Families of parameters for SRNT graphs

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Abstract

The feasiblity conditions obtained in a previous report are refined, and used to determine several infinite families of feasible parameters for strongly regular graphs with no triangles. The methods are also used to improve the lower bound for the number of vertices, and to derive yet another interpretation of the Krein bound.

1. Introduction

This paper is a continuation of my report on *Strongly Regular Graphs with No Triangles*, which will be referred to as [SRNT1]. The graphs X considered in that paper are characterized by two parameters k and c, according to the rules

- X is regular with degree k;
- any two adjacent vertices have no common neighbours;
- any two non-adjacent vertices have c common neighbours.

It is convenient to rule out the pentagon and complete bipartite graphs, so we impose the conditions $k \geq 3$ and $k > c \geq 1$.

According to the standard theory, the eigenvalues of the adjacency matrix of X are k (with multiplicity 1) and the roots λ_1, λ_2 of the equation $\lambda^2 + c\lambda - (k-c) = 0$. Furthermore, there is an integer $s > c$ such that $c^2+4(k-c) = s^2$, where s and c have same parity, and the eigenvalues are the integers

$$k = \frac{s^2 - c^2}{4} + c \qquad \lambda_1 = \frac{s-c}{2}, \qquad \lambda_2 = \frac{-s-c}{2}.$$

The multiplicities m_1, m_2 of λ_1, λ_2 are

$$m_1 = \frac{k}{2cs}\Big((k-1+c)(s+c) - 2c\Big), \quad m_2 = \frac{k}{2cs}\Big((k-1+c)(s-c) + 2c\Big).$$

In the appendix to [SRNT1] we took the basic parameters to be q and c, where $q = \lambda_1$. In that case k, s, and the number of vertices $n_q(c)$, are given by the formulae

$$k = c(q+1) + q^2, \quad s = c + 2q, \quad n_q(c) = Ac + B + D/c,$$

where $A = (q+1)(q+2)$, $B = 2q^3 + 3q^2 - q$, and $D = q^4 - q^2$.

Given q, it was established that c must be in the range $1 \leq c \leq q(q+1)$, and must satisfy

c is a divisor of $q^4 - q^2$ (S1)

$c + 2q$ is a divisor of $q^4 + 3q^3 + 5q^2 + 3q + q(q^4 - q^2)/c$ (S2).

Using these results it was proved that $n = n_q(c)$ is in the range

$$\left\lceil 2q^3 + 3q^2 - q + 2q(q+1)\sqrt{q^2+q-2} \right\rceil \;\leq\; n \;\leq\; q^2(q+3)^2.$$

We shall now look more closely at the feasibility conditions, and use them to establish the existence of infinite families of feasible parameters. Also, in Section 5, we shall be able to improve the lower bound for n.

2. Feasibility conditions revisited

It turns out that the feasibility conditions (S1) and (S2) can be stated more simply. The formulae for m_1 and m_2 given above imply that

$$m_1 + m_2 = \frac{k(k+c-1)}{c}, \qquad m_1 - m_2 = \frac{k(k+c-3)}{s}.$$

If these expressions are integers, then it follows $n = m_1 + m_2 + 1$ is an integer, but m_1 and m_2 may be half-integers. However, we can obtain a set of conditions equivalent to (S1) and (S2) by adding the condition

$$m_1 + m_2 \equiv m_1 - m_2 \pmod 2.$$

Lemma 1 $m_1 + m_2$ is an integer if and only if

$$c \quad \text{divides} \quad q^4 - q^2.$$

In that case, $m_1 + m_2$ and $(q^4 - q^2)/c$ have opposite parity.

Proof Using the fact that $k = c(q+1) + q^2$,

$$\frac{k(k+c-1)}{c} = c(q+1)(q+2) + (2q+1)(q^2+q-1) + \frac{q^4 - q^2}{c}.$$

For all q, the first summand on the right-hand side is an even integer, and the second is an odd integer. Hence the result. □

Lemma 2 $m_1 - m_2$ is an integer if and only if

$$s = c + 2q \quad \text{divides} \quad q(q+1)(q+2)(q+3).$$

In that case $m_1 - m_2$ and $q(q+1)(q+2)(q+3)/(c+2q)$ have opposite parity.

