CUBIC DISTANCE-REGULAR GRAPHS

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ABSTRACT

It is shown that there are just thirteen finite graphs which are cubic (regular with valency three) and distance-regular.

1. *Introduction*

Let G be a connected graph and let ∂ denote the usual metric on the vertex-set of G. The graph G is said to be *distance-transitive* (DT) if whenever u, v, x, y are vertices of G such that $\partial(u, v) = \partial(x, y)$ there is an automorphism ϕ of G for which $\phi(u) = x$ and $\phi(v) = y$. In 1970 **[5]** it was proved that there are just twelve finite DT graphs which are regular with valency 3 (or, as we shall say, *cubic*). Soon afterwards a similar result for the valency 4 case was proved by Smith **[14, 15, 16]**. More recently, it has been shown that for each $k \geqslant 3$ there are finitely many DT graphs with valency k. The first general proof **[7]** relied upon the classification of finite simple groups, but the need for this has since been removed by various means (see Cameron **[6]**, Ivanov **[12]**, and Weiss **[18]**). Specific values of k are dealt with in papers by Gardiner **[9]**, Gardiner and Praeger **[10]**, and Ivanov, Ivanov and Faradzhev **[13]**.

A DT graph has strong combinatorial properties, arising from the fact that the parameters

$$s_{hi}(u, v) = \text{number of vertices } w \text{ such that}$$

$$\partial(u, w) = h \quad \text{and} \quad \partial(v, w) = i$$

depend only on the distance between u and v, rather than the individual vertices. Consequently, in any DT graph we can define a set of intersection numbers s_{hij} by putting $s_{hij} = s_{hi}(u, v)$ for any pair of vertices with $\partial(u, v) = j$. These numbers satisfy many identities, and in fact it is sufficient to specify only the numbers

$$c_j = s_{j-1,1,j}, \quad a_j = s_{j,1,j}, \quad b_j = s_{j+1,1,j},$$

for all relevant values of j. It is thus natural to study graphs for which we assume only that the numbers c_j, a_j, b_j are independent of the vertices u and v, given that $\partial(u, v) = j$. Such a graph is said to be *distance-regular* (DR). A DT graph is DR, but the converse is not necessarily true; indeed, a DR graph may have only the identity automorphism. Information about the general theory of DR graphs is given in **[2, 4]**.

The problem of classifying DR graphs in the same way as DT graphs is an interesting one, and a discussion of its significance will be found in the book by Bannai and Ito **[2]**. Briefly, we remark that a DR graph is a special kind of association scheme, and that its characteristic property (the P-polynomial property) reflects its

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metrizability. Furthermore, the classification of association schemes is a problem with repercussions in several areas of classical algebra and analysis. In this paper we shall study the simplest case, DR graphs with valency 3.

2. *Preliminaries*

In the case of a cubic DR graph with diameter d the parameters (c_j, a_j, b_j) described in the introduction take the values (1, 0, 2), (1, 1, 1) or (2, 0, 1) for values of j in the range $1 \leqslant j \leqslant d-1$. Furthermore, the monotonicity conditions [**4**, p. 135] for the sequences (c_j) and (b_j) ensure that the types occur (if at all) in the order given. Thus the full *intersection array* for a cubic DR graph takes the form

$$\begin{matrix} & 1 & \ldots & 1 & 1 & \ldots & 1 & 2 & \ldots & 2 & c_d \\ 0 & 0 & \ldots & 0 & 1 & \ldots & 1 & 0 & \ldots & 0 & a_d \\ 3 & 2 & \ldots & 2 & 1 & \ldots & 1 & 1 & \ldots & 1 & \end{matrix},$$

where there are $d+1$ columns altogether, $c_d = 1, 2$ or 3, $c_d \neq 1$ if the number of columns (2, 0, 1) is not zero, and $a_d = 3-c_d$. We denote the numbers of columns of types (1, 0, 2), (1, 1, 1), (2, 0, 1) by α, β, γ respectively, so that $\alpha+\beta+\gamma = d-1$. The array is completely specified by the numbers α, β, γ and c_d; for example $\alpha = 1, \beta = 2, \gamma = 1, c_d = 3$ gives the array of the dodecahedron.

