HOMOLOGICAL COVERINGS OF GRAPHS

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ABSTRACT

The homology group of a graph, with any coefficient ring, can be used to construct covering graphs. The properties of the covering graphs are studied, and it is proved that they admit groups of automorphisms related to the group of the base graph. In the case of cubic graphs the construction throws some light on classification problems and it can be used to show that certain finitely presented groups are infinite.

1. *Introduction*

The idea of labelling the edges of a graph by group elements in order to construct a covering graph is quite well known. An account of the theory may be found in Chapter 19 of the author's book [2] and a similar concept is often known by the name of a 'voltage graph' [11]. If we insist that the covering graph should inherit the symmetry properties of the base graph then it is necessary that the edge labelling be compatible with the action of the automorphism group, and several ways of constructing such a labelling have been described [2, 3, 5].

In this paper the group used for the edge labelling will be the first homology group of the graph. The functorial properties of homology ensure that, in general terms, the compatibility condition is satisfied. However, in graph theory it is usually convenient to deal with a specific representation of the homology group (or cycle space, as it is known) in terms of the basic cycles associated with a spanning tree. We shall give an explicit proof of the compatibility condition in these terms.

As with all constructions of this kind, it is crucial to determine the number of components of the covering graph. In our case, when the homology is taken with integer coefficients, we have the remarkable result that the number of components is equal to the number of spanning trees of the base graph.

The theory is readily applicable to the study of cubic graphs. This has already been done in a very simple and special situation in a previous paper [5]. Here we shall investigate some more general questions and obtain results relating to the classification of cubic graphs with a given symmetry type. We shall also explain how the techniques can be used to show that certain families of finitely presented groups have infinite order.

2. *The construction*

Let X be a finite 2-connected graph with vertex-set V and edge-set E. Each edge e in E is an unordered pair $\{\alpha, \beta\}$ of adjacent vertices, and corresponds to two ordered pairs (α, β) and (β, α), called *sides* of X. The set of all sides of X will be denoted by SX. An *orientation* of X is a choice of one of the two sides corresponding to each edge or, more formally, a pair of functions h and t from E to V such that $e = \{h(e), t(e)\}$.

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We define a *path* in X to be a sequence of vertices $\gamma_0, \gamma_1, \ldots, \gamma_r$ such that γ_i is adjacent to γ_{i+1} $(0 \leqslant i \leqslant r-1)$ and $\gamma_i \neq \gamma_{i+2}$ $(0 \leqslant i \leqslant r-2)$. Note that we do not require that $\gamma_0, \gamma_1, \ldots, \gamma_r$ be distinct vertices: in particular, when $\gamma_0 = \gamma_r$ we have a *closed path*.

Let R be a commutative ring with multiplicative identity 1, and let T be a spanning tree for X (regarded as a subset of E). Denote the edges of X which are not in T by $f_1, f_2, \ldots, f_c$, where $c = |E| - |V| + 1$. We choose a fixed orientation for X and let $t(f_i) = \sigma_i$, $h(f_i) = \tau_i$. There is a unique path $\tau_i, \alpha, \beta, \ldots, \theta, \sigma_i$ in T from τ_i to σ_i and this, together with the side (σ_i, τ_i), defines an oriented cycle C_i whose sides are

$$(\sigma_i, \tau_i), (\tau_i, \alpha), \ldots, (\theta, \sigma_i).$$

The cornerstone of our construction is a labelling of the sides of X with values in the abelian group R^c. We define $\mathbf{x} : SX \to R^c$ as

$$\mathbf{x}(\sigma, \tau) = \big(x_1(\sigma, \tau), \ldots, x_c(\sigma, \tau)\big),$$

where

$$x_i(\sigma, \tau) = \begin{cases} 1, & \text{if } (\sigma, \tau) \text{ is in } C_i, \\ -1, & \text{if } (\tau, \sigma) \text{ is in } C_i, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, we have $\mathbf{x}(\tau, \sigma) = -\mathbf{x}(\sigma, \tau)$.

We construct a covering graph $\tilde{X}$ of X, with respect to the given orientation and spanning tree of X. The vertex-set of $\tilde{X}$ is

$$\tilde{V} = R^c \times V,$$

and the edge-set $\tilde{E}$ contains those pairs $\{(\mathbf{z}, \alpha), (\mathbf{z}', \alpha')\}$ for which

$$\{\alpha, \alpha'\} \in E \qquad \text{and} \qquad \mathbf{z} - \mathbf{z}' = \mathbf{x}(\alpha, \alpha').$$

THEOREM 1. *The graph $\tilde{X}$ is independent (up to isomorphism) of the various choices involved in its definition.*

Proof. Suppose first that the spanning tree T is fixed. When the orientation is also fixed the only remaining choice is the ordering of the edges $f_1, \ldots, f_c$. If a new ordering is obtained by applying the permutation π to the subscripts, the new labels $\mathbf{x}^*$ are related to the old ones by the rule

$$\mathbf{x}^*(\sigma, \tau) = P\mathbf{x}(\sigma, \tau),$$

where P is the permutation matrix corresponding to π. The resulting graphs $\tilde{X}$ and $\tilde{X}^*$ are isomorphic under the mapping

$$(\mathbf{z}, \alpha) \longmapsto (P\mathbf{z}, \alpha).$$

It is clear from the construction that the labelling depends only upon the orientation assigned to the edges not in T. Suppose then that one such edge, say f_1, has its orientation reversed. The new labels are related to the old ones by the rule

$$\mathbf{x}^*(\sigma, \tau) = Q\mathbf{x}(\sigma, \tau),$$

where $Q = \operatorname{diag}(-1, 1, 1, \dots, 1)$, and the graphs $\tilde{X}$ and $\tilde{X}^*$ are isomorphic under the mapping $(\mathbf{z}, \alpha) \mapsto (Q\mathbf{z}, \alpha)$.

