ON THRESHOLD CROSSING INDICES OF THE HARMONIC SERIES AT HALF-INTEGER LEVELS

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Abstract. We study the indices at which the harmonic partial sums $H_n = \sum_{k=1}^{n} \frac{1}{k}$ first meet or exceed successive half-integer levels beginning at 1. For each such threshold t, let $\tau(t)$ denote the minimal integer n with $H_n \geq t$. We prove the existence and uniqueness of each crossing, establish the monotonicity of the resulting index sequence, and show that the ordered crossings partition initial segments of $\mathbb{N}$ into disjoint intervals. Using the classical asymptotic expansion for H_n, we obtain

$$\tau(i/2) = e^{i/2-\gamma+o(1)} \quad \text{as } i \to \infty.$$

implying asymptotically exponential growth of successive interval lengths.

1. Introduction

The harmonic series provides a classical example of a slowly diverging series whose partial sums grow logarithmically. In this paper we study the indices at which the partial sums first meet or exceed prescribed threshold levels.

Specifically, we consider the half–integer thresholds

$$1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$$

and record the minimal index n for which the harmonic sum H_n reaches each level. These crossing indices generate a sequence that encodes the inverse growth behavior of the harmonic series.

We show that the ordered crossings induce a natural partition of initial segments of $\mathbb{N}$ and that the indices grow exponentially with respect to the threshold level. The subsequence corresponding to integer thresholds

$$\tau(1), \tau(2), \tau(3), \ldots = 1, 4, 11, 31, 83, 227, 616, \ldots$$

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coincides with the classical sequence recording the least index k for which $H_k \geq n$, which appears in the On-Line Encyclopedia of Integer Sequences as sequence A004080. A related sequence associated with half-integer thresholds, defined by the least index k such that $H_k > n/2$, appears as OEIS sequence A226161 [4]. The sequence studied here differs from A226161 in that it is defined using the weak inequality $H_k \geq t$ rather than the strict inequality $H_k > t$.

2. Definitions

We define the crossing indices associated with half-integer threshold levels of the harmonic partial sums and the induced marker sets. All objects are defined over the real numbers $\mathbb{R}$ and the natural numbers $\mathbb{N}$.

Definition 2.1 (Harmonic partial sums)**.** For each integer $n \geq 1$, define the nth harmonic partial sum by

$$H_n = \sum_{k=1}^{n} \frac{1}{k}. \tag{1}$$

For convenience, extend (1) by setting $H_0 := 0$. The sequence $(H_n)_{n\geq 1}$ is strictly increasing and satisfies

$$\lim_{n\to\infty} H_n = \infty. \tag{2}$$

Definition 2.2 (Threshold schedule)**.** Define

$$T := \left\{\frac{i}{2} : i \in \mathbb{N},\ i \geq 2\right\} = \left\{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \dots\right\}.$$

For $L \in \mathbb{N}$ with $L \geq 2$, define the truncated set

$$T_L := \{t \in T : t \leq L/2\}.$$

Definition 2.3 (Crossing index)**.** For each threshold $t \in T$, define the crossing index

$$\tau(t) = \min\{n \in \mathbb{N} : H_n \geq t\}. \tag{3}$$

Equivalently, $\tau(t)$ is the unique integer satisfying

$$H_{\tau(t)-1} < t \leq H_{\tau(t)}. \tag{4}$$

Definition 2.4 (Two-sided marker sets)**.** Define the post-crossing marker set

$$M^+(L) := \{\, n \in \mathbb{N} : \exists\, t \in T_L \text{ such that } H_{n-1} < t \leq H_n \,\}.$$

Define the pre-crossing marker set

$$M^-(L) := \{\, n \in \mathbb{N} : \exists\, t \in T_L \text{ such that } H_n \leq t < H_{n+1} \,\}.$$

Equivalently, for each $t \in T_L$, the associated pre-crossing index is the largest $n \in \mathbb{N}$ such that $H_n \le t$.

Remark 2.5. By definition, $n \in M^+(L)$ if and only if there exists $t \in T_L$ such that

$$H_{n-1} < t \le H_n,$$

and $n \in M^-(L)$ if and only if there exists $t \in T_L$ such that

$$H_n \le t < H_{n+1}.$$

Definition 2.6 (Interval sequence)**.** Let $M^+(L) = \{m_1 < m_2 < \cdots < m_r\}$, where $r := |M^+(L)|$. Define the interval sequence $\Delta(L)$ by

$$\Delta_1 = m_1,$$

and for $j \ge 2$,

$$\Delta_j = m_j - m_{j-1}.$$

3. Main Results

We first establish the existence and uniqueness of the crossing indices.

