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GIRTH AND RESIDUAL FINITENESS

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1. Introduction

In this paper we shall study the connection between the residual finiteness of a group G and the existence of a sequence of finite graphs, whose groups are quotients of G, for which the girth tends to infinity. Special cases of this relationship have been observed in papers by Evans [6], Hoare [7], Znoiko [9], and the present author [2]. Here we shall establish a general framework for the theory and obtain some specific consequences. In particular, we shall prove that there are finite graphs of arbitrarily large girth having any given symmetry type.

2. Universal coverings and G-quotients

Let Γ be a finite, connected, k-valent graph, which is simple (that is, there are no loops or multiple edges). A *reduced walk* in Γ is a sequence of vertices $v_0, v_1, \dots, v_r$ (frequently written without the commas) in which consecutive vertices are adjacent in Γ, but $v_{i-1} \neq v_{i+1}$ $(1 \leqq i \leqq r-1)$. Let v be a chosen vertex of Γ. The *universal covering* $U\Gamma$ of Γ (with respect to v) is the graph whose vertices are the reduced walks in Γ starting from v, two being adjacent in $U\Gamma$ whenever one of them is a one-step extension of the other. Clearly, $U\Gamma$ is an infinite k-valent tree, and so if T_k is any realisation of such a tree, there is an isomorphism $\Theta\colon T_k \to U\Gamma$. If we are given a group G of automorphisms of T_k there is an isomorphic group G^* of automorphisms of $U\Gamma$, where an element g of G corresponds to $g^* = \Theta g \Theta^{-1}$.

The *canonical projection* $p\colon U\Gamma \to \Gamma$ is the function which takes a vertex of $U\Gamma$, regarded as a reduced walk in Γ, to its final vertex in Γ. The group of *covering automorphisms* of Γ, written Cov (Γ) is the group of automorphisms g of $U\Gamma$ which commute with p: that is, for any reduced walk ω in Γ and any covering automorphism g, ω and $g(\omega)$ have the same final vertex.

Lemma 1. *Let Γ and* Cov (Γ) *be as above. Then the non-identity elements of* Cov (Γ) *act on $U\Gamma$ with no fixed vertices or edges. (We say that an automorphism fixes the edge $\{\omega_1, \omega_2\}$ if it fixes the pair $\{\omega_1, \omega_2\}$ setwise but not necessarily pointwise.)*

Proof. Suppose first that g is a covering automorphism fixing the edge $\{\omega_1, \omega_2\}$ where, without loss of generality we may take

$$\omega_1 = vv_1v_2\dots v_r, \quad \omega_2 = vv_1v_2\dots v_rv_{r+1}.$$

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If $g(\omega_1)=\omega_2$ we have, since $pg=p$,

$$v_r = p(\omega_1) = pg(\omega_1) = p(\omega_2) = v_{r+1},$$

contradicting the fact that Γ has no loops. Hence we need only consider the case $g(\omega_1)=\omega_1$.

Now if g is an automorphism of $U\Gamma$ fixing ω_1, it follows that g permutes the neighbours of ω_1 in $U\Gamma$. These vertices of $U\Gamma$ correspond to reduced walks in Γ whose final vertices are the neighbours of $v_r=p(\omega_1)$ in Γ. Since $pg=p$, we see that g must fix each neighbour of v_r. Continuing the argument, and using the fact that Γ is connected, we conclude that g is the identity automorphism. ∎

It is a consequence of the famous result of Serre [8] (see also [3]) that Cov (Γ) is a free group. In many applications it is useful to identify Cov (Γ) with the fundamental group of Γ, which is free for the geometrically obvious reason that Γ has no 2-dimensional cells. However, we shall not employ that intuition here.

Let G be any group of automorphisms of T_k. We shall say that a finite graph Γ is a *G-quotient* (of T_k) if there is an isomorphism $\Theta\colon T_k \to U\Gamma$ such that the induced group G^* of automorphisms of $U\Gamma$ permutes the fibres of the covering projection $p\colon U\Gamma \to \Gamma$. That is,

$$p(\omega) = p(\omega') \Rightarrow pg^*(\omega) = pg^*(\omega')$$

for all g in G, where $g^*=\Theta g \Theta^{-1}$.

For example, let G be the free product of k copies of $\mathbf{Z}_2$: $G=\langle a_1, a_2, \ldots, a_k | a_1^2=a_2^2=\ldots=a_k^2=1\rangle$, and realise T_k as the Cayley graph of G with respect to the generating set $\{a_1, a_2, \ldots, a_k\}$. Then G is a group of automorphisms of T_k, acting by left-multiplication on the vertices. Let $\bar{G}$ be a finite group generated by k involutions $c_1, \ldots, c_k$, and let Γ be the Cayley graph of $\bar{G}$ with respect to this generating set. We can verify that Γ is a G-quotient of T_k as follows.