Proof Putting $s = c + 2q$,

$$\frac{k(k+c-3)}{s} = c(q+1)(q+2) - (3q^2 + 7q + 3) + \frac{q(q+1)(q+2)(q+3)}{c+2q}.$$

For all q, the first summand on the right-hand side is an even integer, and the second is an odd integer. Hence the result. □

Theorem 1 The parameters (q,c) are feasible for an SRNT graph if and only if $1 \le c \le q(q+1)$, and the rational numbers

$$\alpha = \frac{q^4 - q^2}{c} \qquad \text{and} \qquad \beta = \frac{q(q+1)(q+2)(q+3)}{c+2q}$$

are integers having the same parity.

$\square$

3. Feasible parameters when c is given

For a given c, the condition $c \le q(q+1)$ implies that

$$q \ge q_{min} = \left\lceil \frac{\sqrt{4c+1}-1}{2} \right\rceil .$$

When c is of the form $r(r+1)$, we have $q_{min} = r$, and these parameters are feasible for all r.

The next theorem shows that in almost all cases there is a corresponding number q_{max}.

Theorem 2 For all values of c except $2, 4, 6$ there are only finitely many q for which (q,c) is feasible. Specifically, if (q,c) is feasible then $q \le c(h-1)/2$, where $h = |(c-2)(c-4)(c-6)|$, $(c \ne 2, 4, 6)$.

If c is odd then $q_m = c(h-1)/2$ is feasible. If $c \ge 8$ is even, then $q_m^* = c(h^* - 1)/2$ is feasible, where h^* is the integer $h/16$.

Proof The following identity is easily verified:

$$\begin{aligned} &c(c-2)(c-4)(c-6) - 16q(q+1)(q+2)(q+3) \\ &\qquad = (c+2q)(c-2q-6)(c^2 - 6c + 8 + 12q + 4q^2). \end{aligned}$$

For a given value of c it follows from the identity that if β is an integer then q must be such that $c+2q$ divides $c(c-2)(c-4)(c-6)$. When c is not one of $2, 4, 6$ this means that

$$c + 2q \ \le \ |c(c-2)(c-4)(c-6)|, \quad \text{that is,} \quad q \le \frac{1}{2}c(h-1),$$

so the finiteness result is proved.

Suppose now that c is odd and $q_m = c(h-1)/2$. In this case α is even, and in order to show that q_m is feasible, we must show that β is also even. We have $c + 2q_m = ch$ and for $e = 1, 2, 3$

$$q_m + e = (c-2e)X_e, \qquad \text{where} \qquad X_e = \frac{1}{2}\left(\frac{ch}{c-2e} - 1\right)$$

is an integer. It follows that

$$\beta = \frac{q_m(q_m+1)(q_m+2)(q_m+3)}{ch} = \frac{(h-1)X_1X_2X_3}{2}.$$

Since $h-1$ and at least one of X_1, X_2, X_3 is even, β must also be even.

Now suppose that c is even, $c = 2c^*$, and $q_m^* = c(h^*-1)/2 = c^*(h^*-1)$. We have $c + 2q_m^* = 2c^*h^*$, and for $e = 1, 2, 3$

$$q_m^* + e = (c^* - e)X_e^* \qquad \text{where} \qquad X_e^* = \frac{c^*h^*}{c^*-e} - 1,$$

is an integer. It follows that

$$\beta = \frac{q_m^*(q_m^*+1)(q_m^*+2)(q_m^*+3)}{c^*h^*} = (h^*-1)X_1^*X_2^*X_3^*.$$

In this case α is even except when $c^* \equiv 2 \pmod 4$, and it can be checked that β has the same parity. Hence q_m^* is feasible when c is even. □

For small even values of $c \geq 8$ it is easy to check by explicit computation that if (q, c) is feasible then $q \leq q_m^*$, so that q_m^* is the actual q_{max}. It should be possible to prove this in general.

The situation for $c = 2, 4, 6$ is well known, and is included here for completeness.