Proposition 3.1. *For each $t \in T$, the crossing index*

$$\tau(t) = \min\{n \in \mathbb{N} : H_n \ge t\}$$

exists and is unique.

Proof. Fix $t \in T$. Since $(H_n)_{n\ge1}$ is strictly increasing and $\lim_{n\to\infty} H_n = \infty$, the set

$$A_t := \{n \in \mathbb{N} : H_n \ge t\}$$

is nonempty. Because $\mathbb{N}$ is well-ordered, A_t has a least element; define $\tau(t)$ to be this least element. Uniqueness is immediate from the uniqueness of the least element. □

Proposition 3.2. *If $t_1, t_2 \in T$ and $t_1 < t_2$, then*

$$\tau(t_1) < \tau(t_2).$$

In particular, the sequence $\{\tau(i/2)\}_{i\ge2}$ is strictly increasing.

Proof. Let $t_1, t_2 \in T$ with $t_1 < t_2$, and set $n := \tau(t_1)$. Then $H_{n-1} < t_1 \le H_n$.

Write $t_r = i_r/2$ with integers $i_r \ge 2$. From $t_1 < t_2$ we have $i_2 - i_1 \ge 1$, hence $t_2 - t_1 \ge \frac{1}{2}$. If $n = 1$, then $t_1 = 1$ and $t_2 \ge \frac{3}{2}$, so $H_1 < t_2$ and thus $\tau(t_2) \ge 2 > \tau(t_1)$.

Assume $n \ge 2$. Then $H_n - H_{n-1} = 1/n \le \frac{1}{2}$, so the interval $(H_{n-1}, H_n]$ has length at most $\frac{1}{2}$. Since distinct elements of T differ by at least $\frac{1}{2}$, the interval $(H_{n-1}, H_n]$ contains at most one element of T, namely t_1. Because $t_2 > t_1 > H_{n-1}$, if $t_2 \leq H_n$ then $t_2 \in (H_{n-1}, H_n]$, a contradiction. Hence $H_n < t_2$, and therefore $\tau(t_2) > \tau(t_1)$. □

Lemma 3.3. *For each $L \in \mathbb{N}$,*

$$M^+(L) = \{\tau(t) : t \in T_L\}.$$

Proof. Let $t \in T_L$ and set $n := \tau(t)$. By definition of τ, $H_{n-1} < t \leq H_n$, hence $n \in M^+(L)$.

Conversely, if $n \in M^+(L)$, then there exists $t \in T_L$ such that $H_{n-1} < t \leq H_n$. By minimality in the definition of τ, this implies $\tau(t) = n$. Hence n lies in the set $\{\tau(t) : t \in T_L\}$. □

Lemma 3.4. *For each $t \in T$, define*

$$m^-(t) := \max\{n \in \mathbb{N} : H_n \leq t\}.$$

Then for each $L \geq 2$,

$$M^-(L) = \{m^-(t) : t \in T_L\}.$$

Moreover,

$$m^-(t) = \begin{cases} \tau(t), & \text{if } H_{\tau(t)} = t, \\ \tau(t) - 1, & \text{if } H_{\tau(t)-1} < t < H_{\tau(t)}. \end{cases}$$

Proof. Let $t \in T$ and write $m := m^-(t)$. By definition, $H_m \leq t$, and by maximality of m we have $t < H_{m+1}$. Hence

$$H_m \leq t < H_{m+1},$$

so in particular $m \in M^-(L)$ whenever $t \in T_L$.

Conversely, if $n \in M^-(L)$, then there exists $t \in T_L$ such that

$$H_n \leq t < H_{n+1}.$$

Since n satisfies $H_n \leq t$ and no index larger than n can do so, it follows that $n = m^-(t)$. Therefore

$$M^-(L) = \{m^-(t) : t \in T_L\}.$$

Now let $n := \tau(t)$, so that

$$H_{n-1} < t \leq H_n.$$

If $H_n = t$, then n is the largest index with $H_n \leq t$, hence $m^-(t) = n = \tau(t)$. Otherwise,

$$H_{n-1} < t < H_n,$$

so $n - 1$ is the largest index with $H_{n-1} \leq t$, and therefore $m^-(t) = n - 1 = \tau(t) - 1$. □

Proposition 3.5. *Let* $M^+(L) = \{m_1 < m_2 < \cdots < m_r\}$*, where* $r := |M^+(L)|$*, and set* $m_0 := 0$*. For* $1 \le j \le r$ *define*

$$I_j := \{n \in \mathbb{N} : m_{j-1} < n \le m_j\}.$$

Then $\{I_j\}_{j=1}^r$ *forms a disjoint partition of* $\{n \in \mathbb{N} : n \le m_r\}$.