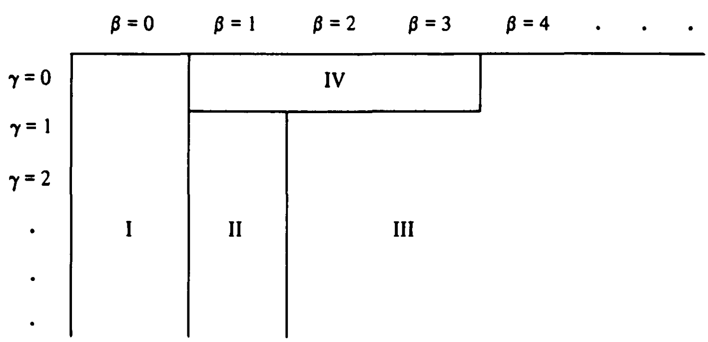

FIG. 1

In the paper [**5**] on the classification of cubic DT graphs it was shown that $\beta \leqslant \alpha+1$ and $\gamma \leqslant \alpha$. These results hold also in the DR case and similar proofs apply. The DT case was completed by appealing to a famous theorem of Tutte [**17**], which implies that $\alpha \leqslant 5$, so that

$$d = \alpha+\beta+\gamma+1 \leqslant 3\alpha+2 \leqslant 17$$

in that case. In the DR case the problem is to replace Tutte's theorem by some combinatorial arguments which will lead to a similar conclusion. In fact, our strategy is somewhat different: we shall study various ranges of values of (β, γ) and establish severe restrictions on α in each range. Specifically, we shall divide the set of possible values of (β, γ) into four regions, as indicated in Figure 1.

In Region I the problem has already been solved by algebraic eigenvalue techniques; the results will be summarised in Lemma 1 below, and there are ten graphs. In Region II it can be shown by elementary arguments that there are no graphs

(Lemma 2). The main part of this paper is devoted to showing that in Region III there is just one graph, the dodecahedron. Our methods fail in Region IV, but fortunately eigenvalue techniques are applicable to the three individual cases $(\beta, \gamma) = (1, 0), (2, 0), (3, 0)$, and these cases will be dealt with in a forthcoming paper by Bannai and Ito [3]. It turns out that there are just two graphs in Region IV. (In all the cases mentioned above it can be shown that each feasible array corresponds to a unique graph.)

On the basis of these results we have the complete list of thirteen finite cubic DR graphs, as given in Table 1.

TABLE 1

Region	β	γ	α	c_d	Graph
I	0	0	0	1	Complete graph K_4
	0	0	1	1	Petersen's graph O_3
	0	0	1	3	Complete bipartite graph $K_{3,3}$
	0	0	2	3	Heawood graph
	0	0	3	3	Tutte's 8-cage T
	0	0	5	3	12-cage, or generalised hexagon
	0	1	1	3	Cube Q_3
	0	1	2	3	Pappus graph
	0	2	2	3	Desargues graph
	0	4	4	3	3-fold cover of T
III	2	1	1	3	Dodecahedron
IV	1	0	2	2	Coxeter's graph
	3	0	3	3	Sextet graph $S(17)$

We conclude the preliminaries by establishing the results for Regions I and II referred to above. We shall denote by $\mathscr{G}(\alpha, \beta, \gamma)$ the set of cubic DR graphs whose intersection arrays have α, β, γ columns of types $(1, 0, 2), (1, 1, 1), (2, 0, 1)$ respectively. Whenever s and t are vertices of a graph G in $\mathscr{G}(\alpha, \beta, \gamma)$ then we say that (with respect to s) t is

(i) an *α-vertex* if $1 \leqslant \partial(s, t) \leqslant \alpha$;
(ii) a *β-vertex* if $\alpha+1 \leqslant \partial(s, t) \leqslant \alpha+\beta$;
(iii) a *γ-vertex* if $\alpha+\beta+1 \leqslant \partial(s, t) \leqslant \alpha+\beta+\gamma$.

LEMMA 1. *Cubic DR graphs with $\beta = 0$ may be classified as follows, according to the value of c_d.*

(i) *If $c_d = 1$ then the graph is a* Moore graph *and the only possibilities are K_4 and O_3.*

(ii) *If $c_d = 2$ there are no graphs.*

(iii) *If $c_d = 3$ then the graph is* bipartite *and there are eight possibilities as listed in Table* 1.

Proof. (i) Since $c_d = 1$ we must have $\gamma = 0$. The girth of such a graph is $2d+1$; in other words, it is a Moore graph. These graphs have been completely classified by Bannai and Ito [7] and Damerell [8]. In the cubic case the only possibilities occur when $d = 1$ and $d = 2$, and the graphs are K_4 and O_3 respectively.

