CONSTRUCTING 5-ARC-TRANSITIVE CUBIC GRAPHS

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1. *Introduction*

The theory of groups acting transitively on the s-arcs of cubic graphs was inaugurated by Tutte in his fundamental paper [**3**]. He set up the elements of the theory and proved that for a finite cubic graph we must have $s \leqslant 5$. He also gave the first example of a connected 5-arc-transitive cubic graph. Conway (see [**1**, p. 130]) showed that there are infinitely many such graphs, but his construction requires enormous numbers of vertices. In this paper we shall show how to construct 5-arc-transitive graphs from 4-arc-transitive ones. Since infinite families of the latter type are known, we recover Conway's result, but with much smaller graphs.

The stimulus for this work came from a particular example with 2352 vertices [**2**]. The author would like to acknowledge the generosity of Dr. J. H. Conway in allowing him to study unpublished work in which the existence of a 5-arc-transitive graph on 2352 vertices is apparent.

2. *The* 4-*arc-transitive case*

Let $\mathscr{G}$ be a finite connected cubic graph, with vertex-set $V\mathscr{G}$ and edge-set $E\mathscr{G}$. An edge e which joins the vertices v and w will be written $e = vw$. The usual distance function will be denoted by ∂, and for any edge $e = vw$ we shall write

$$\mathscr{G}_i(e) = \{x \in V\mathscr{G} \mid \min\{\partial(x,v), \partial(x,w)\} = i\}.$$

An *s-arc* in $\mathscr{G}$ is a sequence of vertices $(v_0, v_1, \ldots, v_s)$ such that $v_i v_{i+1}$ is an edge $(0 \leqslant i \leqslant s-1)$ and $v_i \neq v_{i+2}$ $(0 \leqslant i \leqslant s-2)$. It has two *successors* $(v_1, \ldots, v_s, w')$ and $(v_1, \ldots, v_s, w'')$, where v_{s-1}, w' and w'' are the three vertices adjacent to v_s.

Let G be a group of automorphisms of $\mathscr{G}$. The pointwise stabilizer of an r-arc $(v_0, \ldots, v_r)$ will be denoted by $G(v_0, \ldots, v_r)$. The following basic results go back to Tutte's paper [**3**].

THEOREM 0. (i) *If G acts transitively on the s-arcs of $\mathscr{G}$, but not on the $(s+1)$-arcs (in which case we say that G acts* s-arc-transitively) *then for any s-arc $(v_0, \ldots, v_s)$ we have*

$$|G(v_0, \ldots, v_i)| = 2^{s-i} \quad (1 \leqslant i \leqslant s), \quad |G(v_0)| = 3 \cdot 2^{s-1}.$$

In particular, the pointwise stabilizer of an s-arc is trivial.

(ii) *G acts s-arc-transitively on $\mathscr{G}$ if and only if there are automorphisms g_1 and g_2 in G which take $(v_0, \ldots, v_s)$ onto its two successors and $G = \langle g_1, g_2 \rangle$.*

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The *base* for our construction is a pair $(\mathscr{B}, K)$ consisting of a connected cubic graph $\mathscr{B}$ and a group K of automorphisms acting 4-arc-transitively on $\mathscr{B}$. We begin by examining the special features of this case.

LEMMA 1. *Associated with each edge e of $\mathscr{B}$ there is a unique involution $\bar{e}$ in K such that*

(i) *$\bar{e}$ fixes the vertices in $\mathscr{B}_1(e)$;*

(ii) *if u is in $\mathscr{B}_1(e)$, then $\bar{e}$ switches the pair of vertices in $\mathscr{B}_2(e)$ which are adjacent to u.*

Proof. Suppose that $e = vw$ and (u, v, w, x) is a 3-arc in $\mathscr{B}$. The stabilizer $K(u, v, w, x)$ has order 2, and so it contains a unique involution $\bar{e}$. If u' is the remaining vertex adjacent to v, then since $\bar{e}$ fixes u, v, w it must fix u' also. Hence $\bar{e}$ satisfies (i).

Let t, t' be the vertices in $\mathscr{B}_2(e)$ adjacent to u. Since $\bar{e}$ fixes u and v, it must either fix or switch t and t'. If $\bar{e}(t) = t$, then e is a non-trivial automorphism fixing the 4-arc (t, u, v, w, x), contradicting Theorem 0. Hence (ii) is proved.

