Coxeter and Conway on the Symmetry of Graphs and Maps

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The fascination of symmetry

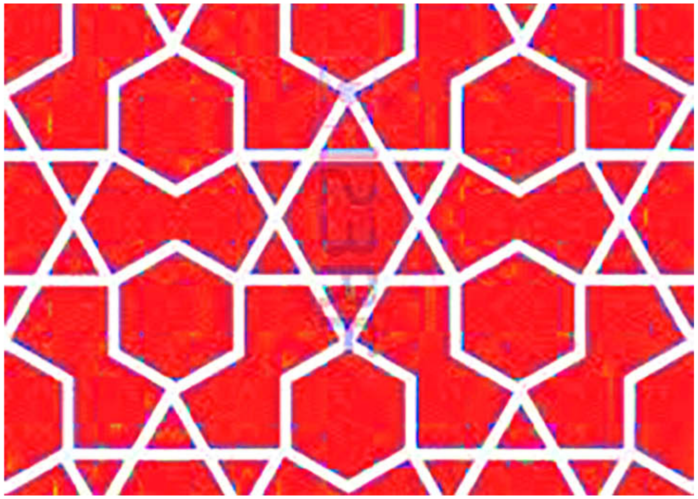
Mirror symmetry

in Nature

and in Art

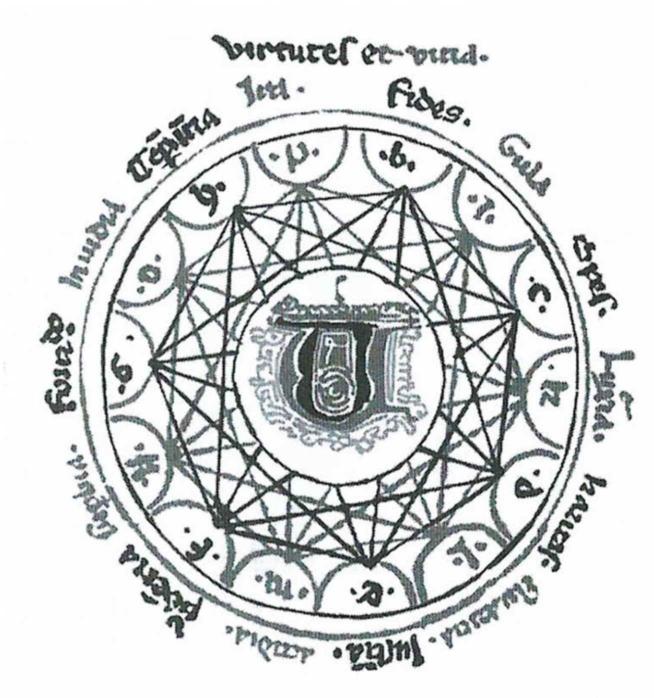
Complex symmetry

in Nature

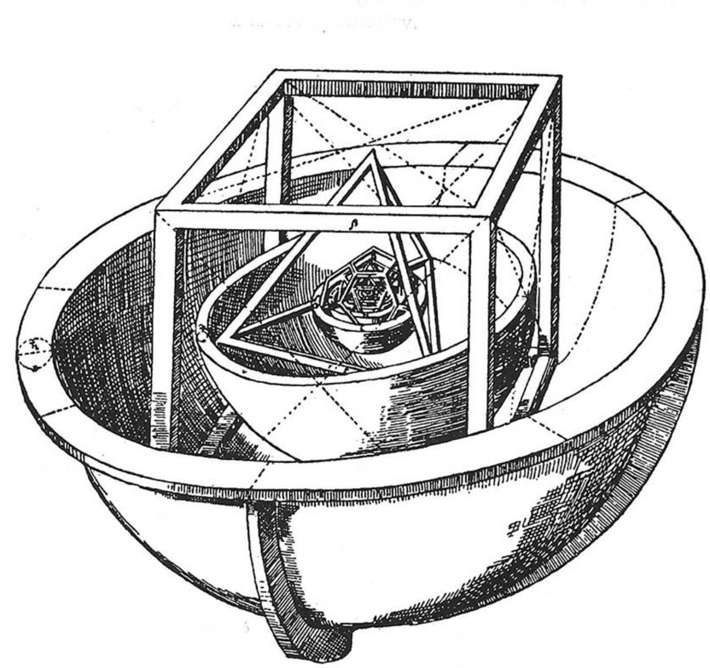


and in Art

The science of symmetry



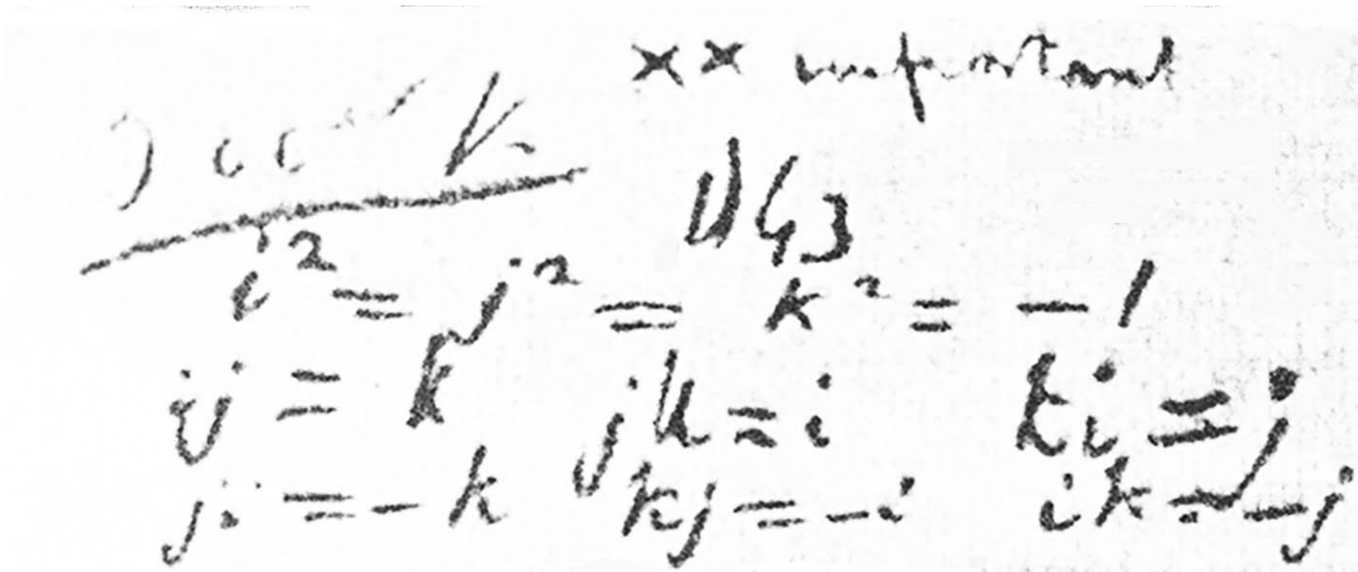
Ramon Llull, c.1280



Johannes Kepler c.1600

19th century algebra – symbols no longer stand for numbers, and so they can obey different rules.

