Europ. J. Combinatorics (1985) **6**, 7–12

Infinite coverings of cages

N. L. Biggs and S. K. Burford

A family of infinite cubic graphs Y_s, $s=2,3,4,5$, is constructed. The vertices of Y_s are lattice points in Euclidean space of dimension 2^{s-1}, the girth of Y_s is $6s-6$, and Y_s has a group of automorphisms which acts regularly on the s-arcs.

1. Introduction

All graphs considered in this paper will be *cubic*, that is, regular with valency three. A famous theorem of Tutte [5] concerns groups of automorphisms which act transitively on the 1-arcs (ordered pairs of adjacent vertices) of a cubic graph. The theorem asserts that, provided the graph is not the infinite cubic tree, the group acts *regularly* on the s-arcs of the graph for some s in the range $1\leq s\leq 5$. [An *s-arc* is an ordered $(s+1)$-tuple of vertices $(\omega_0, \omega_1, \ldots, \omega_s)$, with ω_i adjacent to ω_{i+1} $(0\leq i\leq s-1)$ and $\omega_i\neq\omega_{i+2}$ $(0\leq i\leq s-2)$.] Furthermore, the *girth* g of the graph must satisfy $g\geq 2s-2$.

In the extreme case when $g=2s-2$ the graph is uniquely determined for each of the relevant values $s=2,3,4,5$. These graphs are the *cages* X_2, X_3, X_4, X_5; their structure will be examined in detail in Section 2.

The aim of this paper is the construction of graphs Y_s, $s=2,3,4,5$, with the following properties:

(1) Y_s is an infinite graph,
(2) the vertices of Y_s are lattice points in Euclidean space of dimension 2^{s-1},
(3) Y_s has a group of automorphisms acting regularly on the s-arcs,
(4) the girth of Y_s is $6s-6$,
(5) Y_s is a covering graph of X_s.

The graph Y_2 is a thinly disguised version of the familiar hexagonal lattice in $\mathbb{R}^2$. The graphs Y_3, Y_4, Y_5 are apparently the first examples of infinite cubic graphs with groups acting regularly on the s-arcs for $s=3$, 4, 5, apart from the universal example of the initite cubic tree.

The methods of the paper can be applied more generally, but the work is considerably simplified by the special structure of the cages. The generalisation has some interesting consequences in combinatorial group theory, and this will be the subject of a separate paper.

2. Structure of the Cages

Let X be a cubic graph satisfying $g=2s-2$. Proofs of the assertions about X which follow can be extracted from Tutte's paper [5].

Denote by α, β any pair of adjacent vertices of X, and let d be the usual distance function in X. The edges of X which have at least one vertex ξ for which

$$d(\xi,\alpha)\leq s-2 \quad \text{or} \quad d(\xi,\beta)\leq s-2$$

form a spanning tree T for X (Figure 1). The number of vertices of T is $2m$, where

$$m=1+2+\cdots+2^{s-2}=2^{s-1}-1,$$

and consequently X also has $2m$ vertices. The edges of X not in T link the end-vertices of T, and so they form a subgraph U whose components are cycles.

