Strongly Regular Graphs with No Triangles

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Research Report - September 2009

Abstract

A simplified version of the theory of strongly regular graphs is developed for the case in which the graphs have no triangles. This leads to (i) direct proofs of the Krein conditions, and (ii) the characterization of strongly regular graphs with no triangles such that the second subconstituent is also strongly regular. The method also provides an effective means of listing feasible parameters for such graphs.

1. Introduction

We consider graphs that are *strongly regular* and have *no triangles*, abbreviated to 'SRNT graphs'. Such a graph X is characterized by two parameters k and c, according to the rules

- X is regular with degree k;
- any two adjacent vertices have no common neighbours;
- any two non-adjacent vertices have c common neighbours.

We shall discuss only cases where

$$k \geq 3, \qquad k > c \geq 1.$$

These conditions rule out the the pentagon ($k = 2, c = 1$) and the complete bipartite graphs $K_{k,k}$, which have $c = k$. Thus, by an SRNT graph we mean a non-bipartite connected graph, with diameter 2 and degree at least 3. Only six such graphs are known. There are other pairs (k, c) for which an SRNT graph may exist, but they are quite rare. Brouwer's list [**2**] contains all possibilities with up to 280 vertices, and the Appendix to this paper contains all possibilities with up to 6025 vertices.

If v is a vertex of an SRNT graph X we denote the subgraphs induced by the sets of vertices at distances 1 and 2 from v by $X_1(v)$ and $X_2(v)$ respectively. Clearly, $X_1(v)$ has no edges, for any v. It is worth remarking that the graphs $X_2(v)$ need not all be isomorphic, although we often use the abbreviation X_2 to denote any one of them.

We shall need some standard notation and theory [**7**]. The sizes of X_2 and X are given by

$$\ell = \frac{k(k-1)}{c}, \qquad n = 1 + k + \ell = 1 + \frac{k}{c}(k - 1 + c).$$

The adjacency matrix A of X satisfies the equations $AJ = kJ$ and $A^2 + cA - (k - c)I = cJ$, where I is the identity matrix and J is the all-1 matrix. It follows that the eigenvalues of A are k (with multiplicity 1) and the roots λ_1, λ_2 of the equation $\lambda^2 + c\lambda - (k - c) = 0$. Furthermore, there is an integer $s > c$ such that $c^2 + 4(k - c) = s^2$, where s and c have same parity, and the eigenvalues are the integers

$$k = \frac{s^2 - c^2}{4} + c \qquad \lambda_1 = \frac{s - c}{2}, \qquad \lambda_2 = \frac{-s - c}{2}.$$

The multiplicities m_1, m_2 of λ_1, λ_2 are given by

$$m_1 = \frac{k}{2cs}\Big((k - 1 + c)(s + c) - 2c\Big), \quad m_2 = \frac{k}{2cs}\Big((k - 1 + c)(s - c) + 2c\Big).$$

Note that the algebra exhibits *s-symmetry*: replacing s by $-s$ fixes k but switches λ_1 and λ_2, m_1 and m_2.

The conditions that n and m_1 (and consequently ℓ and m_2) must be integers are known as 'feasibility conditions'. They severely restrict the possible parameters (k, c), for example, they imply that when $c \neq 2, 4, 6$ there are only finitely many possible pairs (k, c).

In Section 3 we shall discuss another feasibility condition, which (among other things) greatly simplifies the calculation of feasible parameters. If we take the basic parameters to be the positive integers c and $\lambda_1 = q$, then it follows that $s = c + 2q$ and $k = (q + 1)c + q^2$. The conditions that n and m_1 are integers also take a fairly simple form in terms of c and q (details are given in the Appendix).

As well as the arithmetical conditions summarized above, it is possible to derive some 'graph-theoretical' conditions. For example, a general result on distance-regular graphs [**1**, **3**] implies that $k > 2c$, a result that can be proved in this case by simple arguments. However, this result is not the best-possible. The feasibility condition given in Section 3 implies that $k \geq 3c - 1$; indeed, apart from the known examples, we must have $k \geq \frac{7}{2}c + \frac{25}{4}$.

2. The second subconstituent and its eigenspaces

It is clear from the definition of an SRNT graph X that X_2 is a regular graph of degree $k - c$. Simple arguments provide more specific information.

Theorem 1 X_2 is a connected graph with diameter 2 or 3.