Hamilton's Quaternions 1843:



Hamilton's Icosian Calculus 1856: $\iota^2 = 1, \kappa^3 = 1, \lambda^5 = 1, \lambda = \iota\kappa$.

Cayley's groups 1854:

Suppose that α is such a root, the group must clearly be of the form

$$1,\ \alpha,\ \alpha^2,\ \gamma,\ \alpha\gamma,\ \alpha^2\gamma,\quad (\alpha^3=1);$$

and multiplying the entire group by γ as nearer factor, it becomes $\gamma, \alpha\gamma, \alpha^2\gamma, \gamma^2, \alpha\gamma^2, \alpha^2\gamma^2$; we must therefore have $\gamma^2=1, \alpha$, or α^2. But the supposition $\gamma^2=\alpha^2$ gives $\gamma^4=\alpha^4=\alpha$, and the group is in this case $1, \gamma, \gamma^2, \gamma^3, \gamma^4, \gamma^5$ $(\gamma^6=1)$; and the supposition $\gamma^2=\alpha$ gives also this same group. It only remains, therefore, to assume $\gamma^2=1$; then we must have either $\gamma\alpha=\alpha\gamma$ or else $\gamma\alpha=\alpha^2\gamma$. The former assumption leads to the group

$$1,\ \alpha,\ \alpha^2,\ \gamma,\ \alpha\gamma,\ \alpha^2\gamma,\quad (\alpha^3=1,\ \gamma^2=1,\ \gamma\alpha=\alpha\gamma),$$

which is, in fact, analogous to the system of roots of the ordinary equation $x^6-1=0$; and by putting $\alpha\gamma=\lambda$, might be exhibited in the form $1, \lambda, \lambda^2, \lambda^3, \lambda^4, \lambda^5$, $(\lambda^6=1)$, in which this system has previously been considered. The latter assumption leads to the group

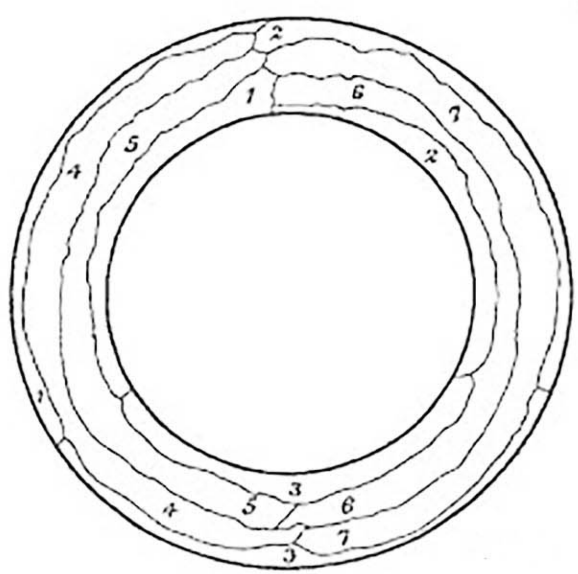
$$1,\ \alpha,\ \alpha^2,\ \gamma,\ \alpha\gamma,\ \alpha^2\gamma,\quad (\alpha^3=1,\ \gamma^2=1,\ \gamma\alpha=\alpha^2\gamma).$$

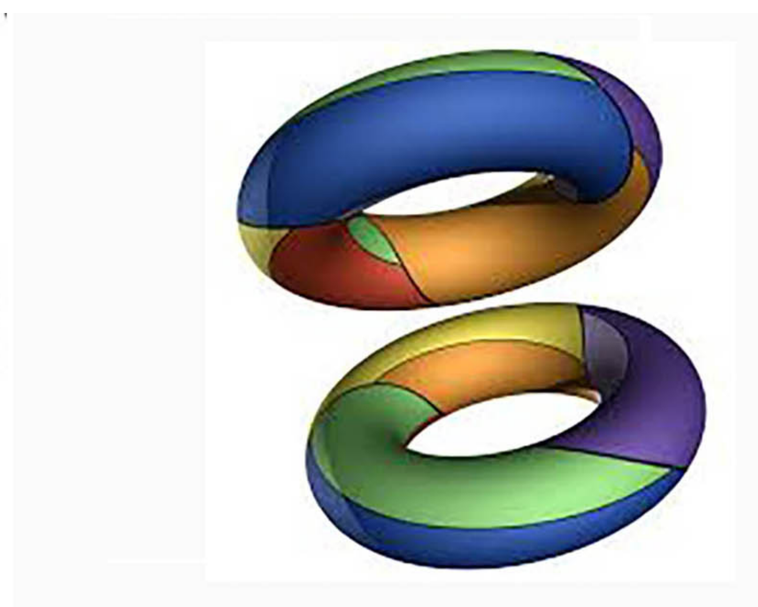
And we have thus two, and only two, essentially distinct forms of a group of six.

Symmetrical networks

In the 19th century, symmetry was studied by many including Klein, Dehn, Burnside, . . .

Also (inadvertently) Heawood: seven mutually adjacent hexagons on a torus 1890:





In 1920, two electrical engineers noticed that in some electrical networks all the edges are 'equivalent'. In 1932 one of them, RONALD FOSTER, made a list.