Theorem 3 The parameters (q, c) are feasible for all q when $c = 2, 4, 6$, except in the cases $c = 2$, $q \equiv 3 \pmod 4$, and $c = 6$, $q \equiv 1 \pmod 4$.

Proof Suppose $c = 2$. Since $\alpha = (q^4 - q^2)/2$ is always an even integer, and $c + 2q = 2(q+1)$, we require that $\beta = q(q+2)(q+3)/2$ is also an even integer. It is easy to check that this holds except when $q \equiv 3 \pmod 4$.

The other cases are similar. □

On the basis of the preceding results, it is easy to list the possible values of q for each c. The values for small $c \neq 2, 4, 6$ are collected here for reference.

$c = 1:$ $q = 1, 2, 7.$
$c = 3:$ $q = 3.$
$c = 5:$ $q = 5.$
$c = 7:$ $q = 7, 14, 49.$
$c = 8:$ $q = 8.$
$c = 9:$ $q = 3, 6, 9, 18, 27, 48, 63, 90, 153, 468.$
$c = 10:$ $q = 5, 10, 15, 25, 55.$
$c = 11:$ $q = 11, 12, 22, 33, 44, 77, 110, 187, 242, 341, 572, 1727.$
$c = 12:$ $q = 3, 4, 6, 9, 12, 14, 24, 30, 39, 54, 84, 174.$

$c = 13:\quad q = 13, 25, 39, 52, 65, 130, 208, 403, 494, 637, 1495, 4498.$
$c = 14:\quad q = 7, 8, 13, 14, 21, 28, 35, 63, 77, 98, 133, 203, 413.$

It will be noted that in many cases q is a multiple of c. For example, $q = 7c$ is feasible for $c = 1, 2, 4, 6, 7, 9, 11, 12, 14, \ldots$. The following theorem covers several of these cases.

Theorem 4 (1) For any positive integer b, $q = bc$ is feasible whenever $c \equiv 2, 4, 6$ modulo $2b + 1$.

(2) For $b \equiv 1, 7 \pmod 9$, $q = bc$ is feasible whenever $c \equiv 2, 4, 6$ modulo $(2b + 1)/3$.

Proof (1) Clearly $\alpha = b^2c(b^2c^2 - 1)$ is always an even integer. Since $c + 2q = (2b + 1)c$ we have

$$\beta = \frac{b(bc + 1)(bc + 2)(bc + 3)}{2b + 1}.$$

Let $c = (2b + 1)f + 2e$, where $e \in \{1, 2, 3\}$. Then

$$bc + e = b((2b + 1)f + 2e) + e = (2b + 1)(bf + e).$$

Thus $2b + 1$ divides one of the factors in the numerator of β. Since $2b + 1$ is odd, β must be even.

(2) As before, α is an even integer, and putting $t = (2b + 1)/3$,

$$\beta = \frac{b(bc + 1)(bc + 2)(bc + 3)}{3t}.$$

Let $c = tf + 2e$, where $e \in \{1, 2, 3\}$. Then

$$bc + e = b(tf + 2e) + e = t(bf + 3e).$$

Thus t divides one of the factors in the numerator of β. Since t is not divisible by 6, the quotient $(bc + 1)(bc + 2)(bc + 3)/t$ is divisible by 6, so β must be an even integer. $\square$

The first part of theorem shows, for example, that $q = 7c$ is feasible for $c \equiv 2, 4, 6 \pmod{15}$, and the second part strengthens this to $c \equiv 2, 4, 6 \pmod 5$.

Theorem 4 can be reformulated in terms of the feasibility of $c = q/b$ for a given value of q. This will be done in the next section.

4. Feasible parameters when q is given

We consider first the factors of $q^4 - q^2$ in the ring $\mathbb{Z}[q]$. Since $q^4 - q^2 = (q-1)q^2(q+1)$ and we require that $c \le q(q+1)$, the relevant factors are

$$q-1,\ q,\ q+1,\ q^2-q,\ q^2-1,\ q^2,\ q^2+q.$$

Theorem 5 The parameters (q, c) are feasible for all q when c takes any one of the values q, q^2-q, q^2, q^2+q.