Proof We show first that every path uvw in X is part of a 5-cycle. For any such path $v \in X_1(u)$ and $w \in X_2(u)$. Let x be any one of the $k - c$ vertices in $X_2(u)$ that is adjacent to w, and let y be any one of the c vertices in $X_1(u)$ that is adjacent to x. Since X has no triangles, $y \neq w$, and hence $uvwxy$ is a 5-cycle.

Now let d and d_2 denote the distance functions in X and $X_2(u)$ respectively. If p, q are vertices in $X_2(u)$ such that $d_2(p, q) > 2$, then $d(p, q) = 2$. All c vertices adjacent to p in $X_1(u)$ must also be adjacent to q, since p and q have c common neighbours.

Any path qvp in X with $v \in X_1(u)$ is part of a 5-cycle $qvpab$. If a were in $X_1(u)$, then a would also be adjacent to q and we should have a triangle abq. Similarly if b were in $X_1(u)$ we should have a triangle abp. Hence the path $pabq$ must be in $X_2(u)$, and $d_2(p, q) = 3$. □

In the case $c = 1$ (known as the *Moore graph* case) X_2 must have diameter 3: in fact, X_2 is an antipodal $(k-1)$-fold covering of the complete graph K_k [**5**]. The known SRNT graphs with $k = 3$ (Petersen) and $k = 7$ (Hoffman-Singleton) exemplify this situation. For some other known SRNT graphs (Clebsch and Higman-Sims) X_2 has diameter 2, and indeed it is strongly regular and thus an SRNT graph.

The algebraic theory of the subconstituents for any strongly regular graph is well-known [**7**, pp227-230]. In our case, when X_1 is trivial, it is possible to give a streamlined version.

Fix a vertex v in X and partition the adjacency matrix according to the vertex-partition $\{v\} \cup X_1(v) \cup X_2(v)$:

$$A = \begin{pmatrix} 0 & J & O \\ J & O & B^T \\ O & B & A_2 \end{pmatrix}.$$

(Here, and in what follows, the J's denote all-1 matrices of the appropriate sizes.)

For any $x \in \mathbb{R}^n$ let $[x_0\ x_1\ x_2]^T$ denote the corresponding column vector, partitioned in the same way as A, so that $x_0 \in \mathbb{R}$, $x_1 \in \mathbb{R}^k$, $x_2 \in \mathbb{R}^\ell$. By elementary matrix algebra it follows that if x is an eigenvector of A with eigenvalue $\lambda \neq k$ then

$$Jx_1 = \lambda x_0, \quad x_0 J + B^T x_2 = \lambda x_1, \quad Bx_1 + A_2 x_2 = \lambda x_2.$$

Theorem 2 For any SRNT graph, $m_1 \geq k$ and $m_2 \geq k$.

Proof Let P be the eigenspace of A for the eigenvector λ_1. If $x \in P$ then $Jx = 0$, and since $Jx_1 = \lambda_1 x_0$, it follows that

$$0 = x_0 + Jx_1 + Jx_2 = (1 + \lambda_1)x_0 + Jx_2.$$

Let Q be the space comprising those x for which $x_0 = 0$ and $x_2 = 0$. Since any $x \in Q$ also satisfies the equation $(1 + \lambda_1)x_0 + Jx_2 = 0$, it follows that the dimension of the space $P + Q$ is at most $n - 1$.

On the other hand, if $x \in P \cap Q$, then $x_0 J + B^T x_2 = \lambda_1 x_1$, so $x = 0$. Thus, by a standard theorem, $\dim(P + Q) = \dim P + \dim Q$, and

$$n - 1 \geq \dim(P + Q) = \dim P + \dim Q = m_1 + k.$$

Since $n - 1 = m_1 + m_2$, it follows that $m_2 \geq k$.

Similarly, or by s-symmetry, $m_1 \geq k$. □

Since $k-c$ is the degree of X_2, it is an eigenvalue of A_2, and since X_2 is connected, $k-c$ has multiplicity 1. We now determine the other possible eigenvalues of X_2. Substituting the partitioned form of A in the equation $A^2+cA-(k-c)I = cJ$ we obtain three significant equations:

$$B^TB = (c-1)J + (k-1)I \tag{1}$$

$$A_2^2 + cA_2 - (k-c)I + BB^T = cJ \tag{2}$$

$$A_2B = -cB + cJ. \tag{3}$$

Theorem 3 Suppose $\mu \neq k-c$ is an eigenvalue of X_2. Then either $\mu = \lambda_1$, $\mu = \lambda_2$, or $\mu = -c$.