(ii) When $\beta = 0$ and $c_d = 2$ any such graph has the property that its shortest odd cycle has length $2d+1$. Let v and w be any two adjacent vertices both at distance d from a given vertex x. The fact that $c_d = 2$ means that two neighbours of x are at distance $d-1$ from v, and also that two of them are at distance $d-1$ from w. Since there are only three neighbours altogether, one of them is at distance $d-1$ from both v and w. Thus we have an odd cycle of length $2d-1$ or less, contradicting the fact that $\beta = 0$.

(iii) The conditions $\beta = 0$ and $c_d = 3$ mean that the graph has no odd cycles and so it is bipartite. The cubic bipartite DR graphs have been classified by Ito [**11**]; there are just eight of them, as claimed.

LEMMA 2. *There are no cubic DR graphs with $\beta = 1$ and $\gamma \geqslant 1$.*

Proof. Let G be a graph in $\mathscr{G}(\alpha, \beta, \gamma)$ with $\beta \geqslant 1$ and $\gamma \geqslant 1$. Choose vertices x, w such that $\partial(x, w) = \alpha+\beta+1$ and let $x, p, \ldots, u, y, w$ be a path of length $\alpha+\beta+1$ from x to w. Let v be the unique vertex adjacent to y with $\partial(x, v) = \alpha+\beta$.

Since $\partial(x, w) = \alpha+\beta+1$ there exists a unique vertex r adjacent to x with $\partial(r, w) = \alpha+\beta+2$. Hence $\partial(r, y) = \alpha+\beta+1$, so that y is a γ-vertex with respect to r and we have

$$\{\partial(u, r), \partial(v, r), \partial(w, r)\} = \{\alpha+\beta, \alpha+\beta, \alpha+\beta+2\}.$$

But $\partial(u, r) = \alpha+\beta$ and $\partial(w, r) = \alpha+\beta+2$, so that $\partial(v, r) = \alpha+\beta$.

Now consider a path $r = r_0, r_1, r_2, \ldots, r_{\alpha+\beta} = v$. Since $\partial(x, r) = 1$ and $\partial(x, v) = \alpha+\beta$ there must be two vertices r_i, r_{i+1} on the path such that $\partial(x, r_i) = \partial(x, r_{i+1})$. Let us call the edge $\{r_i, r_{i+1}\}$ a *sidestep*, noting that sidesteps can only involve β-vertices.

In this case the sidestep cannot be $\{v, y\}$, for $\partial(y, r) = \alpha+\beta+1$ and the path $v, y, \ldots, r$ has length $\alpha+\beta+2$. So there must be a sidestep at distance less than $\alpha+\beta$ from x, and hence $\beta > 1$. Thus $\gamma \geqslant 1$ implies that $\beta > 1$, so that there are no cubic DR graphs with $\beta = 1$ and $\gamma \geqslant 1$.

This result also follows from [**13**, Lemma 18].

3. *Projections and transitions*

Let G be any member of $\mathscr{G}(\alpha, \beta, \gamma)$ and fix a vertex x of G. Give the neighbours of x the labels y_1, y_2, y_3 in some arbitrary, but fixed, order. For each vertex $v \neq x$ in G we define (following Ivanov [**12**]) the *projection* of v on x to be the vector

$$\mathbf{p}(v) = (p_1(v), p_2(v), p_3(v)),$$

whose components are

$$p_i(v) = \partial(v, y_i) - \partial(v, x) \qquad (i = 1, 2, 3).$$

Clearly the possible values of $p_i(v)$ are -1, 0, $+1$. If $\partial(v, x)$ lies in the range $1, 2, \ldots, \alpha$ then there is one component equal to -1 and two equal to $+1$; in the range $\alpha+1, \alpha+2, \ldots, \alpha+\beta$ there is one -1, one 0 and one $+1$; and in the range $\alpha+\beta+1, \alpha+\beta+2, \ldots, \alpha+\beta+\gamma$ there are two -1 components and one $+1$.

We now examine the relationship between $\mathbf{p}(v)$ and $\mathbf{p}(w)$ when v and w are adjacent vertices of G. It will be convenient to define an *edge-transition* (with respect to given values of α, β, γ) to be an ordered pair of positive integers (r, s) such that for any vertex x of a graph in $\mathscr{G}(\alpha, \beta, \gamma)$ there is an edge $\{v, w\}$ such that

$$\partial(x, v) = r \quad \text{and} \quad \partial(x, w) = s.$$

Clearly, if G has an edge-transition (r, s) and x, v are any vertices such that $\partial(x, v) = r$, then there is a vertex w adjacent to v such that $\partial(x, w) = s$. For example, if $\alpha \geqslant 1$ and $\beta \geqslant 1$ then $(\alpha+1, \alpha+1)$ is an edge-transition but (α, α) is not. In all cases we must have $r-s = -1, 0,$ or $+1$.