Proof When $c = q$, $\alpha = q^3 - q$ is always an even integer. In this case $s = 3q$ and so we require that $\beta = (q+1)(q+2)(q+3)/3$ is even, which is clearly true. The other cases are similar. □

Theorem 6 When $c = q-1, q+1, q^2-1$ the parameters (q, c) are feasible for only a finite set of values of q in each case. Specifically,

$c = q-1$ is feasible only when $q = 2, 5, 7, 12, 47$.

$c = q+1$ is feasible only when $q = 1, 3, 13$.

$c = q^2-1$ is never feasible.

Proof When $c = q-1$, $\alpha = (q^4 - q^2)/c = q^3 + q^2$ is always an even integer. In this case $s = 3q-1$, and we have the identity

$$81q(q+1)(q+2)(q+3) = (3q-1)(27q^3 + 171q^2 + 354q + 280) + 280.$$

Hence, in order that $\beta = q(q+1)(q+2)(q+3)/(3q-1)$ should be an even integer, it is necessary that $3q-1$ must evenly divide 280. The possibilities are $3q-1 = 5, 14, 20, 35, 140$, corresponding to $q = 2, 5, 7, 12, 47$.

When $c = q+1$, $s = 3q+1$ and we have a similar identity

$$81q(q+1)(q+2)(q+3) = (3q+1)(27q^3 + 153q^2 + 246q + 80) - 80.$$

Hence, in order that $\beta = q(q+1)(q+2)(q+3)/(3q+1)$ should be an even integer, it is necessary that $3q+1$ must evenly divide 80. The possibilities are $3q+1 = 4, 10, 40$, corresponding to $q = 1, 3, 13$.

When $c = q^2-1$, $s = q^2 + 2q - 1$ and we have the identity

$$q(q+1)(q+2)(q+3) = (q^2 + 2q - 1)(q+2)^2 + (2q+4).$$

So the condition is that $q^2 + 2q - 1$ evenly divides $2q+4$, which holds only for the irrelevant values $q = 1, c = 0$. □

Of course, there are other divisors of $q^4 - q^2$, not covered by the arguments given above. For example, if $c = q(q-1)/b$ is an integer, then $\alpha = bq(q+1)$ is an even integer. Hence in this case (q, c) is feasible provided that

$$\beta = \frac{b(q+1)(q+2)(q+3)}{q+2b-1},$$

is an even integer. We have

$$b(q+1)(q+2)(q+3) = (q+2b-1)Q + R,$$

where

$$Q = b(q^2 - (2b-7)q + +4b^2 - 16b + 18), \qquad R = -4b(b-1)(b-2)(2b-3).$$

When $b = 1$ and $b = 2$, Q is even and $R = 0$, so the parameters are feasible for all q. The case $c = q(q-1)$ is already covered in Theorem 5, but $c = q(q-1)/2$ is a new infinite family. For larger values of b we get only finitely many feasible q in each case.

We summarize the results so far by listing the values of c that have been shown to be such that the parameters (q, c) are feasible for infinitely many q.

$c = 2$ for $q \equiv 0, 1, 2 \pmod 4$;

$c = 4$ for all q;

$c = 6$ for $q \equiv 0, 2, 3 \pmod 4$;

$c = q/b$ with $b \equiv 1, 7 \pmod 9$ for $q \equiv 0 \pmod b$ and $q \equiv 2b, 4b, 6b \pmod{(2b+1)/3}$;

$c = q/b$ with $b \equiv 0, 2, 3, 4, 5, 6, 8 \pmod 9$ for $q \equiv 0 \pmod b$ and $q \equiv 2b, 4b, 6b \pmod{2b+1}$;

$c = q$ for all q;

$c = q(q-1)/2$ for all q;

$c = q(q-1)$ for all q;

$c = q^2$ for all q;

$c = q(q+1)$ for all q.

5. The bounds for n

On the basis of the foregoing theory, it is possible to improve the lower bound for n given in the Appendix to [SRNT1].