Convention. From now on the same symbol e will denote an edge of $\mathscr{B}$ and its associated involution.

LEMMA 2. *If e, f, g are the three edges incident at a vertex of $\mathscr{B}$, then efg is the identity automorphism (we shall write $efg =$ id).*

Proof. Let $e = vw_1$, $f = vw_2$, $g = vw_3$, and denote by w_{i1}, w_{i2} the remaining vertices ($\neq v$) adjacent to w_i ($i = 1, 2, 3$). Using both parts of Lemma 1, we have

$$efg(w_{11}) = ef(w_{12}) = e(w_{11}) = w_{11},$$

and so on. Thus efg fixes the 4-arc $(w_{11}, w_1, v, w_2, w_{21})$, and so $efg = \text{id}$.

LEMMA 3. *For any k in K and any edge e of $\mathscr{B}$, $k(e) = kek^{-1}$.*

(According to our convention, $k(e)$ here denotes the involution associated with the edge $k(e)$.)

Proof. Since k is an automorphism, kek^{-1} is an involution in K fixing $\mathscr{B}_1(k(e))$ pointwise, and hence it is the unique involution $k(e)$.

Let $(v_0, v_1, v_2, v_3, v_4, v_5)$ be a 5-arc in $\mathscr{B}$. Since K acts 4-arc-transitively there is a unique element b in K taking $(v_0, v_1, v_2, v_3, v_4)$ onto $(v_1, v_2, v_3, v_4, v_5)$. We shall require the following additional notation for vertices and edges of $\mathscr{B}$, as displayed in figure 1: $v_i = b^i(v_0)$ $(-3 \leqslant i \leqslant 6)$, $e_i = v_{i-1}v_i$ $(-2 \leqslant i \leqslant 6)$, u_i is the remaining vertex ($\neq v_{i-1}$ or v_{i+1}) adjacent to v_i, $f_i = u_iv_i$ $(-2 \leqslant i \leqslant 5)$.

LEMMA 4. *Let $a = e_3b$. Then a takes the 4-arc $(v_0, v_1, v_2, v_3, v_4)$ onto $(v_1, v_2, v_3, v_4, u_4)$, and $K = \langle a, b \rangle$.*

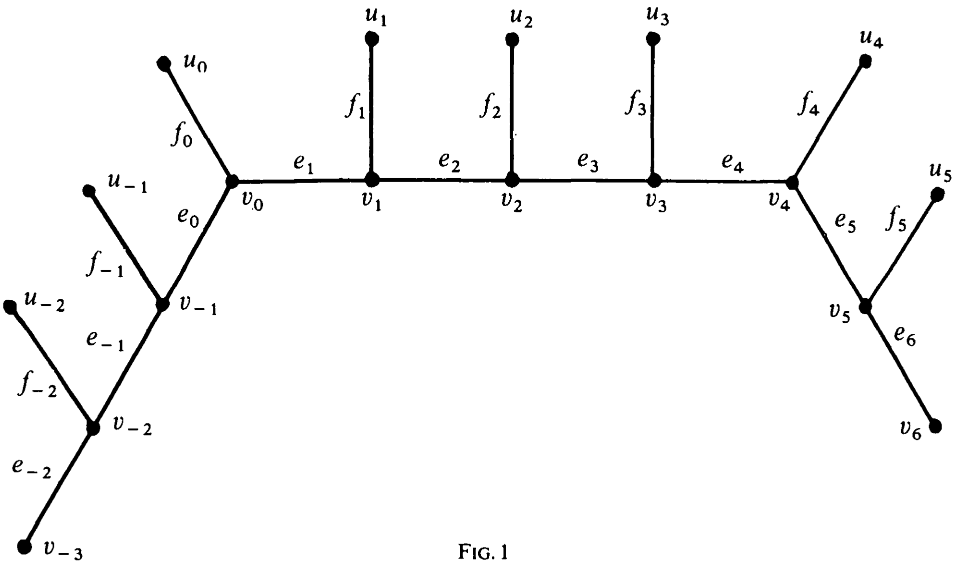

FIG. 1

Proof. By Lemma 1, e_3 fixes v_1, v_2, v_3, v_4, and switches u_4 and v_5. Hence a acts as asserted, and a, b generate K, by part of Theorem 0.