Define $\Theta\colon T_k \to U\Gamma$ to be the isomorphism which takes the vertex $a_\alpha \ldots a_\lambda$ of T_k to the reduced walk $(1, c_\alpha, \ldots, c_\alpha \ldots c_\lambda)$, regarded as a vertex of $U\Gamma$ (with respect to the base vertex 1). Suppose that

$$\omega = (1, c_\alpha, \ldots, c_\alpha \ldots c_\lambda), \quad \omega' = (1, c_\beta, \ldots, c_\beta \ldots c_\mu)$$

are two vertices of $U\Gamma$, such that $p(\omega)=p(\omega')$: in other words $c_\alpha \ldots c_\lambda = c_\beta \ldots c_\mu$ in $\bar{G}$. For any g in G we have

$$g^*(\omega) = \Theta g \Theta^{-1}(\omega) = \Theta(g a_\alpha \ldots a_\lambda)$$

since G acts by left-multiplication. Let the expressions for $g a_\alpha \ldots a_\lambda$ and $g a_\beta \ldots a_\mu$ as reduced words in G be

$$g a_\alpha \ldots a_\lambda = a_\pi \ldots a_\varrho, \quad g a_\beta \ldots a_\mu = a_\sigma \ldots a_\tau.$$

Then $g^*(\omega)$ is the reduced walk $(1, c_\pi, \ldots, c_\pi \ldots c_\varrho)$ in Γ and similarly $g^*(\omega')$ is the reduced walk $(1, c_\sigma, \ldots, c_\sigma \ldots c_\tau)$. But if $\bar{g}$ is the image of g under the homomorphism $G \to \bar{G}$ which takes a_i to c_i $(1 \leqq i \leqq k)$, we have

$$c_\pi \ldots c_\varrho = \bar{g} c_\alpha \ldots c_\lambda, \quad c_\sigma \ldots c_\tau = \bar{g} c_\beta \ldots c_\mu.$$

Since $c_\alpha \ldots c_\lambda = c_\beta \ldots c_\mu$ we conclude that $pg^*(\omega)=pg^*(\omega')$, as required.

Of course, the machinery is quite unnecessary if we simply wish to prove results about this particular example. Its advantage is that it covers other significant examples as well.

3. Residual finiteness

A group G is said to be *residually finite* if, for each g in $G\setminus\{1\}$ there is a normal subgroup N of finite index in G such that $g\notin N$. (Actually, it is not necessary to insist that N is normal, but there is no loss in doing so.)

Before stating our main theorem we need a Lemma.

Lemma 2. *Let G be a group acting on T_k such that the stabilizer G_v of a vertex v is finite. Then for each $r\geqq 0$ the set*

$$S(r) = \{g\in G \mid \delta(v, g(v)) \leqq r\}$$

is finite.

Proof. $S(r)$ is the set of g for which $g(v)=w$ and $\delta(v,w)\leqq r$. For each w with $\delta(v,w)\leqq r$ and such that w is in the same G-orbit as v, let g_w be a fixed element of G satisfying $g_w(v)=w$. Then the set of all g taking v to w is just the coset g_wG_v, and by the hypothesis this set is finite. The set $S(r)$ is the union of these cosets taken over a subset of the finite set of w within distance r of v, and hence $S(r)$ is finite. ∎

Theorem. *Let G be a group of automorphisms of T_k such that*

(1) *the stabilizer of each vertex is finite,*

(2) *the number of vertex-orbits is finite.*

Then there is a sequence (Γ_r) of G-quotients of T_k such that girth $(\Gamma_r)\to\infty$.

Proof. It follows from the general theory of Bass and Serre [8, p. 122] that a group G satisfying conditions (1) and (2) is residually finite.

Let $S(r)$ be as in Lemma 2, and for each g in $S(r)$ choose N_g to be a normal subgroup of finite index in G such that $g\notin N_g$. Define

$$N_r = \bigcap_{g\in S(r)} N_g.$$

Since $S(r)$ is finite, N_r is a normal subgroup of finite index in G. Hence it follows from condition (2) that the number of orbits of N_r on the vertex-set of T_k is finite.

Let Γ_r be the graph whose vertex-set is the set of vertex-orbits of N_r, two being adjacent in Γ when they contain adjacent vertices of T_k. Choose a vertex v in T_k and let $[v]$ denote its orbit under N_r. For each vertex w in T_k there is a unique reduced walk $vv_1\dots w$ in T_k and correspondingly a reduced walk $[v], [v_1], \dots, [w]$ in Γ_r. Let $U\Gamma_r$ be the universal covering of Γ_r (with respect to the base vertex $[v]$), and define $\Theta\colon T_k\to U\Gamma_r$ to be the isomorphism taking w to the vertex of Γ_r represented by the reduced walk $[v], [v_1], \dots, [w]$. In other words, $p\Theta(w)=[w]$, where p is the canonical projection from $U\Gamma_r$ onto Γ_r.