Theorem 7 For all q except $q = 2, 4, 5, 6, 7, 8, 12, 47$ the number n of vertices of an SRNT graph with $\lambda_1 = q$ lies in the range

$$4q^3 + 6q^2 \le n \le q^2(q+3)^2.$$

The bounds are attained by feasible parameters when $c = q$ and $c = q(q+1)$ respectively. When $q = 2, 5, 7, 12, 47$ the lower bound is $4q^3 + 6q^2 - 2(q+1)$, and is attained when $c = q-1$. When $q = 4, 6, 8$ the lower bound is $4q^3 + 6q^2 - 2(q-1) + 12/(q-2)$, and is attained when $c = q-2$.

Proof The lower bound given in [SRNT1] is obtained by showing that there is a unique minimum of $n_q(c) = Ac + B + D/c$, which occurs when

$$c = q\left(\frac{q-1}{q+2}\right)^{\frac{1}{2}}.$$

Clearly this is just less than q, so we must examine values of c in the neighbourhood of q. We have shown in Section 4 that $c = q$ is feasible for all q, and in fact

$$n_q(q) = 4q^3 + 6q^2, \qquad n_q'(q) = 3q + 1 > 0.$$

This implies that, if $c = q$ is not the actual minimum, then the minimum must occur at some feasible $c < q$.

Now, when $c = q-3$,

$$n_q(q-3) = 4q^3 + 6q^2 + 18 + 72/(q-3), \qquad n_q'(q-3) < 0,$$

so values of $n_q(c)$ smaller than $4q^3 + 6q^2$ can only occur when $c = q-2$ or $c = q-1$.

When $c = q-2$

$$n_q(q-2) = 4q^3 + 6q^2 - 2q + 2 + 12/(q-2),$$

which shows immediately that $q-2$ can only be feasible when $q-2$ divides 12, so we must check the cases $q = 3, 4, 5, 6, 8, 14$ individually. It turns out that only $q = 4, 6, 8$ are feasible.

The case $c = q-1$ was covered explicitly in Theorem 5, where we found that only the values $2, 5, 7, 12, 47$ are feasible. In these cases we have $n_q(q-1) = 4q^3 + 6q^2 - 2q - 2$. $\square$

6. Further properties of the second subconstituent

The results in [SRNT1] establish that, for an SRNT graph X, the second subconstituent X_2 is a connected graph of degree $k-c$ with diameter 2 or 3. The only numbers that can be eigenvalues of X_2 are: $k-c$, $-c$ and the eigenvalues λ_1, λ_2 of X. In terms of the parameters (q,c), and in strictly decreasing order, these are

$$q(q+c), \quad q, \quad -c, \quad -(q+c).$$

Since X_2 is connected, $q(q+c)$ has multiplicity 1. Denote the multiplicities of $q, -c, -(q+c)$ by x, y, z respectively, and recall the standard formulae for $S_i = \sum \lambda^i$, $i = 0,1,2$. In this case the resulting equations are

$$S_0 = \ell, \qquad S_1 = 0, \qquad S_2 = \ell(k-c).$$

Putting $k = c(q+1) + q^2$, $\ell = k(k-1)/c$, these equations have a unique solution for x, y, z:

$$x = \frac{(q+1)(q^2+qc+c)(q^2+2qc+c^2-2c-q)}{c(c+2q)},$$

$$y = q^2 + qc + c - 1,$$

$$z = \frac{(q+c-1)(q^2+qc+c)(q^2+q-c)}{c(c+2q)}.$$

It can be verified that these values also satisfy the condition $S_3 = 0$, corresponding to the fact that X_2 has no triangles.

Comparison with the formulae for the Krein parameters given in [SRNT1] shows that

$$qc(c+2q)z = (q^2+qc+c)K_2.$$

Hence the fact that z is a non-negative integer is equivalent to $K_2 \geq 0$. The fact that $z = 0$ when $c = q(q+1)$ means that X_2 is an SRNT graph in this case.

References

[SRNT1] N.L. Biggs. Strongly regular graphs with no triangles. *Research Report* September 2009. arXiv:0911.2160v1

The references from [SRNT1] are repeated here for convenience.

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