Proof If $x \neq 0$ is an eigenvector for $\mu \neq k-c$ then we have $Jx = 0$. By (2), $(A_2^2 + cA_2 - (k-c)I)x = -BB^Tx$. Since $A_2x = \mu x$, we have

$$(\mu^2 + c\mu - (k-c))x = (\mu-\lambda_1)(\mu-\lambda_2)x = -BB^Tx.$$

Thus if $B^Tx = 0$, then $\mu = \lambda_1$ or $\mu = \lambda_2$.
Suppose $B^Tx \neq 0$. Transposing (3) we have

$$B^TA_2x = -cB^Tx, \quad \text{that is} \quad \mu(B^Tx) = -c(B^Tx),$$

and so in this case $\mu = -c$. □

3. The Krein conditions and their consequences

Two more feasibility conditions involve parameters K_1 and K_2, known as the *Krein parameters*. They arise in the general theory of distance-regular graphs, as described in [**3**]. After some elementary algebra, in the strongly regular case they can be written in terms of k, λ_1, λ_2 [**7**]:

$$\begin{aligned} K_1 &= \lambda_1\lambda_2^2 - 2\lambda_1^2\lambda_2 - \lambda_1^2 - k\lambda_1 + k\lambda_2^2 + 2k\lambda_2 \\ &= (k+\lambda_1)(\lambda_2+1)^2 - (\lambda_1+1)(k+\lambda_1+2\lambda_1\lambda_2). \end{aligned}$$

$$\begin{aligned} K_2 &= \lambda_1^2\lambda_2 - 2\lambda_1\lambda_2^2 - \lambda_2^2 - k\lambda_2 + k\lambda_1^2 + 2k\lambda_1 \\ &= (k+\lambda_2)(\lambda_1+1)^2 - (\lambda_2+1)(k+\lambda_2+2\lambda_1\lambda_2). \end{aligned}$$

When X is a SRNT graph, we can express K_1 and K_2 in terms of s and c, as follows:

$$K_1 = \frac{1}{16}(s+c)(s-c+2)((s+c)^2 - 2(s+3c))$$
$$K_2 = \frac{1}{16}(s-c)(s+c-2)((s-c)^2 + 2(s-3c)).$$

Yet more elementary algebra leads to alternative, simpler, formulae.

$$K_1 = \frac{1}{4}(s+c)(s-c+2)(\lambda_2^2 + \lambda_2 - c)$$
$$= \frac{1}{2}cs(s+c)\Big(\frac{m_1}{k} - 1\Big).$$

$$K_2 = \frac{1}{4}(s-c)(s+c-2)(\lambda_1^2 + \lambda_1 - c)$$

.

$$= \frac{1}{2}cs(s-c)\Big(\frac{m_2}{k} - 1\Big).$$

These formulae provide a direct proof of the fundamental result on the Krein parameters, in the SRNT case.

Theorem 4 For any SRNT graph, $K_1 \geq 0$ with equality if and only if $m_1 = k$, and $K_2 \geq 0$ with equality if and only if $m_2 = k$.

Proof This follows immediately from Theorem 2 and the formulae given above. □

Corollary 1 The matrix $A^2 + A - cI$ is positive semidefinite.

Proof The formulae show that λ_1 and λ_2 both satisfy the condition $\lambda^2 + \lambda - c \geq 0$, and the third eigenvalue k also does so. (A direct proof of this corollary may be possible.) □

Corollary 2 $k \geq 3c - 1$.

Proof We have shown that $\lambda_1^2 + \lambda_1 - c \geq 0$. Since λ_1 satisfies the equation $\lambda_1^2 + c\lambda_1 - (k-c) = 0$ it follows that

$$-c\lambda_1 + (k-c) + \lambda_1 - c \geq 0, \quad \text{that is,} \quad \lambda_1 \leq \frac{k-2c}{c-1}.$$

Since λ_1 is a positive integer, $k - 2c \geq c - 1$. □

Parameters such as $(k, c) = (9, 4), (21, 10), \ldots$ are usually ruled out by calculating the Krein parameters [**2**, **7**], but Corollary 2 achieves this result without any calculation. Similar methods lead to the following general results.

Corollary 3 The only SRNT graphs with $c+1 \le k \le 3c+4$ are the six currently-known ones.

Proof We have to consider the cases $k = 3c + b$, $b = -1, 0, 1, 2, 3, 4$.