0195-6698/85/010007+06 $02.00/0

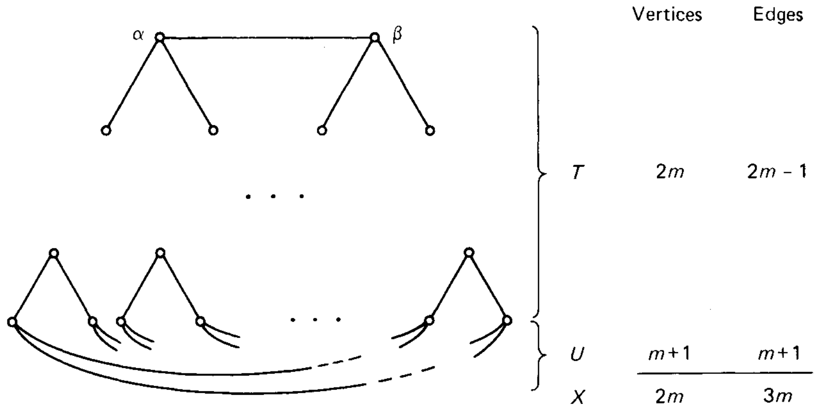

FIGURE 1

We shall require names for the vertices of X. Referring to Figure 1, we label the vertices in descending levels: the two vertices adjacent to, and immediately below, a vertex assigned the symbol σ are given the symbols σ_0 and σ_1. Thus the four vertices at the second level are α_0, α_1, β_0, β_1, and so on. The vertices at the lowest level are those which belong to both T and U; there are $m+1=2^{s-1}$ of them and they have symbols of the form α or β followed by $s-2$ subscripts.

In each of the cases $s=2,3,4,5$ there is essentially only one way to join the vertices of U so that X is a cage. When $s=2$ the vertices α and β are themselves at the lowest level and must be joined by two new edges so that U is a 2-cycle and X_2 is a theta-graph. It is convenient to include this case in the general discussion, but since X_2 is strictly a multigraph we cannot use the convention whereby edges are specified by their vertices.

In the cases $s=3$, 4, and 5 the subgraph U is, respectively, a single 4-cycle, a single 8-cycle, and two 8-cycles. Using the labelling defined above, the cycles are:

$$s=3: \quad \alpha_0, \beta_0, \alpha_1, \beta_1;$$

$$s=4: \quad \alpha_{00}, \beta_{00}, \alpha_{10}, \beta_{01}, \alpha_{01}, \beta_{01}, \alpha_{11}, \beta_{11};$$

$$s=5: \quad \alpha_{000}, \beta_{000}, \alpha_{100}, \beta_{100}, \alpha_{010}, \beta_{010}, \alpha_{110}, \beta_{110};$$

$$\alpha_{001}, \beta_{101}, \alpha_{111}, \beta_{001}, \alpha_{011}, \beta_{111}, \alpha_{101}, \beta_{011}$$

The graph X_3 is just the complete bipartite graph $K_{3,3}$, and the graphs X_4 and X_5 are often called the *Heawood graph* and *Tutte's 8-cage*, respectively.

The first $s+1$ vertices in each of the lists given above will be chosen as the vertices of the *standard s-arc* in X_s. Every s-arc in X_s determines a unique g-cycle ($g=2s-2$). The standard 3-arc in X_3 determines the 4-cycle α_0, β_0, α_1, β_1; the standard 4-arc in X_4 determines the 6-cycle α_{00}, β_{00}, α_{10}, β_{10}, α_{01}, α_0; and the standard 5-arc in X_5 determines the first 8-cycle given above.

It is a consequence of the general theory that a group G acting regularly on the s-arcs of X is generated by the two *shunt* automorphisms a and b which take the standard s-arc onto its two successors. We shall choose a to be that automorphism which shunts the standard s-arc one step around its unique g-cycle, and b to be the other shunt. The automorphisms a and b are uniquely determined by this description; for example, when $s=4$

$$a=(\alpha_{00}, \beta_{00}, \alpha_{10}, \beta_{10}, \alpha_{01}, \alpha_0)(\beta_{11}, \beta_0, \alpha_1, \beta_1, \beta_{01}, \alpha)(\alpha_{11}, \beta);$$

$$b=(\alpha_{00}, \beta_{00}, \alpha_{10}, \beta_{10}, \alpha_{01}, \beta_{01}, \alpha_{11}, \beta_{11})(\alpha_0, \beta_0, \alpha_1, \beta_1)(\alpha, \beta).$$

3. Construction of the Graphs

The method which we shall use to construct covering graphs of the cages is well-known [1, p. 127]. The innovation here is the use of an infinite group, which has to be carefully chosen to fulfil the conditions of the construction.