In order to show that Γ_r is a G-quotient of T_k, suppose ω and ω' are two vertices of $U\Gamma_r$ such that $p(\omega)=p(\omega')$, and let $\omega=\Theta(w)$, $\omega'=\Theta(w')$. Then it follows that $[w]=[w']$, and for each g in G we have $pg^*(\omega)=[g(w)]$, $pg^*(\omega')=[g(w')]$. But since N_r is normal in G,

$$\begin{aligned} [w] = [w'] &\Rightarrow w = n(w') && (n\in N_r)\\ &\Rightarrow g(w) = gn(w') = n'g(w') && (n'\in N_r), \end{aligned}$$

and so $pg^*(\omega)=pg^*(\omega')$ as required.

Furthermore, the girth of Γ_r is greater than r, since a cycle of length s in Γ_r lifts to a reduced walk of length s in $U\Gamma_r$ and the corresponding initial and final vertices in T_k belong to the same N_r-orbit. By definition of N_r two such vertices are separated by a distance exceeding r, and so $s>r$. ∎

4. Some applications of the Theorem

As a first application we shall give a direct proof of the fact that, for given $k \geqq 2$, it is possible to construct a finite group generated by k involutions whose Cayley graph has girth as large as we please. This result can be interpreted as saying that the free product of k copies of $\mathbf{Z}_2$ is a residually finite group. Indeed, the following proof is a graph-theoretical version of the residual finiteness proof given by Baumslag and Tretkoff [1].

Let T_k be realised as the Cayley graph of $G=\mathbf{Z}_2 * \mathbf{Z}_2 * \ldots * \mathbf{Z}_2$, as described in Section 1. Let D_r denote the disc of radius r in T_k, with centre the identity element of G. For each generator a_i of G define a permutation $\bar{a}_i$ of the vertices of D_r by the rule

$$\bar{a}_i(x) = \begin{cases} a_i(x) & \text{if} \quad a_i(x) \in D_r, \\ x & \text{if not.} \end{cases}$$

(The fact that $\bar{a}_i$ is a permutation depends on the property $a_i^2=1$.) Clearly $\bar{a}_i$ is an involution and the k involutions $\bar{a}_i$ $(1 \leqq i \leqq k)$ generate a finite group, a subgroup of the symmetric group on the finite set D_r. The Cayley graph Γ is thus a finite G-quotient of T_k.

Now a cycle of length s in Γ_r corresponds to a reduced word $w_s = w_s(\bar{a}_1, \ldots$ $\ldots, \bar{a}_k)$ which represents the identity permutation. Consider the effect of an arbitrary word of length s, regarded as a permutation of D_r, on the identity vertex of D_r. Each of the first r letters moves this vertex one step nearer the boundary of D_r, and then at least r further letters are required to return to the identity vertex. Hence if $w_s=1$ (and in particular, if w_s fixes the identity vertex) we must have $s \geqq 2r$. Hence girth $(\Gamma_r) \to \infty$.

For our second application we shall turn to the notion of symmetry type introduced by Djokovic [4] (see also [5]).

A pair (A, B) of finite groups is called a *finite simple amalgam of degree* $(k, 2)$ if

(1) $|A: A \cap B| = k$,
(2) $|B: A \cap B| = 2$,
(3) the only subgroup of $A \cap B$ which is normal in both A and B is the identity subgroup.

Let G be the amalgamated free product of A and B, with the subgroup $A \cap B$ amalgamated. We construct a graph whose vertices are the left cosets of A in G, and whose edges are the pairs $\{gA, gyA\}$, where g is any element of G and y is a fixed element of $B \setminus (A \cap B)$. Then this graph is a k-valent tree T_k, and G is a group of automorphisms of it, acting by left multiplication.

Let Γ be any finite connected graph which admits a group of automorphisms H acting *symmetrically:* that is, transitively on the set of ordered pairs of adjacent vertices. Let v_0 and v_1 be adjacent vertices, H_0 the stabilizer of v_0 and $H_{\{0,1\}}$ the stabilizer of $\{v_0, v_1\}$. The pair $(H_0, H_{\{0,1\}})$ is a finite simple amalgam of degree $(k, 2)$, where k is the valency of Γ, and it is known as the *symmetry type* of Γ. The determination of the possible symmetry types is complete for $k \leqq 3$; for other values of k only partial results are known [4].

The techniques developed in this paper have been developed with the following results in mind.