If $k = 3c - 1$ we have $\ell = (3c-1)(3c-2)/c = 9c - 9 + 2/c$, so the only possiblities are $c = 1$ and $c = 2$. These define the pentagon and the Clebsch graph.

Suppose $k = 3c + b$ with $b \ge 0$. Then

$$s^2 = c^2 + 4(k-c) = c^2 + 8c + 4b = (c+4)^2 + 4(b-4).$$

When $b = 0, 1, 2, 3$ this implies that $s^2 < (c+4)^2$, and since s is a positive integer, $s \le c+3$. Hence

$$(c+3)^2 \ge (c+4)^2 - 4(4-b), \quad \text{that is} \quad c \le (9-4b)/2.$$

There are very few possibilities here, and the only one that gives a feasible set of parameters is $b = 0, c = 1, k = 3$, which defines the Petersen graph.

If $k = 3c+4$ we have $\ell = (3c+4)(3c+3)/c = 9c - 21 + 12/c$, so c is a divisor of 12. The only feasible solutions are $c = 1, 2, 4, 6$, which define the graphs known by the names of Hoffman-Singleton, Gewirtz, M_{22}, and Higman-Sims, respectively. □

Corollary 4 An SRNT graph that is not currently-known must have $k \ge \frac{7}{2}c + \frac{25}{4}$.

Proof In the light of the previous theorem, we can assume that $k = 3c + b$ with $b \ge 5$. In this case $s^2 > (c+4)^2$ and hence $s \ge c+5$. Thus

$$(c+5)^2 \le c^2 + 8c + 4b, \quad \text{that is} \quad c \le (4b-25)/2.$$

In other words, $k = 3c + b$ with $b \ge \frac{1}{2}c + \frac{25}{4}$, as claimed. □

4. Linked pairs

We now consider *linked pairs* (X, X') of SRNT graphs, that is, SRNT graphs X and X' such that $X' = X_2(v)$ for every vertex v of X. For comments on this problem, see [**6**].

Theorem 5 The parameters of a linked pair (X, X') are such that

$$k' = k - c, \qquad c' = c - q, \qquad \lambda_1' = \lambda_1,$$

where

$$q = \frac{c^2(k-2)}{k^2 - (c+1)k + c(c-1)}.$$

Proof Clearly, the degree of $X' = X_2$ is $k' = k - c$. Since the number of vertices of X' is equal to ℓ, we have

$$1 + k' + \frac{k'(k'-1)}{c'} = \frac{k(k-1)}{c}.$$

Substituting $k' = k-c$ and solving for c' gives $c' = c-q$, where q is as stated.

Since X' is strongly regular, it has two eigenvalues other than $k-c$, and just one of them (λ_1') is positive. According to Theorem 3, the only possible eigenvalues are λ_1, λ_2 and $-c$, of which only λ_1 is positive. Hence $\lambda_1' = \lambda_1$. □

Combining these equations leads to our main result. Since $\lambda_1' = \lambda_1$ it follows that $s'-c' = s-c$, and hence $s' = s-q$. From the equations $s'^2 = c'^2+4(k'-c')$ and $s^2 = c^2 + 4(k-c)$ we obtain

$$(s-q)^2 = (c-q)^2 + 4(k-c-c+q) = s^2 - 2qc + +q^2 - 4c + 4q,$$

$$\text{that is} \quad s = c - 2 + \frac{2c}{q}.$$

Here both s and q can be written as functions of k and c. This yields the equation

$$c^2(k-2)^2(c^2+4k-4c) = \Big(c(c-2)(k-2) + 2k^2 - 2(c+1)k + 2c(c-1)\Big)^2,$$

which is a quartic in k, and factors conveniently:

$$4(k-1)(k-c)(k^2 - (3c+1)k - c(c^2-4c-1)) = 0.$$

Thus, if there is a linked pair (X, X') and c is given, k must be a positive integer root of the quadratic factor. The discriminant of this factor is

$$\Delta = (3c+1)^2 + 4c(c^2-4c-1) = (c-1)^2(4c+1).$$

So Δ is a perfect square if and only if $4c+1$ is the square of an integer, which must be an odd number $2r+1$. That is, $c = r(r+1)$. The corresponding value of k is

$$\frac{1}{2}(3c + 1 + \sqrt{\Delta}) \ = \ r(r^2 + 3r + 1),$$

and these are the only values for which a linked pair can exist. Furthermore, for these values