Proof. By definition, $I_j = \{n \in \mathbb{N} : m_{j-1} < n \le m_j\}$, so every n with $1 \le n \le m_r$ lies in I_j for the unique index j such that $m_{j-1} < n \le m_j$. In particular,

$$\bigcup_{j=1}^{r} I_j = \{n \in \mathbb{N} : n \le m_r\}.$$

If $i \ne j$, then the inequalities defining I_i and I_j are incompatible, so $I_i \cap I_j = \varnothing$. Hence $\{I_j\}_{j=1}^r$ forms a disjoint partition of $\{n \in \mathbb{N} : n \le m_r\}$. □

We next describe the asymptotic growth of the crossing indices.

Theorem 3.6. *As* $i \to \infty$,

$$\tau(i/2) = \exp\big(i/2 - \gamma + o(1)\big),$$

where γ *denotes the Euler–Mascheroni constant.*

Proof. It is classical that

$$H_n = \log n + \gamma + o(1) \qquad (n \to \infty), \tag{5}$$

where γ is the Euler–Mascheroni constant (see, e.g., [2]). Let $i \ge 2$ and set $t := i/2$, and write $n := \tau(t)$. By definition of τ, we have $H_{n-1} < t \le H_n$. Using (5) for n and $n-1$, this gives

$$\log(n-1) + \gamma + o(1) < t \le \log n + \gamma + o(1) \qquad (i \to \infty),$$

where $n = \tau(t) \to \infty$ as $t \to \infty$. Subtracting γ and absorbing the two $o(1)$ terms into a single $o(1)$ yields

$$\log(n-1) < t - \gamma + o(1) \le \log n.$$

Moreover,

$$\log n - \log(n-1) = \log\Big(1 + \frac{1}{n-1}\Big) = o(1) \qquad (n \to \infty),$$

so $\log(n-1) = \log n + o(1)$. Hence

$$\log n = t - \gamma + o(1) \qquad (i \to \infty).$$

Exponentiating and recalling that $t = i/2$ gives

$$\tau(i/2) = n = \exp\big(i/2 - \gamma + o(1)\big),$$

as claimed. □

Corollary 3.7. *For $i \geq 3$ set $m(i) := \tau(i/2)$ and*

$$\Delta(i) := m(i) - m(i-1).$$

Then

$$\Delta(i) \sim (e^{1/2} - 1)\, m(i-1) \quad \text{as } i \to \infty.$$

In particular, $\Delta(i)$ grows exponentially.

Proof. By the theorem,

$$m(i) = \exp\bigl(i/2 - \gamma + o(1)\bigr), \qquad m(i-1) = \exp\bigl((i-1)/2 - \gamma + o(1)\bigr),$$

so

$$\frac{m(i)}{m(i-1)} = \exp\Bigl(\frac{1}{2} + o(1)\Bigr) = e^{1/2}(1 + o(1)).$$

Therefore,

$$\Delta(i) = m(i) - m(i-1) = m(i-1)\Bigl(\frac{m(i)}{m(i-1)} - 1\Bigr) = m(i-1)\bigl(e^{1/2} - 1 + o(1)\bigr),$$

which yields the claimed asymptotic. □

4. Numerical Illustration

Example 4.1 (Truncation level $L = 14$). Here $T_{14} = \{t \in T : t \leq 7\}$, i.e.

$$T_{14} = \Bigl\{1, \tfrac{3}{2}, 2, \tfrac{5}{2}, 3, \tfrac{7}{2}, 4, \tfrac{9}{2}, 5, \tfrac{11}{2}, 6, \tfrac{13}{2}, 7\Bigr\}.$$

The post-crossing marker set is

$$M^{+}(14) = \{1, 2, 4, 7, 11, 19, 31, 51, 83, 137, 227, 373, 616\}.$$

Note that the subsequence corresponding to integer thresholds $(1, 4, 11, 31, 83, \ldots)$ coincides with OEIS sequence A004080 [3]. The pre-crossing marker set (in the sense of Definition 2.4) is

$$M^{-}(14) = \{1, 2, 3, 6, 10, 18, 30, 50, 82, 136, 226, 372, 615\}.$$

Writing $M^{+}(14) = \{m_1 < \cdots < m_{13}\}$, the interval-length sequence is

$$\Delta(14) = (1, 1, 2, 3, 4, 8, 12, 20, 32, 54, 90, 146, 243).$$

The induced intervals $I_j = \{n \in \mathbb{N} : m_{j-1} < n \leq m_j\}$ (with $m_0 = 0$) are the ranges

$$[1, 1], [2, 2], [3, 4], [5, 7], [8, 11], [12, 19], [20, 31],$$
$$[32, 51], [52, 83], [84, 137], [138, 227], [228, 373], [374, 616].$$