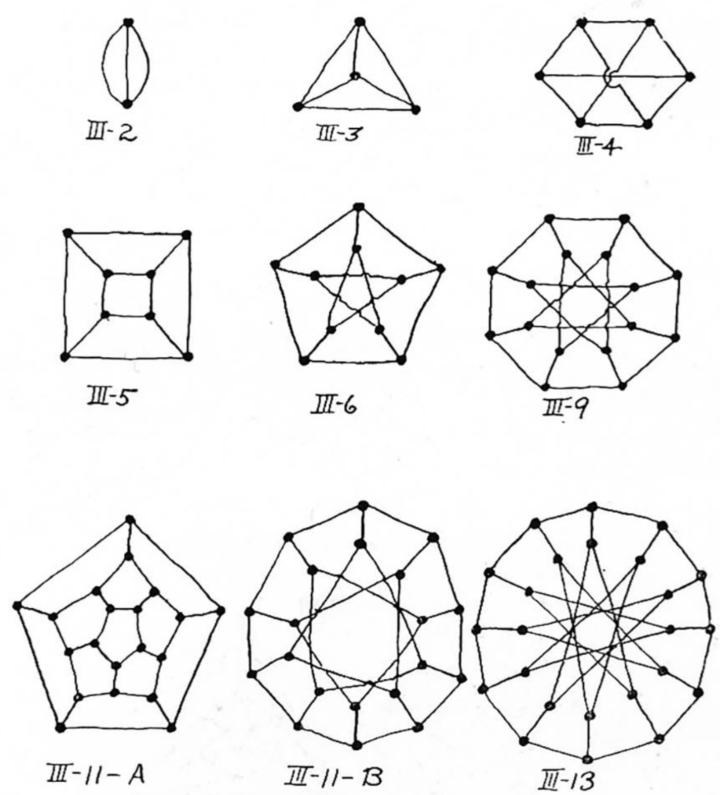
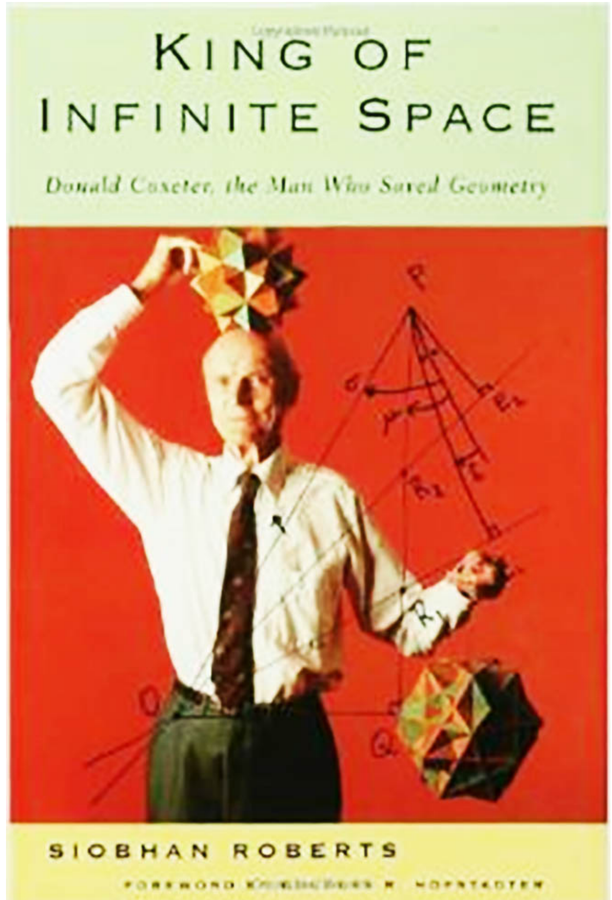
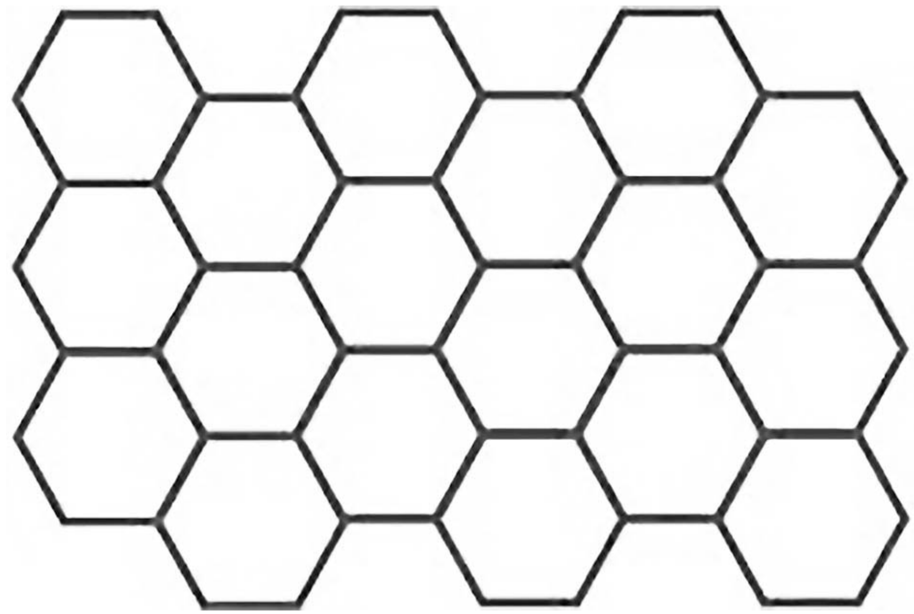


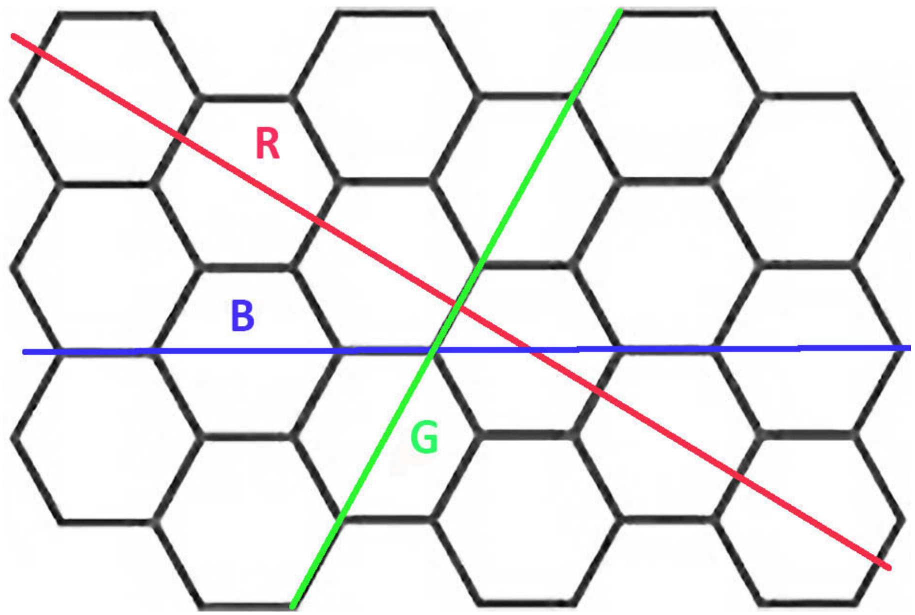
FIG. 9—SYMMETRICAL GEOMETRICAL CIRCUITS WITH THREE BRANCHES AT EACH VERTEX

H.S.M. COXETER (Cambridge 1926-36, Toronto 1936-2003)



He combined geometrical ideas with algebraic operations and made the study of symmetry part of mainstream mathematics.

