph it is necessary that $e > 7/3$. Since e must be an even integer, $e \geqslant 4$ and $n \geqslant 26$. There is at least one vertex-transitive graph with 26 vertices and the required properties: in the notation of Frucht, Graver and Watkins [4], it is the graph $G(13, 5)$.

The proof of the theorem fails when k is even. Since the least positive residue of Y modulo g is $\frac{1}{2}k$ in that case, it follows that g divides $Y-\frac{1}{2}k$, and X can take this value provided only that $e \geqslant 1$.

References

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Royal Holloway College,
Egham,
Surrey TW20 0EX.