Each edge $\sigma\tau$ of X determines two 1-arcs (σ, τ) and (τ, σ), which we shall refer to as *sides* of X. An *orientation* of X is a choice of one of the two sides corresponding to each edge. For the sake of definiteness we shall use the orientation defined as follows. The edge $\alpha\beta$ is given the orientation (α, β); an edge of T on the 'α-side' (Figure 1) of T is oriented towards α; an edge of T on the 'β-side' of T is oriented away from β; and the edges of U are oriented according to the cyclic order of their listing in Section 2.

It will be convenient to use the symbols $\gamma_0, \gamma_1, \ldots, \gamma_{m+1}$ for the vertices of U in the order in which they are listed in Section 2. For example, in X_4 γ_6 denotes α_{11}. The edges of U, oriented as above, are then:

$$s=3: \quad (\gamma_0, \gamma_1), \ldots, (\gamma_3, \gamma_0),$$

$$s=4: \quad (\gamma_0, \gamma_1), \ldots, (\gamma_7, \gamma_0),$$

$$s=5: \quad (\gamma_0, \gamma_1), \ldots, (\gamma_7, \gamma_0), (\gamma_8, \gamma_9), \ldots, (\gamma_{15}, \gamma_8).$$

The $s+1$ vertices of the standard s-arc are $\gamma_0, \gamma_1, \ldots, \gamma_s$ in each case.

Since U is the complement of a spanning tree, the $m+1$ sides representing the edges of U determine a basis for the cycle space of X over the integers $\mathbb{Z}$ as follows. If (γ, γ') is the orientation of an edge of U then we have a well-defined oriented cycle

$$\gamma, \gamma', \theta_1, \theta_2, \ldots, \theta_k, \gamma,$$

where $\gamma', \theta_1, \theta_2, \ldots, \theta_k, \gamma$ is the unique path in T from γ' to γ. The $m+1$ oriented cycles $C_1, C_2, \ldots, C_{m+1}$ so constructed form the required basis.

Let S denote the set of sides of X. We define a function $z: S \to \mathbb{Z}^{m+1}$ in terms of coordinate functions $z_i (1 \leq i \leq m+1)$ as follows:

$$z_i(\sigma, \tau) = \begin{cases} +1, & \text{if } (\sigma, \tau) \text{ is in } C_i, \\ -1 & \text{if } (\tau, \sigma) \text{ is in } C_i, \\ 0, & \text{otherwise.} \end{cases}$$

The columns of the cycle matrix of X with respect to the spanning tree T are the labels $\pm z(\sigma, \tau)$, regarded as column vectors.

Now let G be a group of automorphisms of X. For each g in G we define an $(m+1) \times (m+1)$ matrix of integers, $\hat{g} = (\hat{g}_{ij})$ in the following way. Let (γ, γ') be the jth side of the oriented subgraph U, and set

$$\hat{g}_{ij} = \begin{cases} +1, & \text{if } (g\gamma, g\gamma') \text{ is in } C_i, \\ -1, & \text{if } (g\gamma', g\gamma) \text{ is in } C_i, \\ 0, & \text{otherwise.} \end{cases}$$

The matrix $\hat{g}$ acts on the left of column vectors in the usual way, and we obtain an action of G on $\mathbb{Z}^{m+1}$.

The labelling $z: S \to \mathbb{Z}^{m+1}$ is a special case of a more general construction which assigns to each side (σ, τ) of a connected graph an element $z(\sigma, \tau)$ of $\mathbb{Z}^c$, where $\mathbb{Z}^c$ is the first integral homology group of the graph. It has been shown [3, theorem 5] that the actions of the automorphism on the graph and on its homology are compatible with this labelling. That is,

$$\hat{g}(z(\sigma, \tau)) = z(g\sigma, g\tau)$$

for any automorphism g and any side (σ, τ). In our special case we could verify the result directly for the generating automorphisms a and b in the four graphs X_2, X_3, X_4, X_5.