Proposition 1. *Let (A, B) be a finite simple amalgam of degree $(k, 2)$ and let G and T_k be as described above. Then a finite G-quotient of T_k is a finite graph whose symmetry type is (A, B).*

Proof. Suppose we are given a finite G-quotient Γ of T_k, with respect to some isomorphism $\Theta\colon T_k \to U\Gamma$, the universal covering relative to some base vertex 0 of Γ. The definition of a G-quotient is that $pg^*(\omega) = pg^*(\omega')$ whenever ω and ω' are reduced walks such that $p(\omega) = p(\omega')$. Thus for each g^* in G^* we may define an automorphism $\bar{g}$ of Γ as follows: given a vertex w in Γ let ω be any reduced walk from 0 to w in Γ, and put

$$\bar{g}(w) = pg^*(\omega).$$

Let $\bar{G}$ be the resulting group of automorphisms of Γ. Consider two adjacent vertices, say 0 and 1 of Γ, and the vertices 0 and 01 of $U\Gamma$ which cover them: the restriction of the homomorphism $g^* \to \bar{g}$ yields a homomorphism $G^*_{(0,01)} \to \bar{G}_{01}$, with kernel K^* say. For each k^* in K^*, $\bar{k}$ is the identity, so $pk^*(0\ldots x) = x$ for any reduced walk $0\ldots x$. In particular, $k^*(0y)$ is a reduced walk ending at y, where y is any neighbour of 0 in Γ. But k^* is an automorphism of $U\Gamma$ fixing 0, and $0y$ is adjacent to 0 in $U\Gamma$. Hence $k^*(0y) = 0y$. Similarly we may show that $k^*(01z) = 01z$ for each $z \neq 0$ adjacent to 1 in Γ.

Now we can prove that K^* is normal in G^*_{01}, and also in $G^*_{\{0,1\}}$. For, given any k^* in K^* and g^* in G^*_0, let $g^*(01) = 0y$, so that

$$g^{*-1}k^*g^*(01) = g^{*-1}k^*(0y) = g^{*-1}(0y) = 01,$$

and consequently $g^{*-1}k^*g^* \in K^*$. Similarly the condition $k^*(01z) = 01z$ shows that $K^* \lhd G^*_{\{0,1\}}$.

Since G^*_0 and $G^*_{\{0,1\}}$ may be identified with A and B respectively, the definition of a finite simple amalgam ensures that K^* is the identity. It follows that $\bar{G}_{(0,1)}$ is isomorphic to $G^*_{(0,1)}$, which is $A \cap B$, and that $\bar{G}_0 \approx A$, $\bar{G}_{01} \approx B$. In other words, Γ is a finite graph whose symmetry type, with respect to $\bar{G}$, is (A, B). ∎

Proposition 2. *Given any finite simple amalgam of degree $(k, 2)$ there is a sequence (Γ_r) of finite graphs such that the symmetry type of Γ_r is (A, B) and girth $(\Gamma_r) \to \infty$.*

Proof. As we have seen, the amalgamated free product G of A and B with the subgroup $A \cap B$ amalgamated, is a group of automorphisms of T_k. The group acts transitively, so the number of orbits is one, and the vertex-stabilizer is the finite group A. Hence it follows from the Theorem that there is a sequence of finite G-quotients of T_k with girth $(\Gamma_r) \to \infty$, and by Proposition 1, the symmetry type of Γ_r is (A, B). ∎

It should be possible to establish Proposition 2 by a direct construction of the quotients, analogous to the Cayley graph case. But, as in that case, the construction would almost certainly yield quotients with an astronomical number of vertices. More efficient constructions, such as some of those discussed in [2], would be of great interest.

References

[1] B. Baumslag and M. Tretkoff, Residually finite HNN extensions, *Comm. in Algebra* **6** (1978), 179—194.

[2] N. L. Biggs, Cubic graphs with large girth, *Annals N. Y. Acad. Sci., to appear.*

[3] N. L. Biggs, A proof of Serre's theorem, *Discrete Mathematics, to appear.*

[4] D. Z. Djoković, Automorphisms of regular graphs and finite simple group-amalgams, in: *Algebraic Methods in Graph Theory (ed: L. Lovász and V. T. Sós),* Vol I, 95—118 (North-Holland, 1981).

[5] D. Z. Djoković and G. L. Miller, Regular groups of automorphisms of cubic graphs, *J. Combinatorial Theory Ser. B* **29** (1980), 195—230.

[6] C. W. Evans, Net structure and cages, *Discrete Math.* **27** (1979), 193—204.

[7] M. J. Hoare, On the girth of trivalent Cayley graphs, in: *Graphs and Other Combinatorial Topics:* 109—114. (Teubner, Leipzig, 1982.)

[8] J.-P. Serre, *Trees,* (Springer, Berlin—Heidelberg—New York, 1980.)

[9] D. V. Znoiko, Vertex-symmetric graphs, (in Russian,) *Mat. Zametki* 33 (1983), 781—783.

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