$$q \;=\; \frac{c^2(k-2)}{k^2-(c+1)k+c(c-1)} \;=\; r.$$

It is easy to check that $k = q(q^2+3q+1)$ and $c = q(q+1)$ satisfy all the feasibility conditions for an SRNT graph X, as do the corresponding values for X', $k' = q^2(q+2)$ and $c' = q^2$. Precisely, we have

$$\ell = (q^2+2q-1)(q^2+3q+1), \quad n = q^2(q+3)^2, \quad s = q(q+3),$$
$$\lambda_1 = q, \quad \lambda_2 = -q(q+2), \quad m_1 = (q^2+2q-1)(q^2+3q+1),$$
$$m_2 = q(q^2+3q+1), \quad K_1 = q^2(q+1)(q+2)(q+3)(q^2+q-1), \qquad K_2 = 0.$$
$$\ell' = (q+1)(q+2)(q^2+q-1), \quad n' = (q^2+2q-1)(q^2+3q+1), \quad s' = q(q+2),$$
$$\lambda_1' = q, \quad \lambda_2' = -q(q+1), \quad m_1' = (q^2+3q+1)(q^2+q-1),$$
$$m_2' = (q+1)(q^2+2q-1), \quad K_1' = q^2(q+1)^2(q^3+2q^2-q-1), \quad K_2' = q^2(q^2+q-1).$$

Thus we have the main result.

Theorem 6 The parameters of a linked pair (X, X') of SRNT graphs must be of the form

$$k = q(q^2+3q+1), \quad c = q(q+1), \quad k' = q^2(q+2), \quad c' = q^2,$$

where q is a positive integer. Both sets of parameters are feasible for all $q \geq 1$. (When $q = 1$ we obtain the Clebsch/Petersen pair, and when $q = 2$ we obtain the Higman-Sims/M_{22} pair. These graphs are known to be the unique ones with the relevant parameters.) □

Similar results have been obtained by Cameron [**4**, Theorem 5] and Smith [**11**, Theorem E]. Cameron used a result on partial quadrangles, and Smith considered the case when X admits a group of automorphisms that acts transitively on the vertices, and the stabilizer of a vertex v acts transitively as a group of automorphisms of $X' = X_2(v)$. Her proof involves calculations with the constituents of the permutation characters, which appear similar to the calculations given above.

The values for k and c are the SRNT case of the family known as *negative latin square* parameters, first obtained by Mesner [**8**]. Graphs of this type were also studied by M. Shrikhande [**9**] and S. Shrikhande [**10**].

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Appendix

This Appendix contains calculations from Nimashi Thilakaratne's dissertation (2009) for the MSc in Applicable Mathematics at the LSE. The calculations are based on the following result (see also Cameron [**4**]).

Theorem The number n of vertices of an SRNT graph with $\lambda_1 = q$ is in the range

$$\left\lceil 2q^3 + 3q^2 - q + 2q(q+1)\sqrt{q^2+q-2} \right\rceil \;\le\; n \;\le\; q^2(q+3)^2.$$

The parameter c is in the range $1 \le c \le q(q+1)$, and must be such that

c is a divisor of $q^4 - q^2$, and

$c + 2q$ is a divisor of $q^4 + 3q^3 + 5q^2 + 3q + q(q^4 - q^2)/c$.

Proof Given the values of $\lambda_1 = q$ and c, the parameters s, k and n are

$$s = c + 2q, \qquad k = (q+1)c + q^2, \qquad n = Ac + B + D/c,$$

where $A = q^2 + 3q + 2$, $B = 2q^3 + 3q^2 - q$, $D = q^4 - q^2$.

According to the formulae given in Section 3, the condition $K_2 \geq 0$ implies that $\lambda_1^2 + \lambda_1 - c \geq 0$. Hence c lies in the range $1 \leq c \leq q(q+1)$. As a function of c, n has only one extreme point, a minimum, at the point where

$$c^2 = D/A, \quad \text{that is} \quad c = q\left(\frac{q-1}{q+2}\right)^{\frac{1}{2}}.$$

Substituting this value of c gives the minimum value n_{min}, and since n must be an integer, we get the result as stated above.

The maximum value n_{max} must therefore occur at one of the ends of the range, and a simple calculation shows that the values at $c = 1$ and $c = q(q+1)$ respectively are

$$q^4 + 2q^3 + 3q^2 + 2q + 2 \qquad \text{and} \qquad q^2(q+3)^2.$$

So the maximum occurs when $c = q(q+1)$.