We now have the basic facts needed for the construction of a covering graph $\tilde{X}$ of X and a group $\tilde{G}$ of automorphisms of $\tilde{X}$. We shall assume from now on that G is the full group acting regularly on the s-arcs of X.

The vertices of $\tilde{X}$ are the ordered pairs

$$(\boldsymbol{x}, \omega) \qquad \boldsymbol{x} \in \mathbb{Z}^{m+1}, \quad \omega \in V(X),$$

and $(\boldsymbol{x}, \omega)$ is adjacent to $(\boldsymbol{x}', \omega')$ whenever

$$\omega\omega' \in E(X) \qquad \text{and} \qquad z(\omega, \omega') = \boldsymbol{x}' - \boldsymbol{x}.$$

[$V(X)$ and $E(X)$ denote the vertex-set and edge-set of X.] The group $\tilde{G}$ is the semidirect product $\mathbb{Z}^{m+1} \rtimes G$, whose elements are the pairs

$$(\boldsymbol{x}, g), \qquad \boldsymbol{x} \in \mathbb{Z}^{m+1}, \quad g \in G,$$

with the group operation defined by

$$(\boldsymbol{x}, g)(\boldsymbol{x}', g') = (\boldsymbol{x} + \hat{g}(\boldsymbol{x}'), gg').$$

The action of $\tilde{G}$ on the vertices of $\tilde{X}$ is given by

$$(\boldsymbol{x}, g)(\boldsymbol{x}', \omega') = (\boldsymbol{x} + \hat{g}(\boldsymbol{x}'), g\omega').$$

It follows from the general theory of this construction [1, p. 129] that $\tilde{G}$ acts transitively on the s-arcs of $\tilde{X}$. However, it must be stressed that $\tilde{X}$ is not connected. For example, the covering of the theta-graph X_2 has three components, each isomorphic to the hexagonal lattice in $\mathbb{R}^2$. Since X_2 has a 2-arc-transitive group, each component of $\tilde{X}_2$ (which is a simple graph, unlike X_2) also admits a 2-arc-transitive group.

Let Y denote the component of $\tilde{X}$ containing the vertex $(\boldsymbol{0}, \gamma_0)$. The definition of z implies that

$$z(\gamma_{i-1}, \gamma_i) = \boldsymbol{u}_i \ (1 \leqslant i \leqslant s),$$

where $\boldsymbol{u}_i$ is the column vector with 1 in row i and 0 elsewhere. Consequently, if we let $\boldsymbol{v}_i = \boldsymbol{u}_1 + \cdots + \boldsymbol{u}_i$, we have an s-arc

$$(\boldsymbol{0}, \gamma_0), (\boldsymbol{v}_1, \gamma_1), \ldots, (\boldsymbol{v}_s, \gamma_s)$$

in Y. If a and b are the generating shunt automorphisms in X, the generating shunt automorphisms for a group acting regularly on the s-arcs of Y are

$$\tilde{a} = (\boldsymbol{u}_1, a), \qquad \tilde{b} = (\boldsymbol{u}_1, b).$$

4. Properties of the Covering Graphs

The shunt automorphism $\tilde{a} = (\boldsymbol{u}_1, a)$ has the property that $\tilde{\gamma}_0 = (\boldsymbol{0}, \gamma_0)$ is adjacent to $\tilde{a}(\tilde{\gamma}_0) = (\boldsymbol{u}_1, \gamma_1)$. It follows that $\tilde{a}^{n-1}(\tilde{\gamma}_0)$ is adjacent to $\tilde{a}^n(\tilde{\gamma}_0)$ for all $n \geqslant 1$, and so $\tilde{a}^n(\tilde{\gamma}_0)$ is in the component Y of $\tilde{X}$ for all $n \geqslant 1$.