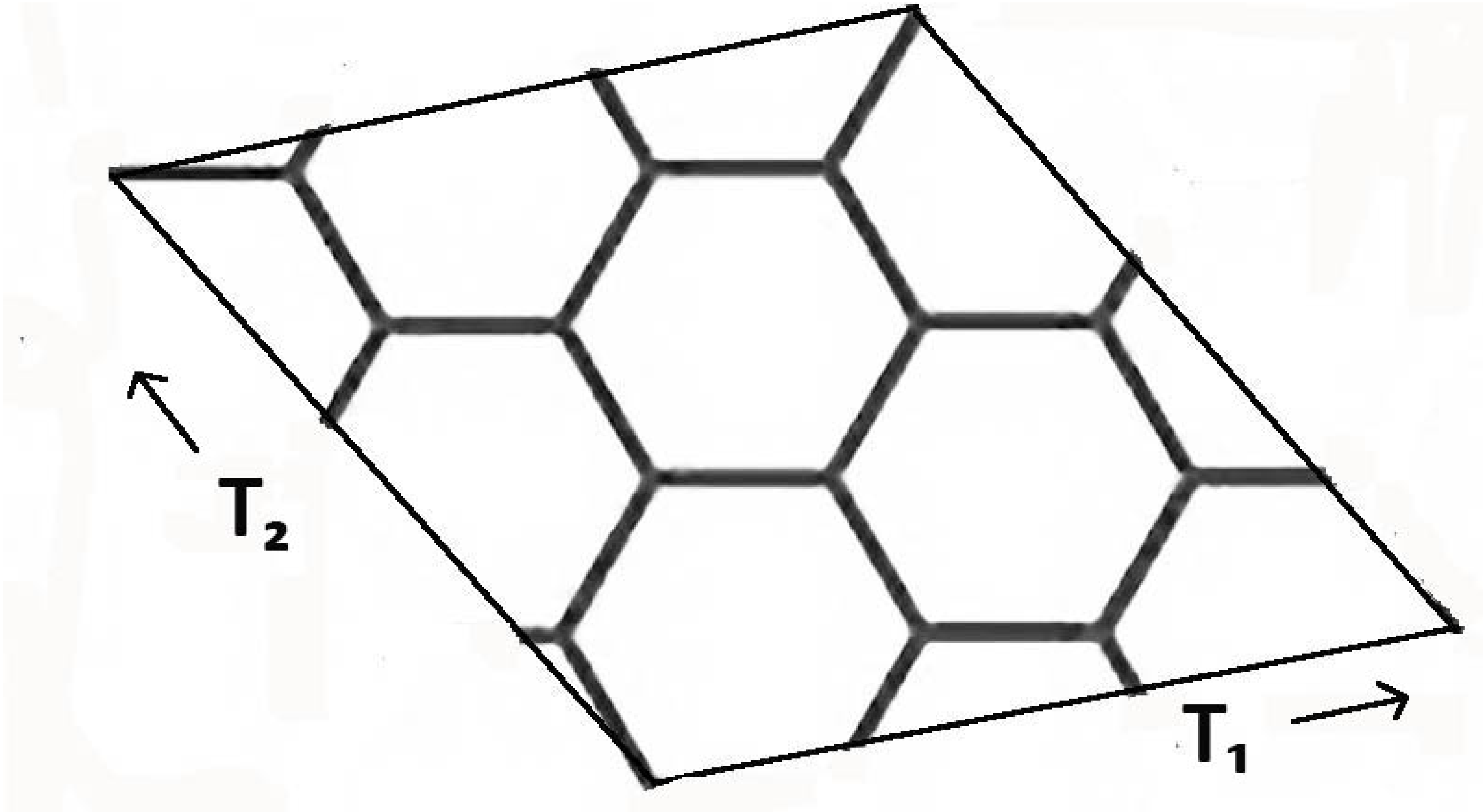




$R^2 = B^2 = G^2 = 1$, and since the combination of reflections in lines meeting at angle θ is a rotation through 2θ.