We now consider the effect of changing the spanning tree T. It is clear that any two spanning trees are related by a sequence of operations of the following kind. Let e be any edge in T and let $Z(e)$ be the cutset of edges of X which it defines; $Z(e)$ contains e and some of the edges f_i not in T. Without loss of generality we may suppose that f_1 is in $Z(e)$, so that e (with the appropriate orientation) is in C_1. Define T^* to be the spanning tree obtained from T by deleting e and adding f_1.

Let $\tilde{X}$ and $\tilde{X}^*$ be the covering graphs constructed using T and T^*. The edges of X not in T^* are

$$f_1^* = e,\ f_2^* = f_2, \dots, f_c^* = f_c.$$

The oriented cycle C_1^* is the same as C_1. If f_j is not in $Z(e)$, then $C_j^* = C_j$. On the other hand, if f_j $(j \neq 1)$ is in $Z(e)$ then C_j^* is the 'difference' $C_j - C_1$. It follows that

$$x_j^*(\sigma, \tau) = \begin{cases} x_j(\sigma, \tau) - x_1(\sigma, \tau), & \text{if } j \neq 1 \text{ and } f_j \in Z(e), \\ x_j(\sigma, \tau), & \text{otherwise}. \end{cases}$$

Hence

$$\mathbf{x}^*(\sigma, \tau) = A\mathbf{x}(\sigma, \tau),$$

where A is a triangular matrix with diagonal entries equal to 1. Hence A is invertible and the mapping $(\mathbf{z}, \alpha) \mapsto (A\mathbf{z}, \alpha)$ defines an isomorphism from $\tilde{X}$ to $\tilde{X}^*$.

3. *The components of the covering graph*

Although the graph $\tilde{X}$ is, in general, not connected, there is a simple way to describe its components algebraically.

When $P = (\gamma_0, \gamma_1, \dots, \gamma_r)$ is a path in X we write

$$\mathbf{x}(P) = \sum_{j=0}^{r-1} \mathbf{x}(\gamma_j, \gamma_{j+1}).$$

Regarding R^c as a $\mathbb{Z}$-module (abelian group), we denote by Λ the submodule generated by the values of $\mathbf{x}(P)$ for closed paths P; that is

$$\Lambda = \langle \mathbf{x}(P) \mid P \text{ is a closed path} \rangle,$$

so that Λ consists of all $\mathbb{Z}$-linear combinations of labels on closed paths. Choose any vertex σ_0 of X and set

$$\nu(\sigma_0) = 0,$$

where 0 indicates the zero coset $\mathbf{0} + \Lambda$ in R^c/Λ. If σ is any vertex and S is a path in X from σ_0 to σ, set

$$\nu(\sigma) = \mathbf{x}(S) + \Lambda.$$

If S' is any other path from σ_0 to σ then $\mathbf{x}(S)-\mathbf{x}(S')$ is in Λ, since S followed by the reverse of S' is the union of some closed paths and some non-closed paths traversed in opposite senses. Hence $v(\sigma)$ is well defined as a member of R^c/Λ.

THEOREM 2. *The number of components of $\tilde{X}$ is equal to the index of Λ in R^c.*

Proof. Define a function $f : \tilde{V} \to R^c/\Lambda$ by the rule

$$f(\mathbf{z}, \sigma) = -\mathbf{z} + v(\sigma).$$

Given any coset $\mathbf{r}+\Lambda$ in R^c/Λ we have

$$f(-\mathbf{r}, \sigma_0) = \mathbf{r} + v(\sigma_0) = \mathbf{r} + \Lambda,$$

and so f is a surjection. We show that $(\mathbf{z}, \sigma)$ and $(\mathbf{z}', \sigma')$ are in the same component of $\tilde{X}$ if and only if $f(\mathbf{z}, \sigma) = f(\mathbf{z}', \sigma')$.