5. Discussion

The results above describe a simple but structured relationship between harmonic partial sums and discrete threshold levels. Although the harmonic series itself grows only logarithmically, Theorem 3.5 shows that the indices at which fixed vertical increments are attained grow exponentially, satisfying

$$\tau(i/2) \sim e^{i/2-\gamma}.$$

Consequently the spacing between successive crossings expands geometrically with asymptotic ratio $e^{1/2}$.

The subsequence corresponding to integer thresholds,

$$\tau(1), \tau(2), \tau(3), \tau(4), \ldots = 1, 4, 11, 31, 83, 227, 616, \ldots,$$

coincides with the classical sequence recording the least index k for which $H_k \geq n$, which appears in the On-Line Encyclopedia of Integer Sequences as sequence A004080 [3]. The present work refines this classical construction by resolving the intermediate half-integer thresholds

$$1, \tfrac{3}{2}, 2, \tfrac{5}{2}, 3, \ldots$$

and recording the associated crossing indices defined by the weak inequality $H_k \geq t$. A related sequence based on the strict inequality $H_k > t$ appears in OEIS as sequence A226161 [4]. The sequence $\tau(i/2)$ studied here differs from A226161 at threshold values where the harmonic partial sums attain the threshold exactly.

The pre-crossing marker sequence

$$1, 2, 3, 6, 10, 18, 30, 50, 82, 136, 226, 372, 615, \ldots$$

is obtained by taking, for each threshold $t \in T$, the largest index

$$m^-(t) := \max\{n \in \mathbb{N} : H_n \leq t\}.$$

Equivalently,

$$m^-(t) = \begin{cases} \tau(t), & \text{if } H_{\tau(t)} = t, \\ \tau(t) - 1, & \text{if } H_{\tau(t)-1} < t < H_{\tau(t)}. \end{cases}$$

This sequence records the largest indices whose harmonic partial sums do not exceed each threshold level. Unlike the post-crossing sequence defined using the strict inequality $H_k > t$, which appears in OEIS as sequence A226161 [4], the weak-inequality formulation used here produces a distinct half-integer crossing sequence. The corresponding pre-crossing sequence does not appear to be listed in the OEIS database at the time of writing.

The two-sided marker formulation provides a convenient description of the threshold-crossing structure. The set $M^+(L)$ records the indices immediately after each threshold is reached, while $M^-(L)$ records the indices immediately preceding each crossing.

Lemma 3.4 shows that the pre-crossing markers satisfy

$$M^-(L) = \{m^-(t) : t \in T_L\}, \qquad m^-(t) = \max\{n \in \mathbb{N} : H_n \le t\},$$

so the two marker sets describe the right and left limits of the harmonic partial sums relative to the threshold schedule.

Finally, the ordered crossings induce a natural partition of initial segments of $\mathbb{N}$. If $m_j = \tau(t_j)$ denotes the crossing indices, then the induced intervals

$$I_j = \{n \in \mathbb{N} : m_{j-1} < n \le m_j\}$$

form a disjoint partition whose lengths

$$|I_j| = m_j - m_{j-1}$$

are precisely the successive differences of the threshold-hitting sequence. In this sense the partition structure reflects the spacing between consecutive harmonic threshold crossings. While the integer-threshold hitting times themselves are classical, this two-sided marker formulation and the induced partition viewpoint do not appear to be part of the standard presentation of these sequences.

Although closely related threshold-crossing sequences have appeared previously, the weak-inequality formulation and the resulting two-sided marker structure considered here do not appear to have been explicitly recorded in this form.

Possible extensions include the study of alternative threshold schedules, such as levels of the form i/k, and analogous crossing structures for other slowly diverging series.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.

[2] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 6th ed., Oxford University Press, 2008.

[3] N. J. A. Sloane et al., *The On-Line Encyclopedia of Integer Sequences*, Sequence A004080, `https://oeis.org/A004080`.

[4] N. J. A. Sloane et al., *The On-Line Encyclopedia of Integer Sequences*, Sequence A226161, `https://oeis.org/A226161`.

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