If such a graph exists, n and m_1 must be integral. Another calculation gives

$$m_1 = Ac + E + \frac{Fc + qD}{c(c+2q)},$$

where $E = q^3 - 4q - 2$, $F = q(q+1)(q^2 + 2q + 3)$, and A, D are as above.

If n is integral c must divide D. In that case c must also divide $Fc + qD$ and hence the condition that m_1 is an integer reduces to the fact that $c + 2q$ must divide $F + q(D/c)$. □

This theorem enables feasible parameters to be calculated systematically. The method is to fix q and find those c in the range $1 \leq c \leq q(q+1)$ such that c and $c + 2q$ satisfy the divisiblity conditions. For example, when $q = 4$ we require the integers c such that $1 \leq c \leq 20$, c divides 240, and $c + 8$ divides $540 + 960/c$. It is easy to check that the only possibilities are $c = 2, 4, 6, 12, 16, 20$.

The theorem also gives bounds n_{min} and n_{max}, and these provide an effective method of tabulating the results. For $1 \leq q \leq 11$ the bounds are as follows:

q	1	2	3	4	5	6	7	8	9	10	11
n_{min}	4	50	154	342	638	1066	1650	2413	3381	4577	6025
n_{max}	16	100	324	784	1600	2916	4900	7744	11664	16900	23716

Suppose we wish to list all the feasible parameters for SRNT graphs with at most 1000 vertices. According to the table, we need only carry out the calculation for $1 \leq q \leq 5$, since $n_{min}(6)$ is greater than 1000. Similarly, if we list the feasible parameters for $1 \leq q \leq 10$, the list will contain all possibilities with fewer than 6025 vertices. The results of these calculations are tabulated below.

n	k	c	s	ℓ	λ_1	λ_2	m_1	m_2	K_1	K_2
10	3	1	3	6	1	−2	5	4	4	1
16	5	2	4	10	1	−3	10	5	24	0
50	7	1	5	42	2	−3	28	21	45	20
56	10	2	6	45	2	−4	35	20	120	24
77	16	4	8	60	2	−6	55	21	468	20
100	22	6	10	77	2	−8	77	22	1200	0
162	21	3	9	140	3	−6	105	56	648	135
176	25	4	10	150	3	−7	120	55	1064	144
210	33	6	12	176	3	−9	154	55	276	144
266	45	9	15	220	3	−12	209	56	5904	99
324	57	12	18	266	3	−15	266	57	11880	0
352	26	2	10	325	4	−6	208	143	840	360
352	36	4	12	315	4	−8	231	120	2080	448
392	46	6	14	345	4	−10	276	115	4200	504
552	76	12	20	475	4	−16	437	114	18240	480
638	49	4	14	588	5	−9	406	231	3672	1040
650	55	5	15	594	5	−10	429	220	5100	1125
667	96	16	24	570	4	−20	551	115	36400	304
704	37	2	12	666	5	−7	407	296	1680	840
784	116	20	28	667	4	−24	667	116	63840	0
800	85	10	20	714	5	−15	595	204	18000	1400

Table 1: Feasible parameters for SRNT graphs with at most 1000 vertices

$q = 5$ (including those listed in Table 1)

c	2	4	5	10	20	25	30
k	37	49	55	85	145	175	205
n	704	638	650	800	1190	1394	1600

$q = 6$

c	2	4	6	9	15	30	36	42
k	50	64	78	99	141	246	288	330
n	1276	1073	1080	1178	1458	2256	2585	2916

$q = 7$

c	1	4	6	7	14	21	28	42	49	56
k	57	81	97	105	161	217	273	385	441	497
n	3250	1702	1650	1666	2002	2450	2926	3906	4402	4900

$q = 8$

c	2	4	6	8	14	24	28	56	64	72
k	82	100	118	136	190	280	316	568	640	712
n	3404	2576	2420	2432	2756	3536	3872	6320	7031	7744

$q = 9$

c	2	4	9	12	15	18	27	36	72	81	90
k	101	121	171	201	231	261	351	441	801	891	981
n	5152	3752	3402	3552	3774	4032	4902	5832	9702	10682	11664

$q = 10$

c	2	4	6	10	20	45	90	100	110
k	122	144	166	210	320	595	1090	1200	1310
n	7504	5293	4732	4600	5425	8450	14280	15589	16900

Table 2: Feasible parameters for SRNT graphs with $\lambda_1 = q = 5, 6, 7, 8, 9, 10$