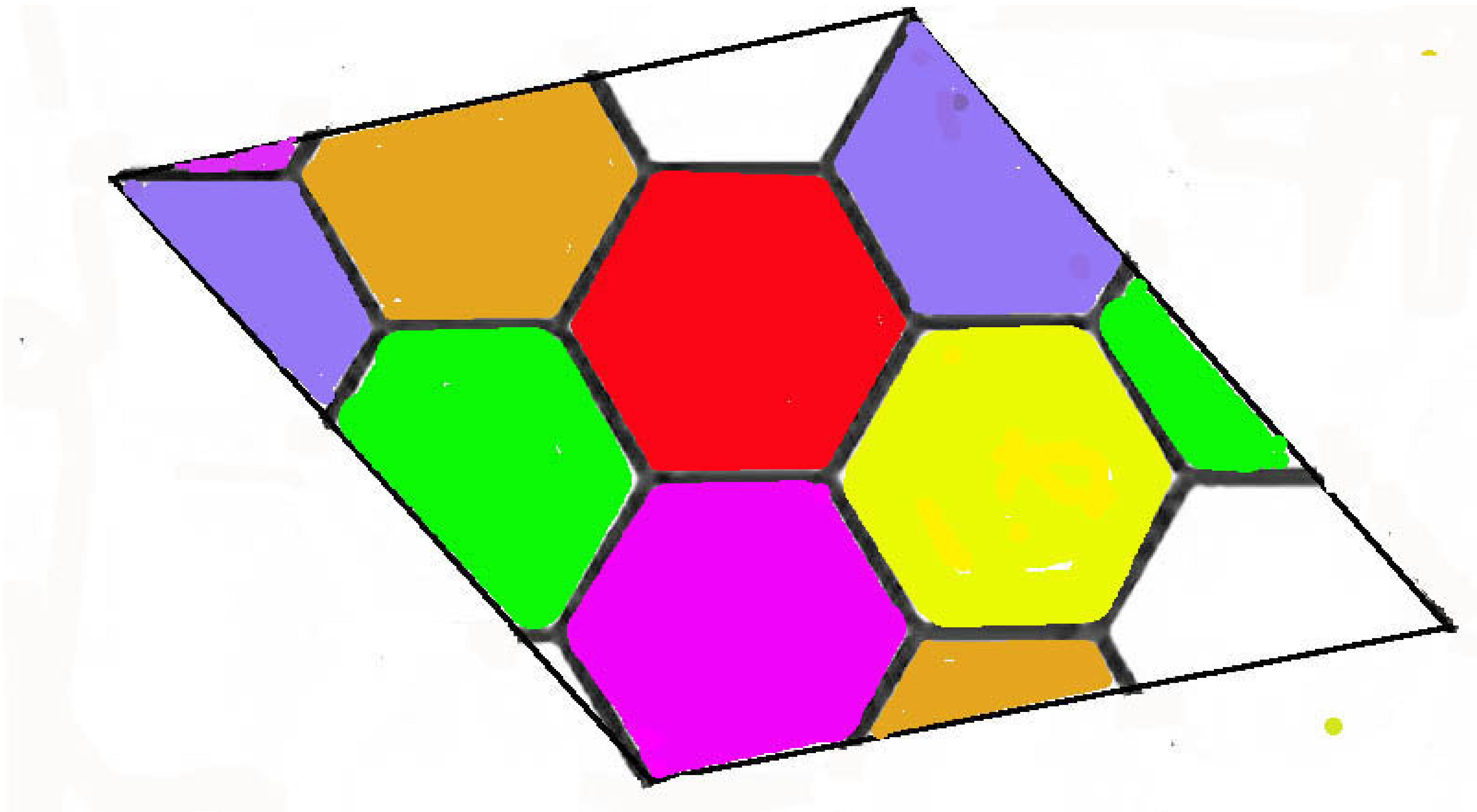
$$(RB)^6 = (BG)^3 = (GR)^2 = 1.$$

This group is *infinite*.

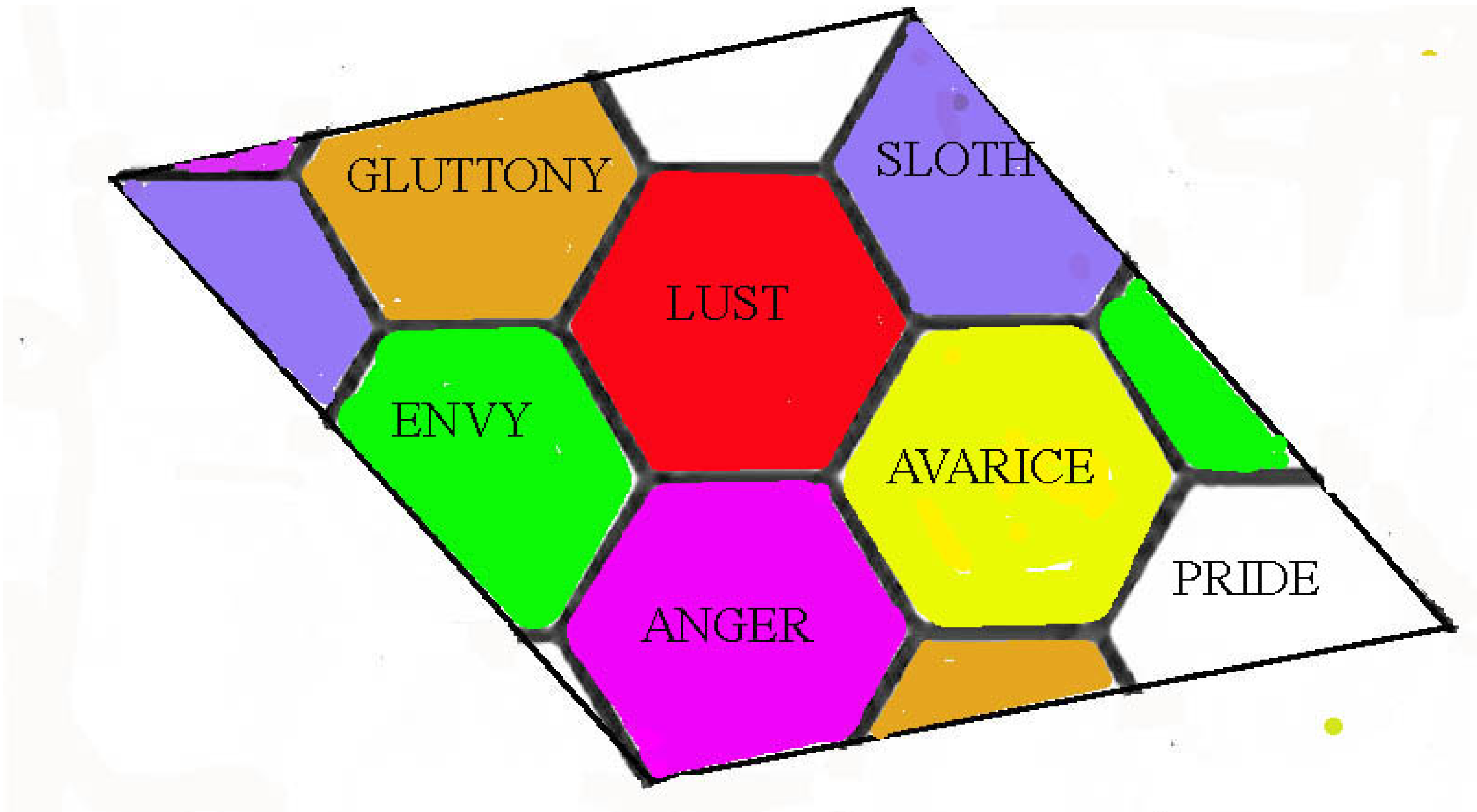


Translational symmetries T_1 and T_2 can be expressed in terms of R, B, G.