Table 2 shows how the values of the components of the projection vector change for various edge-transitions in G. Several of the results (A2, B1, B3, B5, C1, C3, C5) are similar to Ivanov's lemma [**12**, Lemma 1]; the proofs of others are indicated in Theorem 1. It is assumed that $\alpha \geqslant 1$, $\beta \geqslant 2$ and $\gamma \geqslant 1$ throughout; other restrictions for specific types are noted.

TABLE 2

Type	Edge-transition		$-$	0 changes to	$+$
A1	$(\alpha+1, \alpha+1)$		0	$-$	$+$
A2 ($\beta \geqslant 3$)	$(\alpha+i, \alpha+i)$	$(2 \leqslant i \leqslant \beta-1)$	$-$	0	$+$
A3	$(\alpha+\beta, \alpha+\beta)$		$-$	$+$	0
B1 ($\alpha \geqslant 2$)	$(i, i+1)$	$(1 \leqslant i \leqslant \alpha-1)$	$-$		$+$
B2	$(\alpha, \alpha+1)$		$-$		0 or $+$
B3	$(\alpha+i, \alpha+i+1)$	$(1 \leqslant i \leqslant \beta-1)$	$-$	0	$+$
B4	$(\alpha+\beta, \alpha+\beta+1)$		$-$	$-$	$+$
B5 ($\gamma \geqslant 2$)	$(\alpha+\beta+i, \alpha+\beta+i+1)$	$(1 \leqslant i \leqslant \gamma-1)$	$-$		$+$
C1 ($\alpha \geqslant 2$)	$(i+1, i)$	$(1 \leqslant i \leqslant \alpha-1)$	$-$		$+$
C2	$(\alpha+1, \alpha)$		$-$	$+$	$+$
C3	$(\alpha+i+1, \alpha+i)$	$(1 \leqslant i \leqslant \beta-1)$	$-$	0	$+$
C4	$(\alpha+\beta+1, \alpha+\beta)$		$-$ or 0		$+$
C5 ($\gamma \geqslant 2$)	$(\alpha+\beta+i+1, \alpha+\beta+i)$	$(1 \leqslant i \leqslant \gamma-1)$	$-$		$+$

THEOREM 1. *Let G be a graph in $\mathscr{G}(\alpha, \beta, \gamma)$, where $\alpha \geqslant 1$, $\beta \geqslant 2$ and $\gamma \geqslant 1$,* and *let x, y_1, y_2, y_3 be vertices as specified above. Then we have the following rules.*

(*Case* A1) *If v and w are adjacent vertices of G such that $\partial(x, v) = \partial(x, w) = \alpha+1$ then*

$$\mathbf{p}(v) = (-1, 0, +1) \Rightarrow \mathbf{p}(w) = (0, -1, +1).$$

(*Case* A3) *If v and w are adjacent vertices of G such that $\partial(x, v) = \partial(x, w) = \alpha+\beta$ then*

$$\mathbf{p}(v) = (-1, 0, +1) \Rightarrow \mathbf{p}(w) = (-1, +1, 0).$$

(*Case* B2) *If v is a vertex of G such that $\partial(x, v) = \alpha$, and u, w are vertices of G both adjacent to v such that $\partial(x, u) = \partial(x, w) = \alpha+1$ then*

$$\mathbf{p}(v) = (-1, +1, +1) \Rightarrow \{\mathbf{p}(u), \mathbf{p}(w)\} = \{(-1, 0, +1), (-1, +1, 0)\}.$$

Proof. (Case A1) Since $\mathbf{p}(v) = (-1, 0, +1)$, $\partial(v, y_2) = \alpha+1$ so there is a path from v to y_2 with one sidestep, which must be $\{v, w\}$. Hence $\partial(w, y_2) = \alpha$, so $p_2(w) = -1$. Further, $\partial(v, y_1) = \alpha$ and hence $\partial(w, y_1) = \alpha+1$. Thus $p_1(w) = 0$, and so $p_3(w) = +1$, as required.