Suppose that $(\mathbf{z}, \sigma)$ and $(\mathbf{z}', \sigma')$ are in the same component. A path from $(\mathbf{z}, \sigma)$ to $(\mathbf{z}', \sigma')$ in $\tilde{X}$ contains vertices

$$(\mathbf{z}, \sigma), (\mathbf{z}+\mathbf{x}(\sigma, \alpha), \alpha), \ldots, (\mathbf{z}+\mathbf{x}(S), \sigma'),$$

where $S = (\sigma, \alpha, \ldots, \sigma')$ is a path in X, and $\mathbf{x}(S) = \mathbf{z}'-\mathbf{z}$. Let S_0, S_0' be paths in X from σ_0 to σ and σ' respectively, so that

$$v(\sigma) = \mathbf{x}(S_0)+\Lambda, \qquad v(\sigma') = \mathbf{x}(S_0')+\Lambda.$$

We have

$$f(\mathbf{z}, \sigma) = (-\mathbf{z}+\mathbf{x}(S_0))+\Lambda,$$

$$f(\mathbf{z}', \sigma') = (-\mathbf{z}'+\mathbf{x}(S_0'))+\Lambda.$$

But the union of S_0, S, and the reverse of S_0' is the union of closed paths and some non-closed paths traversed in opposite senses. Hence

$$(-\mathbf{z}+\mathbf{x}(S_0))-(-\mathbf{z}+\mathbf{x}(S_0')) = \mathbf{x}(S)+\mathbf{x}(S_0)-\mathbf{x}(S_0')$$

is in Λ, and $f(\mathbf{z}, \sigma) = f(\mathbf{z}', \sigma')$.

Conversely, if the equality holds then

$$\mathbf{z}'-\mathbf{z} = \mathbf{x}(S_0)+\mathbf{x}(S_0')\ (\mathrm{mod}\ \Lambda)$$

and so

$$\mathbf{z}'-\mathbf{z} = \mathbf{x}(P),$$

where P is a path from σ to σ' in X. Hence the covering path in $\tilde{X}$ goes from $(\mathbf{z}, \sigma)$ to $(\mathbf{z}', \sigma')$, and these vertices are in the same component of $\tilde{X}$.

Theorem 2 can be given a more concrete form, especially when $R = \mathbb{Z}$.

Given a spanning tree T and an orientation of X we define the *cycle matrix* K (over R) as follows. The rows of K correspond to the basic cycles C_i ($1 \leqslant i \leqslant c$), and the columns correspond to edges of X, which may be arranged in such a way that the edges $f_1, f_2, \ldots, f_c$ come first. The column $\mathbf{k}_{*e}$ of K corresponding to an edge $e = \{\sigma, \tau\}$ is defined by

$$\mathbf{k}_{*e} = \begin{cases} \mathbf{x}(\sigma, \tau), & \text{if } h(e) = \tau, \\ -\mathbf{x}(\sigma, \tau), & \text{if } h(e) = \sigma. \end{cases}$$

Now let us focus on the case when $R = \mathbb{Z}$. For any path P and any edge e, suppose that $t(e) = \alpha$, $h(e) = \beta$ and define

$$u_e(P) = (\text{number of times } (\alpha, \beta) \text{ occurs in } P) - (\text{number of times } (\beta, \alpha) \text{ occurs in } P).$$

We have a row vector $\mathbf{u}(P)$, whose columns can be labelled in the same order as those of K. For the basic cycles C_i, it follows from the definitions that

$$\mathbf{u}(C_i) = \mathbf{k}_{i*},$$

the i-th row of K. Since these rows form a basis for the cycle space over $\mathbb{Z}$, for any closed path C there are integers y_i $(1 \leqslant i \leqslant c)$ such that

$$\mathbf{u}(C) = \sum_{i=1}^{c} y_i \mathbf{u}(C_i).$$

Using the preceding expression for $\mathbf{u}(C_i)$, we see that

$$\mathbf{u}^t(C) = K^t \mathbf{y}(C),$$

where t denotes the transpose and $\mathbf{y}(C) = (y_1, \ldots, y_c)^t$.

Furthermore, the label $\mathbf{x}(C)$ is, by definition, the sum of the labels on the sides of C, thus

$$\mathbf{x}(C) = \sum_e u_e(C)\mathbf{k}_{*e} = K\mathbf{u}^t(C).$$

Combining these results we obtain

$$\mathbf{x}(C) = KK^t\mathbf{y}(C).$$

THEOREM 3. *When $R = \mathbb{Z}$ the number of components of $\tilde{X}$ is equal to the number of spanning trees of X.*

Proof. Given any $\mathbf{y}$ in $\mathbb{Z}^c$ we can construct a closed path C such that $\mathbf{y}(C) = \mathbf{y}$, and hence $\mathbf{x}(C) = KK^t\mathbf{y}$. In other words, the submodule Λ of $\mathbb{Z}^c$ has a generating matrix $L = KK^t$. It is a standard result [7, p. 14] that the index of Λ in $\mathbb{Z}^c$ is equal to $|\det L|$, provided L is invertible.

On the other hand, it is known (see [6; 14, p. 18]) that $\det KK^t$ is the number of spanning trees of X. Hence L is indeed invertible and the result follows.

Theorem 3 is useful because there are alternative ways of calculating the number of spanning trees of X. When X is regular there is a formula involving the eigenvalues and their multiplicities [2, p. 36], and when X is distance-regular these numbers are determined by the intersection array. Some typical results may be found in [5].

When $R \neq \mathbb{Z}$ it is possible that $\det L = 0$ in R. In such cases we must use more direct means to calculate the index of Λ in R^c. For example, when $R = \mathbb{Z}_3$ and X is the complete bipartite graph $K_{3,3}$, it turns out that L is the 4×4 matrix each of whose entries is 1. Thus the order of Λ is 3 and its index in $(\mathbb{Z}_3)^4$ is 27. Hence $\tilde{X}$ has 27 components and each of them has $6 \times 3^4/27 = 18$ vertices. In fact, each component is a copy of the well-known Pappus graph [8]. We shall discuss these matters in greater detail in Section 5.

