Election Modeling Project Proposal

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1 Abstract

An attempt at modeling election systems using ordinary differential equations. The election systems used are plurality, ranked choice, and Borda count. Voters and potential candidates were randomly generated. We derive several differential equations to represent how voters and candidates change their policy positions over time. We found that voters tend to align themselves with other voters, independent of the election system, and candidate behavior depends upon the election system.

2 Introduction

This project will attempt to model a system of how an electorate and candidates react to elections and the general political environment over time, as well as analyzing how the dynamics of their movement may change depending on the system of voting used for those elections. We will do this by creating a set of n random k-dimensional vectors on $[-1,1]^k$, each of which represents an individual voter's personal ideal positions on k different policy issues, with 1 representing complete support of a policy, and -1 representing complete opposition to a policy. Similarly, to represent a set of candidates, we will create another set of m random k-dimensional vectors on $[-1,1]^k$. With these in place, we then have a graph on $[-1,1]^k$ (for ease of visualization and computation, we will use $k=2$, but the results should be equivalent for an arbitrary $k \geq 2$) that shows how voters and candidates are distributed on different issues.

From these initial conditions, we can model a series of elections and changes in voter and candidate positions as a result of those elections. While there are many ways to determine an individual voter's preferred ranking of candidates and which candidate is most preferred by the electorate as a whole, we will use only one method of ranking a voter's candidate preferences and three distinct systems of aggregating those individual rankings into the results of those elections. Specifically, we assume that voters have single-peaked preferences with an ideal point at the vector randomly determined as above, that they equally prefer any candidate that is an equal distance from them as determined by the Euclidean norm on $\mathbb{R}^k$, and they vote sincerely according to how close their policy preferences are to candidate policy positions. So, if a given voter has preferences $v_i = [0, 0]$, then candidates with policy positions $c_1 = [\sqrt{2}/2, \sqrt{2}/2], c_2 = [-0.5, 0.5], c_3 = [-1, 0]$ would be ranked as $c_2 \succ c_1 \sim c_3$. We can then create a list of these preference rankings for each voter, from which we can model an election. The results of this election will then be combined with the relative position of each voter and candidate to other voters and candidates to describe an update vector that is added to their initial policy position to model how each voter or candidate adjusts their policy positions in response to the observed political environment around them, particularly the results of an election and the existence of political allies near them.

Mathematically, the voting systems we will describe below can be represented as a function from an $n + m \times k$-dimensional vector space to a k-dimensional vector space: that is a function mapping a huge set of vectors that represent voter and candidate policy positions to a single vector that represents the winning candidate's policy positions. As such, these voting systems are in themselves attempts to model a single ideal or majority position of the electorate as a whole. The imperfections that we discuss coming from each of them, then, should be expected considering the degree to which the result of the election is a simplification of a massive and complex set of preferences. This paper will therefore be an attempt to identify how these simplifications may affect the actual behavior of the electorate, from which normative judgments may be made about which imperfections are acceptable in a voting system and which are not.

3 Election Systems

The first election system we will use is a standard Plurality rule (also known as “First Past the Post”). Under this system, we just count how many times each candidate is the most preferred option for any voter, and whichever candidate is most preferred the most often wins the election. In real world terms, this represents a system where voters are presented with a list of candidates, they respond with a single candidate whose policy positions are most similar to theirs, and whichever candidate has the largest number of votes for them – which would be the candidate with the plurality of votes, hence the name – wins the election. This system is relatively simple, requiring minimal computational power to model or to perform in real life, and also has a number of properties that theoretically optimize the probability that the “best” candidate is elected when there are only two candidates against each other, which is largely why this is the system most often used in elections around the world. However, when the number of candidates increases beyond two, this system becomes deeply inadequate, as it often results in the winner of the election being someone who would lose against other candidates in a one-on-one scenario, and there is distrust of the results of an election when the winner doesn’t have the support of the majority of the electorate. The Plurality rule also has been shown to lead to two-party electorates, which increases polarization between the candidates and voters in the election, but this is typically the result of strategic patterns of voting, which we do not attempt to model here, so we don’t necessarily expect to see this pattern emerge in our model. Regardless, Plurality voting has properties that make it unappealing in a complicated political situation like we are modelling here, and as such we will compare it to alternative systems that have been proposed by political analysts throughout history.

A second and slightly more complicated system we will analyze has been proposed by several people throughout history, but most famously by French mathematician Jean-Charles de Borda in 1770, from which it gets the name “Borda Count”. This system, unlike the Plurality rule, uses the full ranking of candidates for each voter, assigning a weight to each candidate depending on where they fall in the ranking. Specifically, for m candidates, the Borda Count would have each voter assign a number from 1 to m for each candidate, which under our model will be precisely where the candidate falls in the preference ranking, with the least preferred candidate being assigned 1, the second least preferred candidate 2, and so on up to the most preferred candidate receiving m. This is aggregated simply by summing the ranking that a candidate receives from each voter, and whichever candidate has the largest total wins the election. While it is not used by very many political organizations around the world, the increased information on voter preferences that it provides makes it attractive, especially because it doesn't significantly increase the computational power required to perform the election.