(Case A3) Since $\mathbf{p}(v) = (-1, 0, +1)$, it follows that $\partial(v, y_1) = \alpha+\beta-1$. Let u, t be vertices adjacent to v with $\partial(u, y_1) = \alpha+\beta-2$, $\partial(u, x) = \alpha+\beta-1$ and $\partial(t, y_1) = \alpha+\beta$, $\partial(t, x) = \alpha+\beta+1$. Now $\partial(t, y_i) \leqslant \partial(v, y_i)+1$ $(i = 1, 2, 3)$ so $p_i(v) \geqslant p_i(t)$ $(i = 1, 2, 3)$. But t is a γ-vertex with respect to x, so $\mathbf{p}(t)$ has one $+1$

coordinate and two -1 coordinates, so that $\mathbf{p}(t) = (-1, -1, +1)$. Similarly $\mathbf{p}(u) = \mathbf{p}(v)$.

Now $\partial(v, y_1) = \alpha+\beta-1$ so that

$$\{\partial(u, y_1), \partial(w, y_1), \partial(t, y_1)\} = \{\alpha+\beta-2, \alpha+\beta-1, \alpha+\beta\}.$$

But $\partial(u, y_1) = \alpha+\beta-2$, $\partial(t, y_1) = \alpha+\beta$ and so $\partial(w, y_1) = \alpha+\beta-1$. Similar considerations based on y_2, y_3 complete the proof.

(Case B2) Let u, w be distinct vertices adjacent to v and both at distance $\alpha+1$ from x. Clearly $\partial(u, y_1) = \partial(w, y_1) = \alpha$, that is, $p_1(u) = p_1(w) = -1$. Now if $\mathbf{p}(u) = \mathbf{p}(w) = (-1, 0, +1)$ (say) then we would have $\partial(v, y_3) = \alpha+1$ and $\partial(u, y_3) = \partial(w, y_3) = \alpha+2$, contradicting the fact that v is a β-vertex with respect to y_3. Hence $\mathbf{p}(u) \neq \mathbf{p}(w)$, so that $\{\mathbf{p}(u), \mathbf{p}(w)\} = \{(-1, 0, +1), (-1, +1, 0)\}$.

There are two cases (B2 and C4) where the change in the projection vector is not uniquely determined by the corresponding edge-transition. However, it is possible to make some further useful remarks about these cases, as in the following theorems.

THEOREM 2. *Let G be a graph in $\mathscr{G}(\alpha, \beta, \gamma)$ where $\alpha \geq 1$ and $\beta \geq 1$. Let v be a vertex of G such that $\partial(x, v) = \alpha$, and u, w distinct vertices of G both adjacent to v such that $\partial(x, u) = \partial(x, w) = \alpha+1$.*

If $\mathbf{p}(u) = (-1, 0, +1)$ then $\mathbf{p}(v) = (-1, +1, +1)$ and $\mathbf{p}(w) = (-1, +1, 0)$.

Proof. This is essentially covered in the proof of Theorem 1 (Case B2), as given above.

THEOREM 3. *Suppose that $\beta \geq 2$, $\gamma \geq 1$ and let v be a vertex of G such that $\partial(x, v) = \alpha+\beta+1$, and u, w be distinct vertices adjacent to v such that $\partial(x, u) = \alpha+\beta = \partial(x, w)$.*

If $\mathbf{p}(u) = (-1, 0, +1)$ then $\mathbf{p}(v) = (-1, -1, +1)$ and $\mathbf{p}(w) = (0, -1, +1)$.

Proof. As in the proof of Theorem 1 (Case A3) we have $\mathbf{p}(v) = (-1, -1, +1)$. Let t be the vertex adjacent to v such that $\partial(t, x) = \alpha+\beta+2$. Then $\partial(t, y_1) = \partial(t, y_2) = \alpha+\beta+1$. Now v is a β-vertex with respect to both y_1 and y_2, so that

$$\{\partial(t, y_i), \partial(u, y_i), \partial(w, y_i)\} = \{\alpha+\beta+1, \alpha+\beta, \alpha+\beta-1\} \qquad (i = 1, 2).$$

This gives $\partial(w, y_1) = \alpha+\beta$ and $\partial(w, y_2) = \alpha+\beta-1$. Thus $\partial(w, y_3) = \alpha+\beta+1$, and we have $\mathbf{p}(w) = (0, -1, +1)$.