We now return to the case when $R = \mathbb{Z}$ and establish a result which will be useful in later sections of the paper. Let us say that a closed path C is *reverting* if $\mathbf{u}(C) = \mathbf{0}$. In other words, C traverses each edge the same number of times in both directions.

THEOREM 4. *When $R = \mathbb{Z}$ and C is a closed path in X we have $\mathbf{x}(C) = \mathbf{0}$ if and only if C is a reverting closed path.*

Proof. Clearly, if C is reverting then $\mathbf{x}(C) = \mathbf{0}$ (this is true for any R). Conversely, we have

$$\mathbf{x}(C) = KK'\mathbf{y}(C),$$

and when $R = \mathbb{Z}$, KK' is invertible. Hence $\mathbf{x}(C) = \mathbf{0}$ implies that $\mathbf{y}(C) = \mathbf{0}$ and since $\mathbf{u}'(C) = K'\mathbf{y}(C)$ we have $\mathbf{u}(C) = \mathbf{0}$ also. Thus C is reverting.

Theorem 4 does not hold when $R \neq \mathbb{Z}$. For example, when $R = \mathbb{Z}_m$ the closed path C obtained by going m times around any cycle has $\mathbf{x}(C) = \mathbf{0}$, but it is not reverting.

4. *Action of graph automorphisms*

Throughout this section we continue to assume that an orientation and a spanning tree for X have been chosen.

Let g be any automorphism of X. We define a $c \times c$ matrix $\hat{g} = (g_{ij})$ over R, representing an action of g on R^c, as follows. Let f_j $(1 \leqslant i \leqslant c)$ be an edge of X not in T, and suppose that $t(f_j) = \sigma_j$, $h(f_j) = \tau_j$; then we set

$$g_{ij} = \begin{cases} 1, & \text{if } (g\sigma_j, g\tau_j) \text{ is in } C_i, \\ -1, & \text{if } (g\tau_j, g\sigma_j) \text{ is in } C_i, \\ 0, & \text{otherwise}. \end{cases}$$

THEOREM 5. *The action of g on SX defined by $g(\sigma, \tau) = (g\sigma, g\tau)$ and the action of $\hat{g}$ on R^c as a matrix operating on column vectors are compatible with the labelling $\mathbf{x} : SX \to R^c$. That is*

$$\hat{g}(\mathbf{x}(\sigma, \tau)) = \mathbf{x}(g\sigma, g\tau)$$

for all $(\sigma, \tau) \in SX$.

Proof. We have to show that for each $i = 1, 2, \ldots, c$ and each $(\sigma, \tau) \in SX$,

$$\sum_j g_{ij} x_j(\sigma, \tau) = x_i(g\sigma, g\tau)$$

where, according to the definition in Section 2, $x_i(g\sigma, g\tau)$ is 1 if $(g\sigma, g\tau)$ is in C_i, -1 if $(g\tau, g\sigma)$ is in C_i, and 0 otherwise.

Suppose first that the edge $\{\sigma, \tau\}$ is one of the edges not in T, say f_k. We may assume that the notation has been chosen so that $t(f_k) = \sigma$, $h(f_k) = \tau$. It follows that $x_j(\sigma, \tau) = 0$ for all $j \neq k$, whereas $x_k(\sigma, \tau) = 1$. Thus

$$\sum_j g_{ij} x_j(\sigma, \tau) = g_{ik} = x_i(g\sigma, g\tau)$$

by the definition of g_{ik}.

Suppose now that the edge $\{\sigma, \tau\}$ is in T. Let $V = V_1 \cup V_2$ be the partition of the vertex-set which is defined by deleting $\{\sigma, \tau\}$ from T, where we may take $\sigma \in V_1$, $\tau \in V_2$. The corresponding cutset H, consisting of edges with one vertex in V_1 and one vertex in V_2, contains $\{\sigma, \tau\}$ and some of the edges f_j. Furthermore, f_j is in H if and only if (σ, τ) or (τ, σ) is in C_j.

Fix i $(1 \leqslant i \leqslant c)$ and let C^* be the set of edges $e = \{\alpha, \beta\}$ such that $(g\alpha, g\beta)$ or $(g\beta, g\alpha)$ is in C_i. Now, according to the definitions of g_{ij} and $x_j(\sigma, \tau)$, if

$$g_{ij} = x_j(\sigma, \tau) = \pm 1\,,$$

we must have f_j in $C^* \cap H$, with the orientation of f_j in C^* going from V_1 to V_2. Similarly, if

$$g_{ij} = -x_j(\sigma, \tau) = \pm 1\,,$$

we must have f_j in $C^* \cap H$, with the orientation of f_j in C^* going from V_2 to V_1.

But $C^* \cap H$ contains an even number of edges. If $\{\sigma, \tau\}$ is not in $C^* \cap H$ (that is, if it is not in C^*) then there is an even number of edges f_j in $C^* \cap H$: half of them are oriented in C^* from V_1 to V_2 and the other half from V_2 to V_1. Hence

$$\sum_j g_{ij} x_j(\sigma, \tau) = 0$$

in this case, and also the value of $x_i(g\sigma, g\tau)$ is zero since neither $(g\sigma, g\tau)$ nor $(g\tau, g\sigma)$ is C^*. If $\{\sigma, \tau\}$ is in $C^* \cap H$, there is an odd number of f_j in $C^* \cap H$, and they make a net contribution of ± 1 to the sum, according to the orientation of $\{\sigma, \tau\}$ in C^*. Hence the result.