The order of the automorphism a in X_s is $2s-2$, so that

$$\tilde{a}^{2s-2} = (\boldsymbol{v}_{2s-2}, 1),$$

and consequently

$$\tilde{a}^{r(2s-2)} = (r\boldsymbol{v}_{2s-2}, 1), \qquad r \geqslant 1.$$

Thus the image of $\tilde{\gamma}_0$ under $\tilde{a}^{r(2s-2)}$ is $(r\boldsymbol{v}_{2s-2}, \gamma_0)$, and we conclude that Y contains infinitely many vertices.

In order to compute the girth of Y, we shall use the correspondence between cycles in Y and positive words in the shunt automorphisms which represent the identity [2, lemma 8].

It can be verified that $(\tilde{a}^{s-2}\tilde{b})^6$ is the identity in each of the cases $s=2,3,4,5$. More light is thrown on this remark if we consider the cycles determined by $(a^{s-2}b)^6$ in X_s. Since γ_0 and γ_{s-1} are separated by the maximum distance $s-1$ in X_s, and X_s is a cage, it follows that there are three disjoint paths of length $s-1$ joining γ_0 and γ_{s-1} [Figure 2(a)].

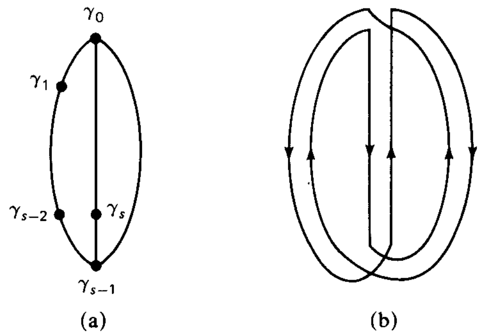

FIGURE 2

Recall that the automorphism a represents the shunt in the direction of the unique $(2s-2)$-cycle containing the standard s-arc, and b represents the alternative shunt. Using the details of the correspondence between words and cycles [2], we conclude that $(a^{s-2}b)^6$ corresponds to shunting the standard s-arc around the cyclic route depicted in Figure 2(b). This route traverses the edges of the three paths in both directions and so the covering route in Y_s leads from $(\boldsymbol{0}, \gamma_0)$ back to $(\boldsymbol{0}, \gamma_0)$.

We conclude that Y_s has cycles of length $6s-6$. The fact that there are no shorter cycles can be verified directly in each of the four cases $s=2,3,4,5$. Since Y_s is vertex-transitive it is only necessary to check the vertices within distance $3s-2$ of one vertex in order to verify that no cycles of length less than $6s-6$ occur. Alternatively, we could check to see if there are any shorter positive words in $\tilde{a}$ and $\tilde{b}$ which represent the identity. This work is eased by two remarks. First, only a small subset of the positive words can be 'girth words', since there are restrictive 'overlap' conditions. Secondly, if a word $w(\tilde{a}, \tilde{b})$ represents the identity in $\tilde{G}$, then $w(a, b)$ must represent the identity in G. The remaining checks can be carried out mechanically, and the fact that the girth of Y is $6s-6$ is verified again.

Finally, the number of components of $\tilde{X}$ (each of them necessarily isomorphic to Y) can be found by algebraic means. We give only a sketch here, and refer the reader to [3] for general proofs.

Let $\mathbf{K}$ denote the cycle matrix of X, with respect to the spanning tree T and our fixed orientation. As we remarked in Section 3, the column of $\mathbf{K}$ corresponding to an edge $\sigma\tau$ is $\pm z(\sigma, \tau)$, the sign depending upon the orientation of $\sigma\tau$. The $(m+1)\times(m+1)$ matrix $\mathbf{L}=\mathbf{K}\mathbf{K}^{\mathrm{T}}$ may be given the following interpretation. When the oriented cycle C_j is traversed, beginning and ending at a vertex σ in X, the corresponding path in $\tilde{X}$ beginning at $(\boldsymbol{x}, \sigma)$ will end at $(\boldsymbol{x}+\boldsymbol{l}_j, \sigma)$, where $\boldsymbol{l}_j$ is the jth column of $\mathbf{L}$.