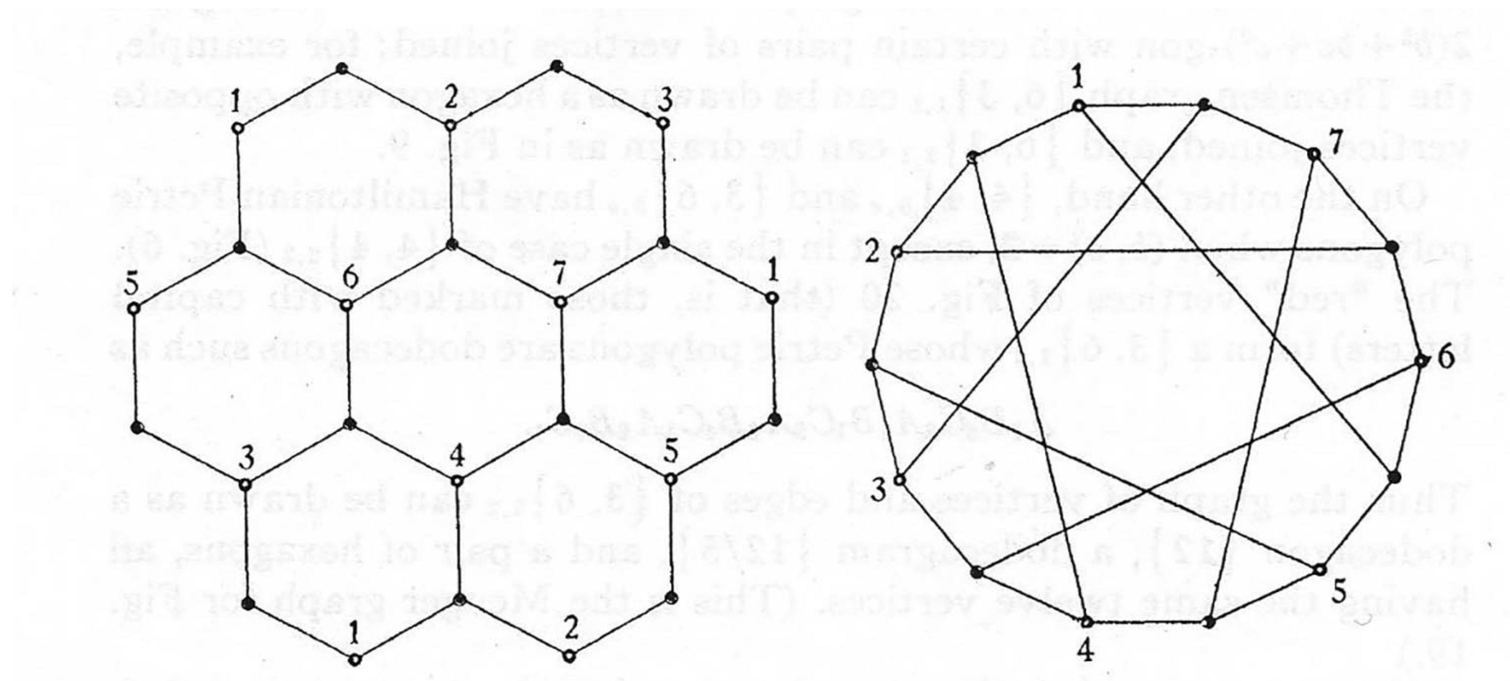
Adding the relations $T_1(R, B, G) = 1$ and $T_2(R, B, G) = 1$ defines a *finite* group.



Heawood's map



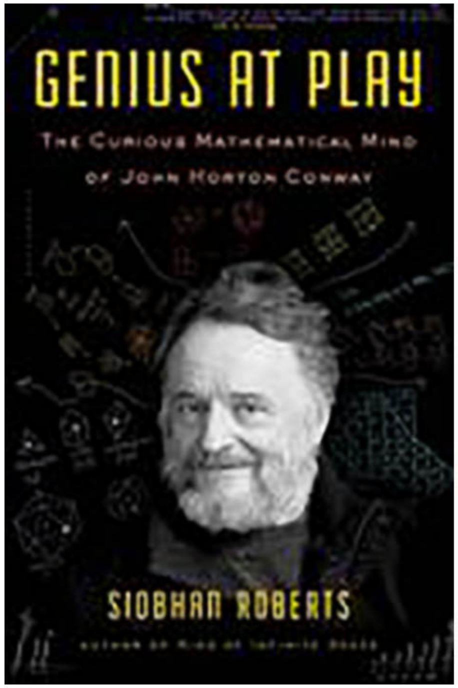
Dual of Llull's graph

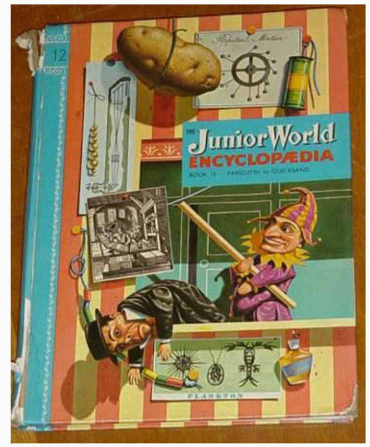


Heawood’s map and graph

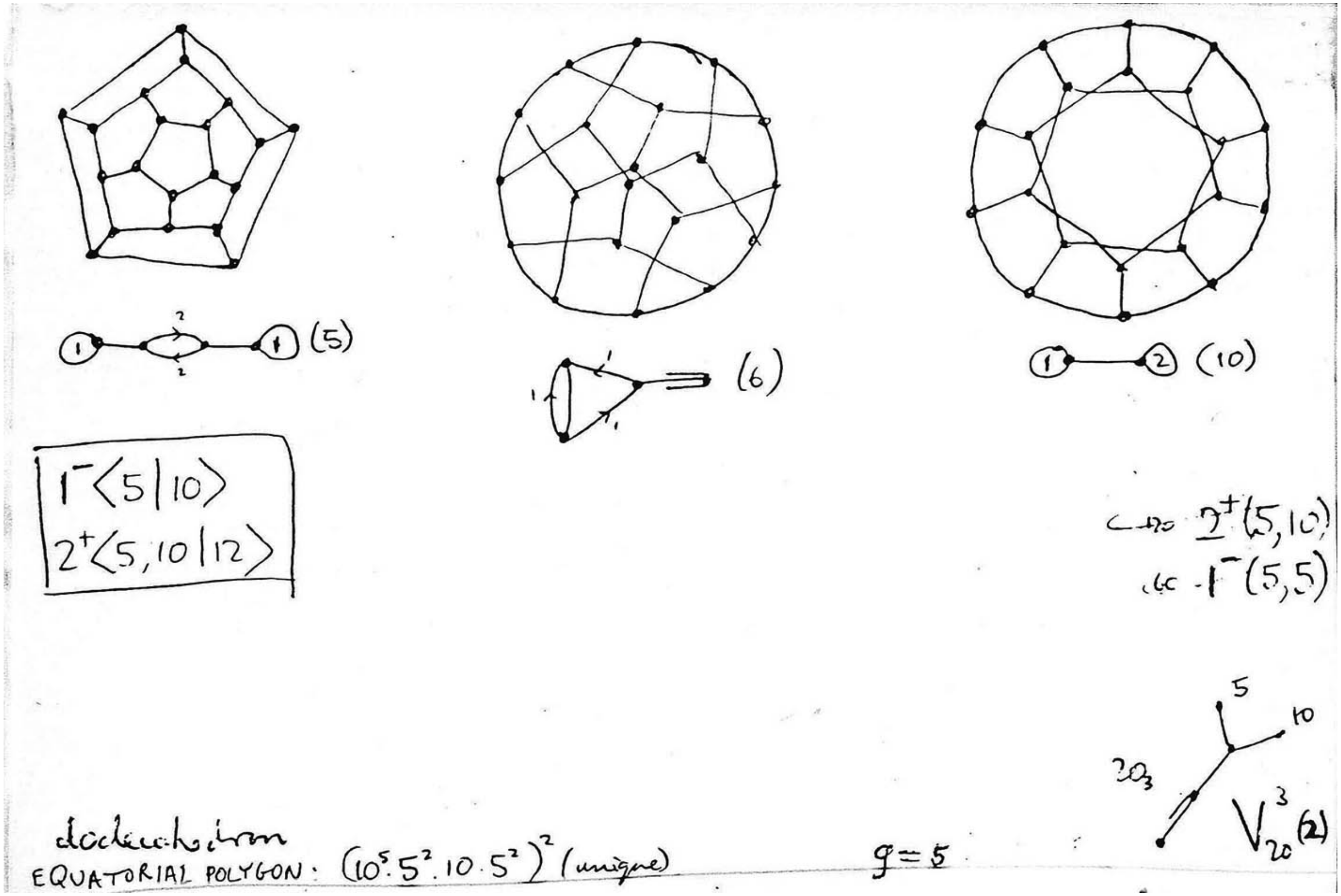
In 1948 BILL TUTTE moved from Cambridge to Toronto. In 1947 he had written about symmetrical graphs, including this one. These diagrams are from a long paper written by Coxeter (1950), in which Tutte’s influence was prominent.