THEOREM 1. *Let L denote the subgroup of K generated by the involutions e $(e \in E\mathscr{B})$. Then*

(i) *if $\mathscr{B}$ is bipartite, $|K : L| = 2$,*

(ii) *if $\mathscr{B}$ is not bipartite, $K = L$.*

Proof. The following elements of K are defined in terms of the edges labelled in figure 1.

$$c_1 = e_2, \quad c_2 = f_1, \quad c_3 = f_1 e_0, \quad c_4 = f_1 f_{-1}, \quad c_5 = f_1 f_{-1} e_{-2}.$$

Let $C_i = \langle c_1, \ldots, c_i \rangle$ $(1 \leqslant i \leqslant 5)$. First, $C_1 = K(v_0, v_1, v_2, v_3)$ since both groups have the same order and fix the specified vertices. Next, c_2 fixes v_0, v_1, v_2 but not v_3, so that C_2 is a subgroup of $K(v_0, v_1, v_2)$ strictly larger than C_1, and by considerations of order, $C_2 = K(v_0, v_1, v_2)$. In the same way, it can be verified that $C_3 = K(v_0, v_1)$ and $C_4 = K(v_0)$.

Let U denote the orbit of v_0 under C_5. We have

$$c_5(v_0) = f_1 f_{-1} e_{-2}(v_0) = f_1 f_{-1}(u_{-1}) = f_1(u_{-1}),$$

and so

$$\partial(v_0, c_5(v_0)) = \partial(v_0, f_1(u_{-1})) = \partial(f_1(v_0), u_{-1}) = \partial(v_0, u_{-1}) = 2.$$

If x is any vertex at distance 2 from v_0 then, since K is 4-arc-transitive, there is some k in K which fixes v_0 and takes $c_5(v_0)$ to x. It follows that x is in U, because $k \in K(v_0) = C_4 \leqslant C_5$.

Now we shall prove (by induction) that U contains every vertex whose distance from v_0 is an even integer $2r$. The result has just been established when $r = 1$. Suppose it is true when $r = l-1$, and let $\partial(y, v_0) = 2l$. We can find a vertex z such that $\partial(y, z) = 2$, $\partial(z, v_0) = 2l-2$, and by the induction hypothesis z is in U, say $z = c(v_0)$, $c \in C_5$. Let $w = cc_5(v_0)$, so that $\partial(w, z) = 2$. Then $K(z) = cK(v_0)c^{-1}$ is contained in C_5 and it contains an element k' fixing z and taking w to y. Hence y is in U.

Denote by V_0 the set of vertices whose distance from v_0 is even; we have shown that $V_0 \subseteq U$. If $\mathscr{B}$ is bipartite, each involution e fixes the two parts setwise, and so $U = V_0$. Now

$$C_5 \leqslant L \leqslant K \quad \text{and} \quad C_5 \geqslant C_4 = K(v_0),$$

so that $C_5(v_0) = K(v_0) = L(v_0)$. Since the orbit of v_0 under K has length $|V\mathscr{B}| = 2|U|$, it follows that $|K : L| = 2$. If $\mathscr{B}$ is not bipartite then V_0 contains an adjacent pair of vertices, and $U = V\mathscr{B}$. In this case the preceding argument shows that $L = K$.

3. *Construction of* 5-*arc-transitive graphs*

When $\mathscr{B}$, K, and L are as in the previous section, we may construct a *covering* graph $\mathscr{C}$ of $\mathscr{B}$ in the following way. The vertex-set $V\mathscr{C}$ is $L \times V\mathscr{B}$, and the vertices (l_1, x_1), (l_2, x_2) are adjacent in $\mathscr{C}$ if there is an edge $e = x_1x_2$ in $\mathscr{B}$ and $l_2 = l_1e$.

Given any vertex (l, x) of $\mathscr{C}$, there are just three vertices adjacent to it, namely (le_1, y_1), (le_2, y_2) and (le_3, y_3), where $e_i = xy_i$ $(i = 1, 2, 3)$ are the three edges of $\mathscr{B}$ incident with x. Thus $\mathscr{C}$ is a cubic graph. At this stage, we make no assertion concerning its connectedness.