Let G be a group of automorphisms of X, and define $\tilde{G}$ to be the semi-direct product $R^c \rtimes G$ whose members are the pairs $(\mathbf{z}, g)$ in $R^c \times G$, with the group operation in $\tilde{G}$ given by

$$(\mathbf{z}_1, g_1)(\mathbf{z}_2, g_2) = \big(\mathbf{z}_1 + \hat{g}_1(\mathbf{z}_2), g_1 g_2\big)\,.$$

It can be checked that, as a consequence of the compatibility condition, the action of $\tilde{G}$ on the vertices of $\tilde{X}$ given by

$$(\mathbf{z}, g)(\mathbf{z}', \sigma) = \big(\mathbf{z} + \hat{g}(\mathbf{z}'), g(\sigma)\big)$$

defines $\hat{G}$ as a group of automorphisms of $\tilde{X}$. In the next section we shall study the relationship between G and $\tilde{G}$ more closely in the case where X is a cubic graph.

5. *Applications to cubic graphs*

Let X be a connected cubic graph admitting a group of automorphisms acting transitively on the vertices and edges. In this case the foregoing theory takes an especially elegant and useful form. We shall describe the framework and its application to classification problems for cubic graphs in this section, and in the following section we shall explain how such ideas can be useful in combinatorial group theory.

Henceforth we shall assume that X is a finite connected cubic graph and G is a group of automorphisms which acts transitively on its vertices and edges. It is known [13] that G acts regularly on the paths of length s for some s in the range $1 \leqslant s \leqslant 5$.

We shall choose an s-path $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ in X and refer to it as the *standard* s-path. Let $\alpha_{s-1}, \beta', \beta''$ be the three vertices adjacent to α_s. Since G acts regularly on the s-paths there are unique automorphisms a, b, t in G such that

$$a(\alpha_0, \alpha_1, \ldots, \alpha_{s-1}, \alpha_s) = (\alpha_1, \alpha_2, \ldots, \alpha_s, \beta'),$$

$$b(\alpha_0, \alpha_1, \ldots, \alpha_{s-1}, \alpha_s) = (\alpha_1, \alpha_2, \ldots, \alpha_s, \beta''),$$

$$t(\alpha_0, \alpha_1, \ldots, \alpha_{s-1}, \alpha_s) = (\alpha_s, \alpha_{s-1}, \ldots, \alpha_1, \alpha_0).$$

It can be shown that the *shunt* automorphisms a and b generate G (see [2, p. 115]). The automorphism t is introduced in order to describe the underlying structure more intuitively.

When s is given, the automorphisms a, b, t satisfy some canonical relations. First, since t^2 fixes the standard s-path and G acts regularly on s-paths, it follows that $t^2 = 1$. Similarly, examining the action of tat, we see that it must be either a^{-1} or b^{-1}. The first possibility can occur when $s = 2, 3, 4, 5$, but not when $s = 1$, and for the purposes of this paper we shall restrict ourselves to these cases. (In fact, $tat = b^{-1}$ can occur only when $s = 1, 2, 4$.) In each case an analysis of the action of suitable combinations of a, b and t on the basic s-path provides a set $R(s)$ of $s+3$ relations which must hold in G. If we denote by R_0 the relations

$$t^2 = (at)^2 = (bt)^2 = (ab^{-1})^2 = 1,$$

then the full sets of relations are as follows

$R(2)$: $R_0 \cup \{abta^2 = b^2\}$,

$R(3)$: $R_0 \cup \{(a^2b^{-2})^2 = 1, a^2bta^3 = bab\}$,

$R(4)$: $R_0 \cup \{(a^2b^{-2})^2 = 1, a^3b^{-3}a^3 = bab, a^3bta^4 = ba^2b\}$,

$R(5)$: $R_0 \cup \{(a^2b^{-2})^2 = 1, a^3b^{-3}a^3 = b^3, a^4b^{-4}a^4 = ba^2b, a^4bta^5 = ba^3b\}$.

The relations in this form originate from unpublished work of J. H. Conway. Equivalent sets of relations can be found in the paper of Djokovic and Miller [10].

These relations will be satisfied in any group acting regularly on the s-paths of a cubic graph. They are proved by a purely local analysis of the action of a, b and t and so they hold even when the graph is the infinite cubic tree. It follows that the groups

$$U_s = \langle a, b, t \mid R(s) \rangle \qquad (s = 2, 3, 4, 5)$$

are infinite groups, and a group G acting in the appropriate way on a finite cubic graph is a proper quotient of the relevant U_s [**10**, Theorem 1]. (In the application of the relations to finite cases it must be remembered that a and b are not necessarily interchangeable.)