4. *Cycles and their periods*

Let G be any member of $\mathscr{G}(\alpha, \beta, \gamma)$, $\beta \geq 2$, and let

$$C: x_0, x_1, \ldots, x_{2\alpha+\lambda-1}$$

be an oriented cycle of length $2\alpha+\lambda$ in G. The girth of G is $2\alpha+3$ (since we are assuming that $\beta \neq 0$) and hence $\lambda \geq 3$. The distances in G from x_0 to the other vertices of C take the general form

$$1, 2, \ldots, \alpha, \alpha+1, \ldots, \alpha+1, \alpha, \ldots, 1.$$

The distinctive features of C are indicated by the $\lambda-1$ values in the middle of the sequence, which comprise a $(\lambda-1)$-vector $\pi^{(0)}(C)$ with components

$$\pi_i^{(0)}(C) = \partial(x_0, x_{\alpha+i}) \qquad (1 \leqslant i \leqslant \lambda-1).$$

In general, we shall say that a $(\lambda-1)$-vector π with positive integer components is a *profile* for $\mathscr{G}(\alpha, \beta, \gamma)$ if

(i) $\pi_1 = \pi_{\lambda-1} = \alpha+1$,

(ii) for each value of i in the range $1 \leqslant i \leqslant \lambda-2$, (π_i, π_{i+1}) is an edge-transition with respect to the given α, β, and γ.

We now study how the profile vector associated with a given oriented cycle changes as we move from the base vertex x_0 to the adjacent vertex x_1. Consider the projections $\mathbf{p}(x_{\alpha+i})$ of the vertices $x_{\alpha+i}$ $(1 \leqslant i \leqslant \lambda-1)$ and let $p_1(x_{\alpha+i})$ be the component of the projection vector corresponding to the neighbour x_1 of x_0 in each case. When $i = 1$ we have

$$p_1(x_{\alpha+1}) = \partial(x_1, x_{\alpha+1}) - \partial(x_0, x_{\alpha+1}) = \alpha-(\alpha+1) = -1.$$

The remaining projection values $p_1(x_{\alpha+2}), p_1(x_{\alpha+3}), \ldots, p_1(x_{\alpha+\lambda-1})$ are obtained from the initial value -1 by a sequence of edge-transitions, according to the rules formulated in Section 3. For example, the profile vector

$$(\alpha+1, \alpha+1, \alpha, \alpha+1, \alpha+1)$$

corresponds to a cycle of length $2\alpha+6$ for which the projection values are

$$-1, \quad 0, \quad +1, \quad +1, \quad +1.$$

Note that in order to determine the fourth value uniquely it is necessary to use Theorem 3.

When the edge-transition $(\alpha, \alpha+1)$ follows $(\alpha-1, \alpha)$ there will be an ambiguity in the projection value which cannot be resolved by the rules given in Section 3. However, in some cases it is possible to remove such ambiguities by remarking that the final value $p_1(x_{\alpha+\lambda-1})$ cannot be -1. This is because $\partial(x_0, x_{\alpha+\lambda-1}) = \alpha+1$ so that components of the projection of $x_{\alpha+\lambda-1}$ on x_0 take the values $+1$, 0, -1 in some order. Since C is a $(2\alpha+\lambda)$-cycle, the unique neighbour of x_0 giving the value -1 must be $x_{2\alpha+\lambda-1}$ rather than x_1. Thus $p_1(x_{\alpha+\lambda-1})$ is either 0 or $+1$. For example, consider the $(2\alpha+8)$-cycle with profile vector

$$(\alpha+1, \alpha+1, \alpha, \alpha-1, \alpha, \alpha+1, \alpha+1).$$

Starting from -1, the first five projection values must be $-1, 0, +1, +1, +1$, but (according to the rules given in Section 3) the sixth value could be either 0 or $+1$. However, the value 0 gives a final value of -1, in contradiction to the remarks in the preceding paragraph. Hence the values must be $-1, 0, +1, +1, +1, +1, +1$.

We shall say that a profile vector π is *good* if it determines a unique sequence of projection values $z_i = p_1(x_{\alpha+i})$ $(1 \leqslant i \leqslant \lambda-1)$, subject to the conditions $z_1 = -1$, $z_{\lambda-1} \neq -1$.