J.H. CONWAY (Cambridge 1956-1986, Princeton 1986-)





Conway and the *Junior World Encylopedia*

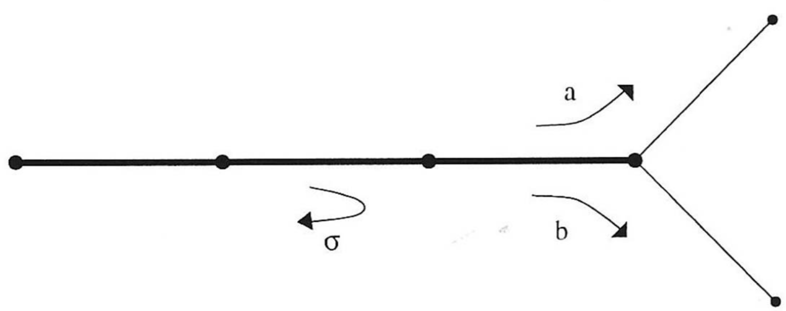


The dodecahedron according to Conway's *Junior World Encyclopedia*

Tutte (1947): In a finite symmetric graph with degree 3, there is a number s such that given any two s-arcs there is a unique automorphism which transforms one into the other. And

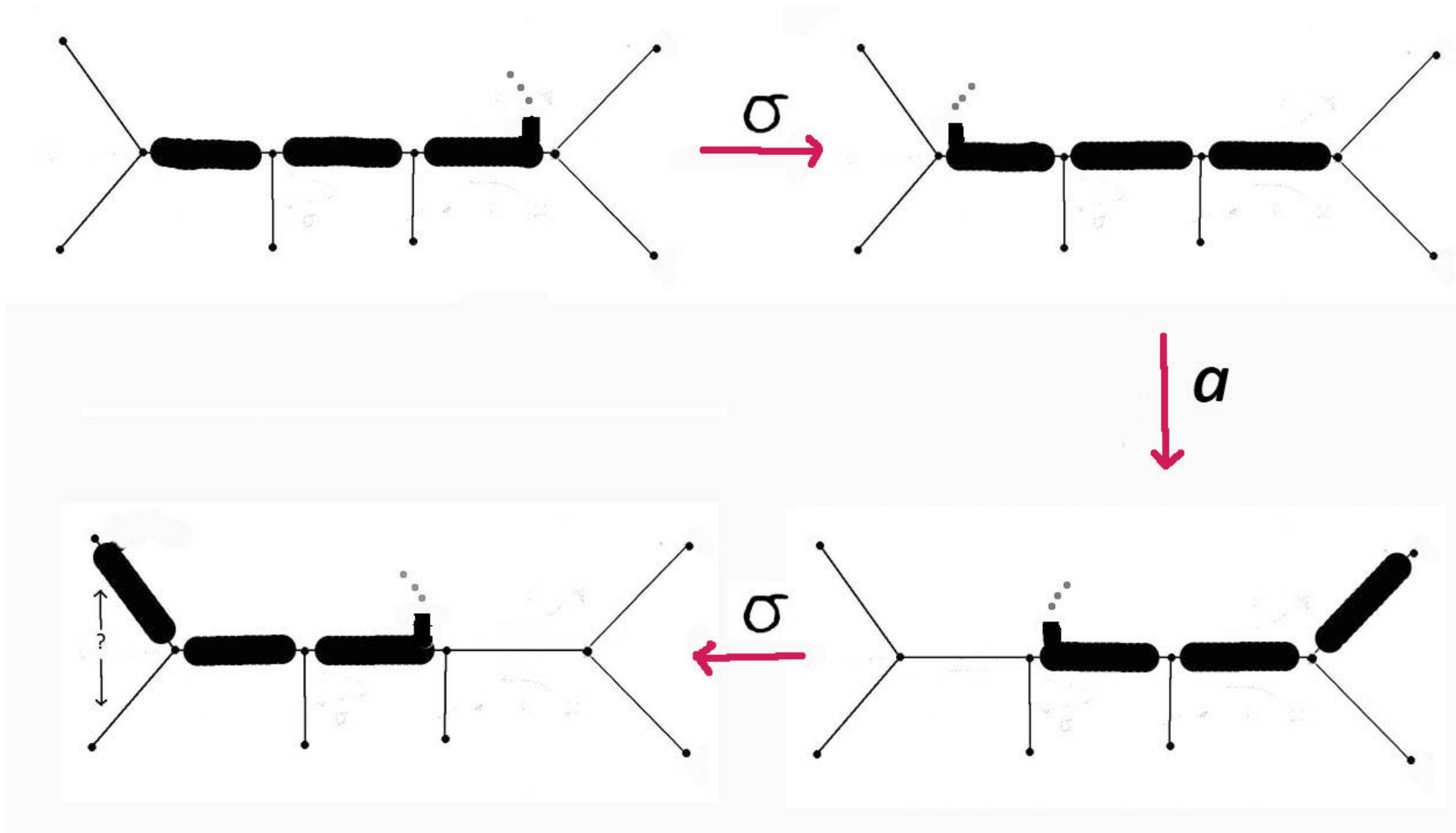
s cannot exceed 5.