It is a consequence of Lemma 3 (or Theorem 1) that L is a normal subgroup of K; in other words, K acts by conjugation as a group of automorphisms of L. We have a split extension $H = L \rtimes K$, whose elements are the pairs (l, k), $l \in L$, $k \in K$, with multiplication defined by

$$(l_1, k_1)(l_2, k_2) = (l_1k_1l_2k_1^{-1}, k_1k_2).$$

LEMMA 5. *H acts faithfully as a group of automorphisms of $\mathscr{C}$.*

Proof. We define an action of H on $\mathscr{C}$ as follows:

$$(l_1, k_1)(l, x) = \big(l_1k_1lk_1^{-1}, k_1(x)\big).$$

If (l_1, k_1) acts trivially on $\mathscr{C}$, then it certainly fixes each vertex (id, x), $x \in V$, and this implies that $l_1 = k_1 = \mathrm{id}$. Hence H acts faithfully. It is straightforward to check that the action conforms with the multiplication in H, and that it defines an automorphism of $\mathscr{C}$.

The construction of $\mathscr{C}$ and H is a special case of a well-known technique, and we could now proceed to show that H acts 4-arc-transitively on $\mathscr{C}$ (see, for example [**1**, p. 129]). The unusual feature of the construction is contained in the next simple lemma.

LEMMA 6. *The mapping* ξ *defined by* $\xi(l, x) = (l^{-1}, l(x))$ *is an automorphism of* $\mathscr{C}$.

Proof. Suppose that (l_1, x_1) and (l_2, x_2) are adjacent in $\mathscr{C}$. Then $l_2 = l_1 e$, where e is the involution corresponding to the edge $e = x_1 x_2$ of $\mathscr{B}$. Now

$$\xi(l_2, x_2) = (l_2^{-1}, l_2(x_2)) = (el_1^{-1}, l_1 e(x_2)) = (l_1^{-1} e_1, l_1(x_2)),$$

where e_1 is the involution $l_1(e)$ (by Lemma 3). But

$$\xi(l_1, x_1) = (l_1^{-1}, l_1(x_1)),$$

and since e_1 is an edge of $\mathscr{B}$ joining $l_1(x_1)$ to $l_1(x_2)$, it follows that $\xi(l_1, x_1)$ is adjacent to $\xi(l_2, x_2)$.

Suppose that we are given a configuration of vertices and edges of $\mathscr{B}$ labelled as in figure 1. Using the definition of adjacency in $\mathscr{C}$, and Lemma 2, we may verify that $\mathscr{C}$ contains a configuration labelled as in figure 2. Let $\mathscr{C}_0$ denote the component of $\mathscr{C}$ containing this configuration.

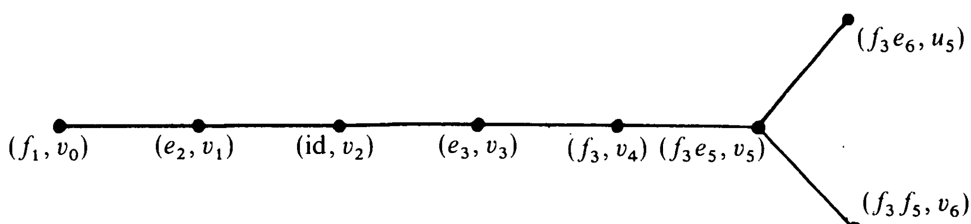

FIG. 2

THEOREM 2. *The group* $G = \langle H, \xi \rangle$ *acts 5-arc-transitively on* $\mathscr{C}_0$.

Proof. Let b be the automorphism of $\mathscr{B}$ defined in Section 2, and let $\beta = (e_3, b)$. For any involution e we have

$$\beta(e, x) = (e_3 b e b^{-1}, b(x)) = (e_3 e', b(x)),$$

where $e' = b(e)$. Now we can check that β takes the 'horizontal' 5-arc in figure 2 onto its 'lower' successor; for example

$$\beta(f_1, v_0) = (e_3 f_2, v_1) = (e_2, v_1),$$

$$\beta(f_3 e_5, v_5) = (e_3 f_4 e_6, v_6) = (e_3 e_4 e_5 e_6, v_6) = (f_3 f_5, v_6).$$

Next we remark that ξ fixes the first five vertices of the horizontal 5-arc, but

$$\xi(f_3 e_5, v_5) = (e_5 f_3, f_3 e_5(v_5)) = (e_5 f_3, u_4).$$

Also

$$\beta(e_5 f_3, u_4) = (e_3 e_6 f_4, u_5) = (e_3 \cdot e_6(f_4) \cdot e_6, u_5) = (e_3 e_4 e_6, u_5) = (f_3 e_3, u_5).$$

Thus the automorphism $\alpha = \beta\xi$ takes the horizontal 5-arc onto its 'upper' successor, and by Theorem 0 we conclude that G acts 5-arc-transitively on $\mathscr{C}_0$.