THEOREM 4. *Let G be a graph in $\mathscr{G}(\alpha, \beta, \gamma)$ and let $x_0, x_1, \ldots, x_{2\alpha+\lambda-1}$ be the vertices of an oriented $(2\alpha+\lambda)$-cycle C in G. Suppose that $\pi^{(0)}(C)$ is a good profile and* *let z_i $(1 \leqslant i \leqslant \lambda-1)$ be the unique sequence of projection values it determines, subject to the conditions $z_1 = -1$, $z_{\lambda-1} \neq -1$. Then the profile $\pi^{(1)}(C)$ of C with respect to x_1 is given by*

$$\pi_i^{(1)}(C) = \begin{cases} \pi_{i+1}^{(0)}(C) + z_{i+1} & (1 \leqslant i \leqslant \lambda-2), \\ \alpha+1 & (i = \lambda-1). \end{cases}$$

Proof. By definition,

$$\begin{aligned} z_{i+1} = p_1(x_{\alpha+i+1}) &= \partial(x_{\alpha+i+1}, x_1) - \partial(x_{\alpha+i+1}, x_0) \\ &= \pi_i^{(1)}(C) - \pi_{i+1}^{(0)}(C) \qquad (1 \leqslant i \leqslant \lambda-2). \end{aligned}$$

Also, $z_{\lambda-1} \neq -1$ so that, by C2, $\partial(x_{\alpha+\lambda}, x_1) = \alpha+1$; that is, $\pi_{\lambda-1}^{(1)}(C) = \alpha+1$.

In certain cases, the profile $\pi^{(1)}(C)$ will itself be a good profile, so that $\pi^{(2)}(C)$ will be determined uniquely, and so on. In such cases, when the profile vector of C with respect to each of its vertices is good, the vectors themselves form a periodic sequence. We shall refer to the period ω of this sequence as the *period* of C, and remark that ω must be a divisor of $2\alpha+\lambda$. For example, when $\beta \geqslant 3$ the calculation of the sequence of good profiles determined by the initial good profile $(\alpha+1, \alpha+1, \alpha, \alpha+1, \alpha+1)$ may be set out as in Figure 2.

$$\begin{array}{cccccccc} \alpha+1 & \alpha+1 & \alpha & \alpha+1 & \alpha+1 & & & \\ -1 & 0 & +1 & +1 & +1 & & & \\ & \alpha+1 & \alpha+1 & \alpha+2 & \alpha+2 & \alpha+1 & & \\ & -1 & 0 & 0 & 0 & 0 & & \\ & & \alpha+1 & \alpha+2 & \alpha+2 & \alpha+1 & \alpha+1 & \\ & & -1 & -1 & -1 & -1 & 0 & \\ & & & \alpha+1 & \alpha+1 & \alpha & \alpha+1 & \alpha+1 \end{array}$$

FIG. 2

In this case we have a sequence with period $\omega = 3$, and we conclude that $2\alpha+6$ must be divisible by 3, that is, $\alpha \equiv 0 \pmod 3$.

5. *The main results*

In this Section we shall apply the theory of profiles to obtain the result stated in Section 2: there is only one cubic DR graph in Region III. We shall have to deal separately with the three cases (i) $\beta = 2$, $\gamma \geqslant 1$; (ii) $\beta = 3$, $\gamma \geqslant 1$; (iii) $\beta \geqslant 4$, $\gamma \geqslant 0$, and in each case we shall obtain two kinds of cycle whose periods lead to contradictory congruence relations for α.

The case in which $\beta = 2$, $\gamma \geqslant 1$. For these values of β and γ any graph G in $\mathscr{G}(\alpha, \beta, \gamma)$ must contain an oriented $(2\alpha+6)$-cycle with profile $(\alpha+1, \alpha+1, \alpha, \alpha+1, \alpha+1)$, since choosing any pair of vertices at distance α to be x_0 and $x_{\alpha+3}$ determines the remaining vertices uniquely. When $\beta = 2$ this profile determines a sequence with period 4 (note the difference from the $\beta \geqslant 3$ case considered as an example in the previous section):

$$\begin{aligned} &(\alpha+1, \alpha+1, \quad \alpha, \quad \alpha+1, \alpha+1) \\ &(\alpha+1, \alpha+2, \alpha+2, \alpha+1, \alpha+1) \\ &(\alpha+1, \alpha+2, \alpha+3, \alpha+2, \alpha+1) \\ &(\alpha+1, \alpha+1, \alpha+2, \alpha+2, \alpha+1). \end{aligned}$$

Thus $2\alpha+6$ must be a multiple of 4, and α must be odd.