Further relations in G arise from the existence of closed paths in the graph. There is no loss of generality in assuming that the initial segment of a path P coincides with the standard s-arc, and so we may suppose that $P = (\alpha_0, \alpha_1, \ldots, \alpha_r)$. Then, for each i in the range $0 \leqslant i \leqslant r-s-1$ there is a unique automorphism v_i such that

$$v_i(\alpha_i, \alpha_{i+1}, \ldots, \alpha_{i+s}) = (\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_{i+s+1}).$$

For $0 \leqslant i \leqslant r-s-1$ define

$$w_i = v_0^{-1} v_1^{-1} \ldots v_{i-1}^{-1} v_i v_{i-1} \ldots v_0 .$$

Each w_i is either a or b, since it shunts the standard s-path onto one of its successors. Furthermore,

$$w_0 w_1 \ldots w_i = v_i v_{i-1} \ldots v_0 ,$$

and so the automorphism $w_0 w_1 \ldots w_i$ (which is a word in positive powers of a and b) shunts the standard s-path through i steps along P. In particular, when P is a closed path we may write $\alpha_r = \alpha_0, \alpha_{r+1} = \alpha_1, \ldots, \alpha_{r+s} = \alpha_s$ and extend the definitions of v_i and w_i to the range $0 \leqslant i \leqslant r-1$. Since the automorphism $v_{r-1} v_{r-2} \ldots v_0$ fixes the standard s-path, it is the identity. Hence

$$w = w_0 w_1 \ldots w_{r-1}$$

is a word in positive powers of a and b which is equal to the identity automorphism, and we have a relation which must hold in G.

For example, let $K_{3,3}$ be the complete bipartite graph with vertex bipartition $\{1, 3, 5\} \cup \{2, 4, 6\}$. The full group of automorphisms acts regularly on the 3-paths, and if $(1, 2, 3, 4)$ is the standard 3-path the automorphisms a, b, t are

$$a = (1234)(56),$$

$$b = (123456),$$

$$t = (14)(23)(56).$$

In addition to the relations $R(3)$, which can be checked, there are relations like

$$a^4 = 1, \quad b^6 = 1, \quad (ab)^6 = 1 .$$

It should be noted that the relations $a^4 = 1$ and $R(3)$ together define G completely, whereas $b^6 = 1$ and $R(3)$ together define the group of a 3-fold covering of $K_{3,3}$, and $(ab)^6 = 1$ and $R(3)$ together define an infinite group. The last assertion will be justified later in this section. In general, it is not easy to determine a complete set of defining relations for G.

These techniques can be easily applied to the covering graphs $\tilde{X}$ constructed in the earlier sections of this paper. If the label assigned to the first side of the standard s-path is

$$\mathbf{x}(\alpha_0, \alpha_1) = \mathbf{x}_0 ,$$

then the compatibility condition (Theorem 4) asserts that

$$\mathbf{x}(\alpha_i, \alpha_{i+1}) = \hat{a}^i(\mathbf{x}_0) = \hat{b}^i(\mathbf{x}_0) \qquad (0 \leqslant i \leqslant s-1).$$

We can therefore choose the vertices of a standard s-path in $\tilde{X}$ to be

$$(\mathbf{0}, \alpha_0), (\mathbf{x}_0, \alpha_1), (\mathbf{x}_0 + \hat{a}(\mathbf{x}_0), \alpha_2), \ldots, (\mathbf{x}_0 + \hat{a}(\mathbf{x}_0) + \ldots + \hat{a}^{s-1}(\mathbf{x}_0), \alpha_s).$$

It can be checked that the automorphisms

$$\tilde{a} = (\mathbf{x}_0, a), \qquad \tilde{b} = (\mathbf{x}_0, b)$$

act as shunts with respect to the standard s-path in $\tilde{X}$. Also, we can define

$$\tilde{t} = (\mathbf{y}_0, t), \qquad \text{where } \mathbf{y}_0 = (1 + \hat{a} + \ldots + \hat{a}^{s-1})(\mathbf{x}_0),$$

and it can be verified that $\tilde{a}, \tilde{b}, \tilde{t}$ satisfy the relations $R(s)$. Thus the group $\tilde{G}$, like G, is a quotient of the universal group U_s.

The following theorem is the main tool for investigating the relationship between closed paths in $\tilde{X}$ and closed paths in X, and the consequent relationship between presentations for the respective groups. Recall that if $P = (\sigma_0, \sigma_1, \ldots, \sigma_r)$ is a path in X, then there is a unique covering path $\tilde{P}$ starting from a given vertex $(\mathbf{z}_0, \sigma_0)$ in $\tilde{X}$: its vertices are

$$(\mathbf{z}_0, \sigma_0), (\mathbf{z}_0 + \mathbf{x}(\sigma_0, \sigma_1), \sigma_1), \ldots, (\mathbf{z}_0 + \mathbf{x}(P), \sigma_r).$$

THEOREM 6. *Let X be a finite connected cubic graph admitting a group of automorphisms G with generators a and b as above. Let C be a closed path in X corresponding to the relation*

$$w(a, b) = 1$$

in G, and let $\tilde{C}$ be a path in $\tilde{X}$ covering C. The following statements are equivalent:

(1) $\mathbf{x}(C) = \mathbf{0}$,

(2) *$\tilde{C}$ is a closed path in $\tilde{X}$*,

(3) *$w(\tilde{a}, \tilde{b}) = 1$ in $\tilde{G}$*.