Let $\boldsymbol{y}=(y_j)$ be any column vector in $\mathbb{Z}^{m+1}$. By following a route in which the oriented cycle C_j is traversed y_j times $(1\leq j\leq m+1)$, we obtain a path in $\tilde{X}$ joining the vertices

$$(\boldsymbol{x}, \sigma) \quad \text{and} \quad (\boldsymbol{x}+\mathbf{L}\boldsymbol{y}, \sigma).$$

A well-known result on integer lattices [4, p. 14] asserts that the number of classes of the sublattice $\{\mathbf{L}\boldsymbol{y} \mid \boldsymbol{y}\in\mathbb{Z}^{m+1}\}$ in $\mathbb{Z}^{m+1}$ is $|\det \mathbf{L}|$. It follows that the number of classes (under the relation of connection) of points $(\boldsymbol{x}, \sigma)$ in $\mathbb{Z}^{m+1}\times V(X)$ is also $|\det \mathbf{L}|$.

Now, it can be shown [6, p. 18] that det $\mathbf{L}$ (that is, det $\mathbf{KK}^T$) is the number of spanning trees of X. It would be interesting to have a constructive (bijective) proof of the fact that the number of components of $\tilde{X}$ is equal to the number of spanning trees of X, but at the moment we have only the algebraic proof outlined above.

The number of spanning trees of a cubic graph X can be computed from the formula [1, p. 36]

$$\kappa = \frac{1}{|V(X)|} \prod_{i=1}^{r} (3-\lambda_i)^{m_i},$$

where $\lambda_1, \ldots, \lambda_r$ are the distinct eigenvalues (except 3) and m_i is the multiplicity of λ_i. Since the cage X_s is a distance-regular graph with diameter $s-1$, the eigenvalues and their multiplicities are determined by the intersection array of X_s, according to the theory expounded in [1, Chap. 21]. The intersection array of X_s is the array of $2(s-1)$ integers

$$\{3, 2, \ldots, 2; 1, \ldots, 1, 3\},$$

and the eigenvalues (except 3) and their multiplicities are:

$$s=2: \quad -3(1);$$

$$s=3: \quad 0(4), -3(1);$$

$$s=4: \quad 2^{1/2}(6), -2^{1/2}(6), -3(1);$$

$$s=5: \quad 2(9), 0(10), -2(9), -3(1).$$

The consequent values of κ are given in Table 1.

TABLE 1

s	2	3	4	5
X_s	Theta	$K_{3,3}$	Heawood	Tutte
Vertices of X_s $(=2m)$	2	6	14	30
Girth of X_s $(=2s-2)$	2	4	6	8
Dimension of cycle space $(=m+1)$	2	4	8	16
Girth of Y_s $(=6s-6)$	6	12	18	24
Components of $\tilde{X}$	3	3^4	$3 \cdot 7^5$	$3^{10} \cdot 5^8$

REFERENCES

1. N. L. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Cambridge, 1974.
2. N. L. Biggs, Constructing 5-arc-transitive cubic graphs, *J. London Math. Soc.* **26**(2) (1982), 193-200.
3. N. L. Biggs, Homological coverings of graphs, *J. London Math. Soc.* **30** (1984, 1-14.
4. J. W. S. Cassels, *An Introduction to the Geometry of Numbers*, Springer-Verlag, Berlin, 1959.
5. W. T. Tutte, A family of cubic graphs, *Proc. Cambridge Philos. Soc.* **43** (1947), 459-474.
6. W. T. Tutte, Lectures on matroids, *J. Res. Nat. Bur. Standards*, B **69** (1965), 1-47.

Received 25 June 1983 and in revised form 10 February 1984

N. L. BIGGS AND S. K. BURFORD
Department of Mathematics, Royal Holloway College,
Egham, Surrey TW20 0EX, U.K.