Similarly, provided $\alpha \geqslant 2$ we can choose any pair of vertices at distance $\alpha-1$, label them x_0 and $x_{\alpha+4}$, and take $x_0, x_1, \ldots, x_{\alpha+3}, x_{\alpha+4}$ to be a path of length $\alpha+4$. Then there is a unique vertex $x_{\alpha+5}$ adjacent to $x_{\alpha+4}$ such that $x_{\alpha+5} \neq x_{\alpha+3}$ and $\partial(x_\alpha, x_{\alpha+5}) = \alpha$. There are two choices for a vertex $x_{\alpha+6}$ adjacent to $x_{\alpha+5}$ for which $\partial(x_0, x_{\alpha+6}) = \alpha+1$ and exactly one of them yields a $(2\alpha+\beta)$-cycle with profile $(\alpha+1, \alpha+1, \alpha, \alpha-1, \alpha, \alpha+1, \alpha+1)$. It turns out that this also determines a sequence of good profiles with period 4:

$$\begin{array}{l}
(\alpha+1, \alpha+1, \quad \alpha, \quad \alpha-1, \quad \alpha, \quad \alpha+1, \alpha+1)\\
(\alpha+1, \alpha+1, \quad \alpha, \quad \alpha+1, \alpha+2, \alpha+2, \alpha+1)\\
(\alpha+1, \alpha+1, \alpha+2, \alpha+3, \alpha+2, \alpha+1, \alpha+1)\\
(\alpha+1, \alpha+2, \alpha+2, \alpha+1, \quad \alpha, \quad \alpha+1, \alpha+1).
\end{array}$$

It follows that $2\alpha+8$ is divisible by 4, so α is even and we have the required contradiction, provided $\alpha \geqslant 2$.

The only remaining possibility is that $\alpha = 1$. Since $\gamma \leqslant \alpha$ we must have $\gamma = 1$ also, and thus $d = 5$ and $c_5 = 2$ or 3. It is easy to check that $c_5 = 2$ is impossible and $c_5 = 3$ yields a unique graph, the dodecahedron.

The case in which $\beta = 3, \gamma \geqslant 1$. Here again we can show the existence of $(2\alpha+6)$-cycles and $(2\alpha+8)$-cycles with initial profiles as in the previous case, but because the projection values change in a different way the periods of the cycles are also different.

In the previous section we noted that when $\beta \geqslant 3$ and $\gamma \geqslant 0$ a $(2\alpha+5)$-cycle with initial profile $(\alpha+1, \alpha+1, \alpha, \alpha+1, \alpha+1)$ has period 3, so that $\alpha \equiv 0 \pmod 3$.

Furthermore, when $\beta = 3$ and $\gamma \geqslant 1$ a $(2\alpha+8)$-cycle with profile $(\alpha+1, \alpha+1, \alpha, \alpha-1, \alpha, \alpha+1, \alpha+1)$ has period 6, the sequence of profiles being as follows:

$$\begin{array}{l}
(\alpha+1, \alpha+1, \quad \alpha, \quad \alpha-1, \quad \alpha, \quad \alpha+1, \alpha+1)\\
(\alpha+1, \alpha+1, \quad \alpha, \quad \alpha+1, \alpha+2, \alpha+2, \alpha+1)\\
(\alpha+1, \alpha+1, \alpha+2, \alpha+3, \alpha+3, \alpha+2, \alpha+1)\\
(\alpha+1, \alpha+2, \alpha+3, \alpha+4, \alpha+3, \alpha+2, \alpha+1)\\
(\alpha+1, \alpha+2, \alpha+3, \alpha+3, \alpha+2, \alpha+1, \alpha+1)\\
(\alpha+1, \alpha+2, \alpha+2, \alpha+1, \quad \alpha, \quad \alpha+1, \alpha+1).
\end{array}$$

Hence 6 is a divisor of $2\alpha+8$, that is $\alpha \equiv 2 \pmod 3$, giving the required contradiction.

The case in which $\beta \geqslant 4, \gamma \geqslant 0$. In this case, the $(2\alpha+6)$-cycle with period 3 exists as in the previous case, and so we must have $\alpha \equiv 0 \pmod 3$. Unfortunately, the $(2\alpha+8)$-cycle now has period 5, which leads only to the conclusion $\alpha \equiv 1 \pmod 5$.

An increasingly despairing search for a type of cycle with a period yielding a contradictory congruence culminated, rather surprisingly, in the discovery of a $(2\alpha+13)$-cycle with period 15. The initial profile is

$$(\alpha+1,\ \alpha+2,\ \alpha+3,\ \alpha+3,\ \alpha+2,\ \alpha+1,\ \alpha+1,\ \alpha+2,\ \alpha+3,\ \alpha+3,\ \alpha+3,\ \alpha+2,\ \alpha+1),$$

and it can be verified by routine calculation that this is the first of a cyclic sequence of 15 good profiles. The fact that $2\alpha+13$ is divisible by 15 yields the congruence $\alpha \equiv 1 \pmod 3$, which is the required contradiction.

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