Proof. (1) $\Rightarrow$ (2) This follows immediately from the definition of $\tilde{C}$.

(2) ⇒ (3) Since $\tilde{a}$ and $\tilde{b}$ act as shunt automorphisms for the standard s-path in $\tilde{X}$, this implication follows from the general theory outlined above.

(3) ⇒ (1) Let v_i and w_i $(0 \leqslant i \leqslant r-1)$ be the automorphisms associated with C as above, so that each w_i is either a or b and

$$w(a,b) = w_0 w_1 \dots w_{r-1} = 1\,.$$

The labels on the sides of C are $\mathbf{x}_0, \hat{v}_0(\mathbf{x}_0), \hat{v}_1\hat{v}_0(\mathbf{x}_0), \dots, (\hat{v}_{r-2}\dots\hat{v}_0)(\mathbf{x}_0)$, and so

$$\begin{aligned}\mathbf{x}(C) &= \{1+\hat{v}_0+\hat{v}_1\hat{v}_0+\dots+(\hat{v}_{r-2}\dots\hat{v}_0)\}(\mathbf{x}_0)\\ &= \{1+\hat{w}_0+\hat{w}_0\hat{w}_1+\dots+(\hat{w}_0\dots\hat{w}_{r-2})\}(\mathbf{x}_0)\,.\end{aligned}$$

But $\tilde{w}_i = (\mathbf{x}_0, \mathbf{w}_i)$, since each $\tilde{w}_i$ is either $\tilde{a}$ or $\tilde{b}$; and we are given that $w(\tilde{a},\tilde{b}) = 1$. Thus

$$\begin{aligned}1 &= (\mathbf{x}_0, w_0)(\mathbf{x}_0, w_1)\dots(\mathbf{x}_0, \mathbf{w}_{r-1})\\ &= \big(\mathbf{x}_0+\hat{w}_0(\mathbf{x}_0)+\dots+(\hat{w}_0\hat{w}_1\dots\hat{w}_{r-2})(\mathbf{x}_0),\, w_0w_1\dots w_{r-1}\big)\,,\end{aligned}$$

and taking the first component we see that $\mathbf{x}(C) = \mathbf{0}$.

We have shown (Theorem 4) that when $R = \mathbb{Z}$ we have $\mathbf{x}(C) = \mathbf{0}$ if and only if C is a reverting closed path. So in this case we have a useful characterization of the closed paths and the relations which are preserved in the covering graph. For example, it is remarked in **[5]** that in the *cages* X_s $(s = 2, 3, 4, 5)$ the closed path corresponding to the relations $(a^{s-2}b)^6 = 1$ is reverting, and hence the same relation holds in the covering group. The cage X_3 is just $K_{3,3}$, and so here $(ab)^6 = 1$ holds in the covering group also. Thus we have an explanation of the fact, mentioned earlier, that this relation (together with the set $R(3)$) does not define the group of $K_{3,3}$: in fact, the group so defined is infinite. We shall return to this topic in the next section.

We shall conclude this section by describing how these methods are related to results of Miller **[12]** and how they suggest natural extensions of that work. Miller obtains a classification of cubic graphs which admit a group of type U_2 and have girth 6 or less. For each $m \geqslant 3$ there is such a graph $G^*(m)$ with $2m^2$ vertices, and when $m = 3m_0$ $G^*(m)$ is a 3-fold covering of another such graph $G_2^*(3, m_0)$ with $6m_0^2$ vertices. These are the only graphs of the required kind.

Let $Y_2(m)$ be a component of the covering graph $\tilde{X}_2(m)$ obtained from the theta-graph (or 2-cage) X_2 using the ring $\mathbb{Z}_m$. Since X_2 has 2 vertices and $c = 2$, we see that $|\tilde{X}_2(m)| = 2m^2$. The number of components of $\tilde{X}_2(m)$ is given by Theorem 2; it is the index of the submodule

$$\Lambda = \langle (2,-1), (-1,2) \rangle$$

in $\mathbb{Z}_m \times \mathbb{Z}_m$. Now, it can be shown that

$$|(\mathbb{Z}_m)^2 : \Lambda| = \begin{cases} 3, & \text{if } 3 \mid m\,,\\ 1, & \text{otherwise}\,.\end{cases}$$

So the order of $Y_2(m)$ is $2m^2$ if m is not divisible by 3, and $6m_0^2$ if $m = 3m_0$; furthermore, it turns out that

$$Y_2(m) = \begin{cases} G_2^*(3, m_0), & \text{if } m = 3m_0, \\ G^*(m), & \text{otherwise}. \end{cases}$$

In other words, the graphs $Y_2(m)$ are the minimal graphs with the properties needed for Miller's classification.

Roughly speaking, Miller's methods involve taking quotients of a 'universal' graph of girth 6 admitting a group of type U_2. This graph is the plane hexagonal tessellation, which we can denote by $Y_2(\infty)$ since it is the result of the construction described above using $R = \mathbb{Z}$. Thus it is clear that the minimal graphs $Y_2(m)$ are obtained by taking quotients with respect to the coefficient ring. On the other hand, it may be more natural to consider them as coverings of X_2 rather than as quotients of $Y_2(\infty)$.