It follows from Theorem 0 that the automorphisms α and β are shunts generating the 5-arc-transitive group G. It is easy to compute their orders in terms of the orders of a and b. We have

$$\beta^k(l, x) = (a^k l b^{-k}, b^k(x))$$

$$\alpha^{2k}(l, x) = ((ab)^k l(ba)^{-k}, (ba)^k(x)),$$

so that the order of β is the least common multiple of the orders of a and b, and the order of α is twice that of ab.

4. *The components of the covering graph*

Each component of $\mathscr{C}$ is a connected cubic graph, isomorphic with the component $\mathscr{C}_0$ which contains the configuration shown in figure 2.

We study first the possibility that $\mathscr{C}_0$ is isomorphic to $\mathscr{B}$. Let Γ be a cycle in $\mathscr{B}$, whose edges are (in cyclic order) $d_0, d_1, \dots, d_{r-1}$. The element $d_0 d_1 \dots d_{r-1}$ of L defined in terms of the associated involutions is altered by conjugation when the initial edge of Γ is chosen differently, and so the statement

$$\partial(\Gamma) = d_0 d_1 \dots d_{r-1}$$

may be regarded as defining $\partial(\Gamma)$ up to conjugacy.

LEMMA 7. *If $\partial(\Gamma)$ is the identity for every cycle Γ in $\mathscr{B}$, then $\mathscr{C}_0$ is isomorphic to $\mathscr{B}$.*

Proof. Define a function $\lambda : V\mathscr{B} \to L$ in the following way: set $\lambda(v_0) = f_1$ and if $\lambda(v)$ has already been defined, and $e = vw$ is an edge, set $\lambda(w) = \lambda(v)e$. Since $\mathscr{B}$ is connected there is a sequence of edges joining v_0 to any given vertex x, and the rule may be applied. To see that it defines a function, suppose that $d_0, \dots, d_k$ and $d'_0, \dots, d'_l$ are two sequences of edges leading from v_0 to x. Then the hypothesis implies that

$$d_0 \dots d_k d'_l \dots d'_0 = \text{id}, \qquad \text{so that} \qquad d_0 \dots d_k = d'_0 \dots d'_l.$$

Thus $\lambda(x)$ is well-defined.

Now the function I defined by $I(v) = (\lambda(v), v)$ maps the vertices of $\mathscr{B}$ onto those of $\mathscr{C}_0$, and the definition of λ ensures that it is an isomorphism.

Suppose that the vertices of the cycle Γ are (in cyclic order) $w_0, w_1, \dots, w_{r-1}$. Since K acts 4-arc-transitively, there are unique elements s_i $(0 \leqslant i \leqslant r-1)$ of K such that

$$s_i(w_i, w_{i+1}, w_{i+2}, w_{i+3}, w_{i+4}) = (w_{i+1}, w_{i+2}, w_{i+3}, w_{i+4}, w_{i+5}),$$

where the subscripts are taken modulo r. Let

$$t_i = \begin{cases} s_i & (i = 0), \\ s_0^{-1} s_1^{-1} \dots s_{i-1}^{-1} s_i s_{i-1} \dots s_1 s_0 & (1 \leqslant i \leqslant r-1). \end{cases}$$

Then it may be verified that each t_i takes the 4-arc $(w_0, \ldots, w_4)$ onto one of its successors in $\mathscr{B}$. Now there is a unique element m in K which takes $(w_0, \ldots, w_4)$ onto the 'standard' 4-arc $(v_0, \ldots, v_4)$, and mt_im^{-1} takes $(v_0, \ldots, v_4)$ onto one of its successors. Hence

$$mt_im^{-1} = a \text{ or } b \qquad (0 \leqslant i \leqslant r-1).$$

We shall define the *K-signature* of Γ to be the word in a and b given by

$$\omega(\Gamma; a, b) = (mt_0m^{-1})(mt_1m^{-1}) \ldots (mt_{r-1}m^{-1}),$$

remarking that, although the K-signature is an element of K, we do not consider it as such. The next lemma explains why.