Conway's method (1960s)



a and b shunt the arc onto its two successors, σ reverses it.

a, b, σ satisfy some relations, for example



$\sigma a \sigma$ must be equal to either a^{-1} or b^{-1}.

For example, when $s = 2$ we find the relations

$$\sigma^2 = 1,\ \sigma a \sigma = a^{-1},\ \sigma b \sigma = b^{-1},\ ab^{-1}a = b,\ ab\sigma ba^2 = b^2.$$

These define an infinite group. For a finite group, we need more relations, like

$$a^n = 1,$$

which corresponds to a cycle of length n in the graph.

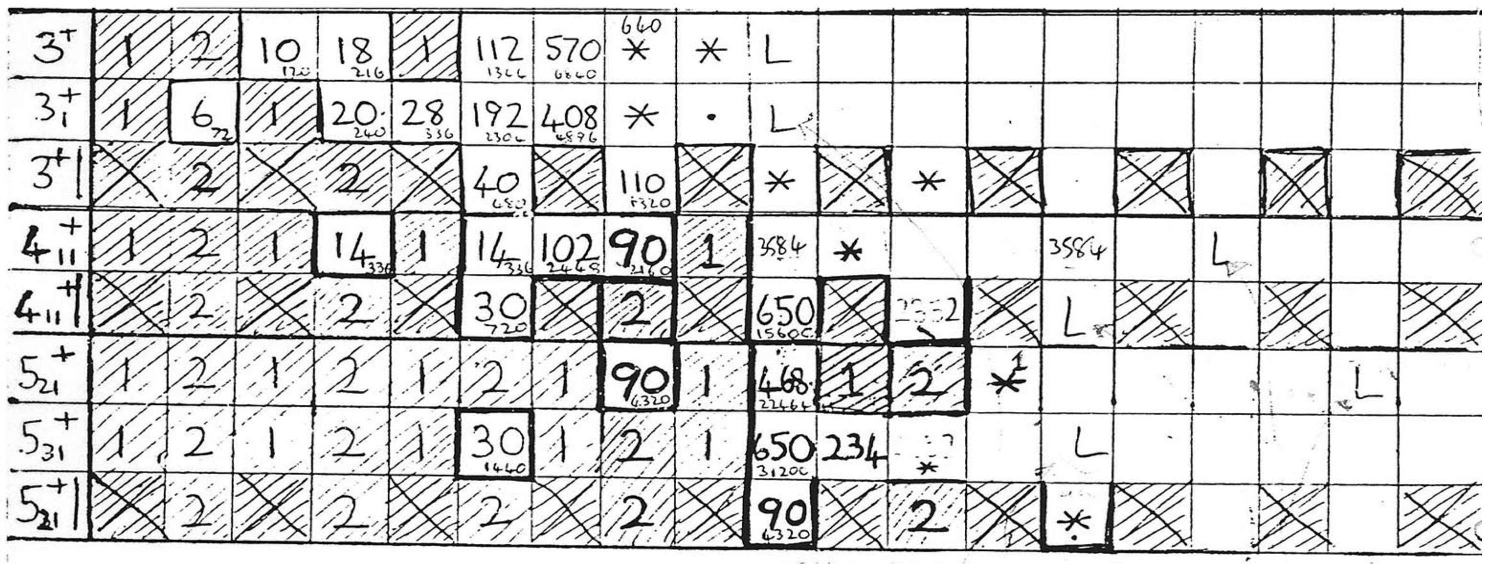
In order to find out if such a group is indeed finite we can try:

coset enumeration.

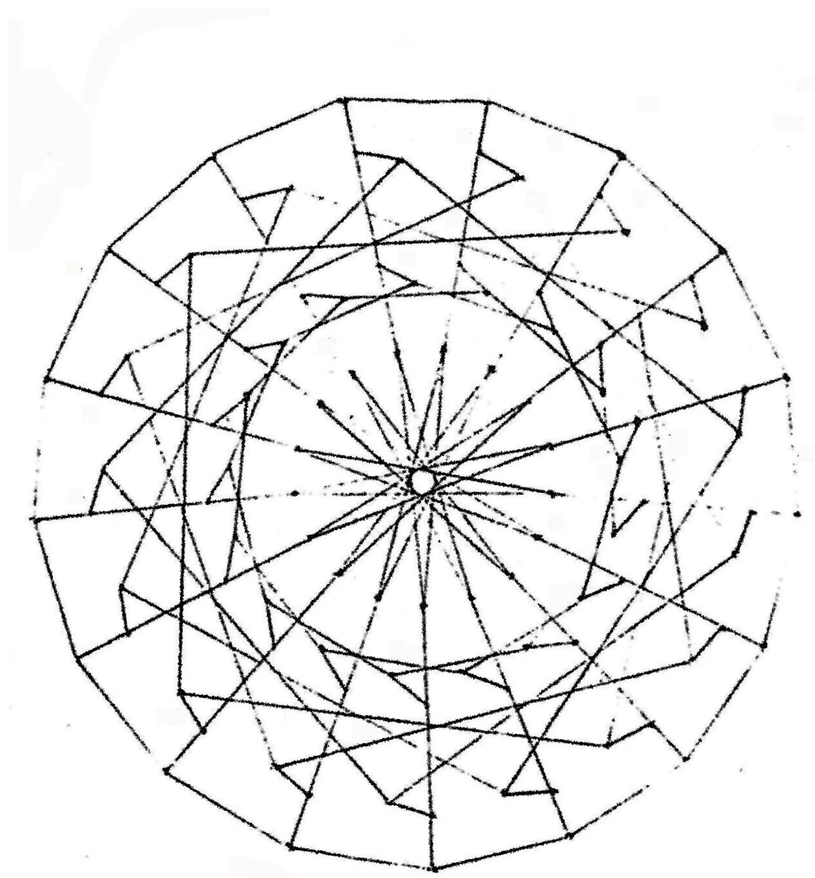
An algorithm, invented by Todd and Coxeter (1936). The objective is to find the number of cosets of a known subgroup in a group defined by generators and relations. It may take a long time.

In the 1960s it was programmed by MJT Guy at the Cambridge Computing Laboratory.

Applied to some of the Conway presentations, with suitable choices, it worked!



Results of coset enumerations, Conway and Guy c.1965



The graph with 102 vertices and a group of order 2448
Note the 17-fold rotational symmetry

This turned out to be fruitful area of research, for several reasons.

H.S.M.Coxeter, ‘My graph’ *Proc. LMS* 46 (1983) 117-136.

W.T.Tutte, *Graph Theory As I Have Known It* (Oxford U.P., 1998).

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