The preceding remarks lead naturally to questions about the analogous coverings of the cages X_s ($s = 3, 4, 5$). The properties of the infinite coverings $Y_s = Y_s(\infty)$ were studied in [5], and the finite coverings $Y_s(m)$ can be investigated in the same way. It is reasonable to conjecture that, for $s = 3, 4, 5$, the graphs $Y_s(m)$ are the minimal graphs admitting a group of type U_s and having girth $6s-6$ or less.

6. *Some infinite groups*

As indicated in the foregoing discussion, the theory can be used to show that certain finitely presented groups are infinite. The basic idea is formalised in the next theorem.

THEOREM 7. *Let $w = w(a, b)$ be a word in positive powers of a and b, and let $U_s(w)$ denote the quotient of U_s obtained by adjoining the relation $w = 1$. If there is a finite connected cubic graph X admitting a group of type U_s, and such that the closed path defined by w is reverting, then $U_s(w)$ is an infinite group.*

Proof. Let $\tilde{X}$ be the covering of X constructed using the ring $\mathbb{Z}$, and let $\tilde{X}_0$ be the component of $\tilde{X}$ containing the standard s-path.

Clearly, $\tilde{X}$ is an infinite graph, and its components are all isomorphic to $\tilde{X}_0$, since $\tilde{X}$ admits the group $\tilde{G}$ acting transitively on vertices. Furthermore, the number of components is finite, by Theorem 3. Hence $\tilde{X}_0$ is an infinite graph.

The automorphism group $\tilde{G}_0$ of $\tilde{X}_0$ is generated by a and b, as defined in Section 5, and they satisfy the relations in U_s. Now it follows from the hypothesis about w, together with Theorems 4 and 6, that $w(\tilde{a}, \tilde{b}) = 1$ also. Hence $\tilde{G}_0$ is a quotient of $U_s(w)$, and since $\tilde{G}_0$ acts transitively on the vertices of the infinite graph $\tilde{X}_0$, both groups must be infinite.

One of the simplest ways of obtaining a reverting closed path is as follows. Choose three edge-disjoint simple paths P_1, P_2, P_3 joining two given vertices, and denote by $\bar{P}_i$ the reverse of P_i. Then the closed path described by $P_1, \bar{P}_2, P_3, \bar{P}_1, P_2, \bar{P}_3$ is reverting. If P_i has length l_i ($i=1,2,3$), the closed path has length $2(l_1+l_2+l_3)$.

EXAMPLE 1. When X_s is the cage of even girth $2s-2$ we may take [5]

$$l_1 = l_2 = l_3 = s-1$$

and obtain the reverting closed path corresponding to the word $(a^{s-2}b)^6$. It follows from Theorem 7 that the groups

$$U_s(a^{s-2}b)^6 \qquad (s = 2, 3, 4, 5)$$

are infinite. Similarly, when $s \geqslant 3$ we may take $l_1 = l_3 = s$, $l_2 = s-2$, which gives the infinite groups

$$U_s(a^{s-2}b^2a^{s-2}b^s)^2 \qquad (s = 3, 4, 5).$$

When $s \geqslant 4$ we can also get infinite groups by taking $l_1 = l_3 = s+1$, $l_2 = s-3$, and when $s = 5$ by taking $l_1 = l_3 = 7$, $l_2 = 1$. All these reverting closed paths are obtained from the union of two overlapping $(2s-2)$-cycles, and of course there are many others obtained from different configurations.

EXAMPLE 2. There are just two cubic graphs which admit a group of type U_s and have girth $2s-1$. They are the complete graph K_4 (with $s = 2$) and Petersen's graph (with $s = 3$).

In the case of K_4 if we take two 3-cycles with a common edge we obtain the reverting closed path corresponding to the word $(ab^2ab)^2$. A component of the covering graph is isomorphic to the graph described by Coxeter [9] as Laves's graph of girth ten. Thus we conclude that the group of this graph is a quotient of the infinite group

$$U_2(ab^2ab)^2.$$

In the case of Petersen's graph we can take two 5-cycles with one or two edges in common. We obtain the infinite groups

$$U_3(a^2b^2a^2bab)^2, \qquad U_3(a^2b^3a^2b)^2.$$

Similar examples can be manufactured very rapidly; indeed, any cubic graph admitting a group of type U_s will yield many infinite groups $U_s(w)$. On the other hand, computations involving 'short' words w tend to yield finite groups $U_s(w)$. It seems possible that for a given s, there are infinitely many words w for which $U_s(w)$ is finite (and of course there are certainly infinitely many for which it is infinite). In this vein, we have our final theorem.

THEOREM 8. *Let W be any finite set of words in positive powers of a and b. Then there are finite quotients of U_s $(s = 2, 3, 4, 5)$ for which the defining relations (additional to $R(s)$) do not belong to W.*

Proof. The analysis of Djokovic and Miller [10] makes it clear that each U_s is a free product, with amalgamation, of two finite groups. It follows from a general result of Baumslag [1] that U_s is residually finite. Consequently, for each $w \in W$ we can find a normal subgroup N_w of finite index in U_s such that $w \notin N_w$. Let N be the intersection of the N_w taken over all $w \in W$. Since W is finite, N has finite index in U_s, and U_s/N is a finite quotient of U_s with the required property.

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