LEMMA 8. *As an element of K, $\omega(\Gamma; a, b)$ is the identity for every cycle Γ.*

Proof. It follows from the definition that $\omega(\Gamma; a, b)$ is conjugate in K to $t_0t_1 \ldots t_{r-1}$. But

$$t_0t_1 \ldots t_{r-1} = s_0 \cdot s_0^{-1}s_1s_0 \cdot \ldots \cdot s_0^{-1}s_1^{-1} \ldots s_{r-2}^{-1}s_{r-1} \ldots s_1s_0,$$

$$= s_{r-1} \ldots s_1s_0.$$

Now the right-hand side is an element of K fixing the 4-arc $(w_0, \ldots, w_4)$, and so it is the identity.

LEMMA 9. *For any cycle Γ in $\mathscr{B}$, $\partial(\Gamma)$ is conjugate in K to $\omega(\Gamma; b, a)$.*

Proof. Suppose that Γ has vertices $w_0, \ldots, w_{r-1}$ and edges $d_0 = w_0w_1$, $d_1 = w_1w_2, \ldots, d_{r-1} = w_{r-1}w_0$, and let m, s_i, t_i $(0 \leqslant i \leqslant r-1)$ be the elements of K defined above. By definition, $\delta(\Gamma)$ is conjugate to

$$d_3d_4 \ldots d_1d_2 = d_3 \cdot s_0(d_3) \cdot s_1s_0(d_3) \cdot \ldots \cdot s_{r-2} \ldots s_0(d_3)$$

$$= d_3 \cdot s_0d_3s_0^{-1} \cdot s_1s_0d_3s_1^{-1}s_0^{-1} \cdot \ldots$$

$$= d_3t_0 \cdot d_3t_1 \cdot \ldots \cdot d_3t_{r-1}.$$

Conjugating by m, we see that $\delta(\Gamma)$ is conjugate to

$$= e_3mt_0m^{-1} \cdot e_3mt_1m^{-1} \cdot \ldots \cdot e_3mt_{r-1}m^{-1}$$

$$= \omega(\Gamma; b, a),$$

since $a = e_3b$ and $b = e_3a$.

THEOREM 3. *The graphs $\mathscr{C}_0$ and $\mathscr{B}$ are isomorphic if and only if $\mathscr{B}$ admits a group of automorphisms acting 5-arc-transitively.*

Proof. If $\mathscr{B}$ does not admit a 5-arc-transitive group, then it cannot be isomorphic with $\mathscr{C}_0$, since $\mathscr{C}_0$ does admit such a group.

Conversely, let G be a group acting 5-arc-transitively on $\mathscr{B}$. It follows from Tutte's theorem that G is the full group of automorphisms of $\mathscr{B}$, so K is a subgroup of index two in G, and consequently normal in G. There is a (unique) element g in G taking the 5-arc $(v_0, \ldots, v_4, v_5)$ onto $(v_0, \ldots, v_4, u_4)$, and, since g^2 fixes the first 5-arc, g is an involution. Now gag acts like b on the 4-arc $(v_0, \ldots, v_4)$, and both elements are in K (since K is normal in G). Thus $gag = b$ and $gbg = a$, and $\omega(\Gamma; b, a)$ is conjugate in G to $\omega(\Gamma; a, b)$. It follows from Lemmas 7, 8 and 9 that $\mathscr{C}_0$ is isomorphic with $\mathscr{B}$.

5. *Examples*

In a sense, the trivial first part of Theorem 3 is the more significant. If we are given a known graph $\mathscr{B}$, which admits a 4-arc-transitive group K, but not a 5-arc-transitive one, then we can be sure that our construction will produce a 'new' 5-arc-transitive graph. For example, when $\mathscr{B}$ is Heawood's graph on 14 vertices it turns out that $\mathscr{C}_0 = \mathscr{C}$ and we obtain a graph with 2352 vertices [**2**]. However, when $\mathscr{B}$ is Tutte's graph on 30 vertices (which is 5-arc-transitive and admits two distinct 4-arc-transitive groups), we know from the second part of Theorem 3 that $\mathscr{C}_0 = \mathscr{B}$ and nothing new is obtained.

There are several infinite families of cubic graphs which are known to admit 4-arc-transitive groups but not 5-arc-transitive ones. They are associated with the octahedral subgroups of linear fractional groups. If p is a prime congruent to $\pm 1 \pmod{16}$, the group $K = \mathrm{PSL}(2, p)$ acts 4-arc-transitively on a cubic graph with $p(p^2-1)/48$ vertices [**4**]; the action is primitive, and so the graph cannot be bipartite and $L = K$. Thus our construction yields a 5-arc-transitive graph with (at most) $p^2(p^2-1)^2/96$ vertices.

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