Third, and most complicated, we will analyze the increasingly popular Ranked Choice or Instant Runoff voting rule. As the problems with the Plurality rule have become more culturally salient in the United States recently, this system has been proposed as an alternative that has more attractive properties, being used more and more often in primaries and even ACME presidential elections around the country. It certainly is true that Ranked Choice avoids several of the most prominent problems of Plurality voting, but it comes at the cost of high complexity and opaqueness in the process of voting and counting votes, as well as extremely high computational requirements (to represent every possible result of a ranking of m candidates, there are m! different combinations possible, which each precinct would need to submit as distinct for the results to be effectively counted, compared to the sums of results for m different combinations required for the other two systems). Regardless, we will consider it as an alternative voting rule for our model and observe how it changes the behavior of voters and candidates over time differently from the other systems, modelled as follows.

The Ranked Choice system again takes the entire ranking of candidates we determined for each voter above, starting with the same process as the Plurality rule, where the number of voters who most prefer a given candidate is counted and compared to other candidates. If there is a candidate who wins more than $\frac{n+1}{2}$ votes in this first round, Ranked Choice is in fact identical to Plurality, as that candidate is considered the winner. If there is no candidate who wins the majority of highest preference votes, though, an instant runoff occurs where the candidate who had the least number of most preferred votes is dropped out of the race, and voters who had that candidate as their most preferred then have their votes cast for their second most preferred candidate. This process continues until a candidate has more than $\frac{n+1}{2}$ votes cast for them, at which point that candidate is the winner of the election.

As a simple example of this system (the complexity of which should demonstrate how complicated this gets with the number of voters that would be in any wide-scale election), consider 5 voters who have preferences $v_1 = [1,1], v_2 = [1,-1], v_3 = [-1,-1], v_4 = [-1,1], v_5 = [0,0]$ and 3 candidates with positions $c_1 = [0.5, 0.5], c_2 = [-0.5, 1], c_3 = [1,1]$. Then we have a ranking of $v_1 : c_3, c_1, c_2, v_2 : c_1, c_3, c_2, v_3 : c_2, c_1, c_3, v_4 : c_2, c_1, c_3, v_5 : c_1, c_2, c_3$, so in the first round there are two votes for c_2, two votes for c_1, and one vote for c_3. As such, no candidate has won the majority of votes, and c_3 has won the fewest votes, so we recast voter 1's vote for their second choice, which is c_1, now giving c_1 three votes, a majority, and so they win. We can consider the results that would occur from other systems as well, where the Borda count would add up the rankings, with c_1 getting $2+3+2+2+3 = 12$ votes, c_2 getting $1+1+3+3+2 = 10$ votes, and c_3 getting $3+2+1+1+1 = 8$ votes, again showing that c_3 is least preferred and c_1 is most preferred. The Plurality rule would give minimal information, however: in this case suggesting that c_1 and c_2 are equally preferred, which is evidently not the case considering that several voters prefer c_2 the least while nobody prefers c_1 the least, even if they are equally likely to be most preferred. The rest of this paper will demonstrate how a higher complexity of voting conditions will allow each of these different systems to affect the electorate in different ways, which should reflect reality fairly well, assuming that sincere voting for candidates most ideologically similar to a voter with single-peaked preferences that fall off in a radial pattern around an ideal point are in fact typically the case.

4 Modeling System

Each voter's movement vector will be a combination of four vectors: 1.) the direction of the center of gravity of the voters around the given voter, 2.) the direction of the center of gravity of candidates around the given voter, 3.) the direction of the winning candidate or the optimal ideology, and 4.) a random direction. Each vector will be scaled by the amount of influence each direction should have. The first vector is based on the idea that each voter's ideology will likely be affected by voters who share similar views as they likely share similar news sources, online groups, and community groups. The second vector is based on the candidates that they agree with and may or may not have been able to vote for based on the voting system being used. This reflects the political factions and politicians who are likely to reach the voter. The third vector reflects the influence of the governing ideology on public discourse and the establishment of the status quo. Finally, the fourth vector accounts for individual changes in taste, environment, education, social groups, and so forth. This should incorporate all smaller non-political influences on voter opinion.

Each candidate's movement vector is a combination of 3 vectors: 1.) the direction of the center of gravity of their voter base, 2.) the direction away from the center of gravity of candidates around them, and 3.) the direction towards the winning ideology. Each vector is again scaled by the amount of influence each direction should have. The first vector reflects candidates trying to maintain popularity with their base and reflects the way candidates cater to their base during the primary elections. The second vector reflects candidates trying to distinguish themselves from other candidates. Finally, the third vector reflects candidates attempting to optimize their strategy by moving towards the winning ideology. This should also reflect how candidates tend to try to appeal to moderates outside their immediate base.

Let $\vec{V}$ be a vector of n voters in $\mathbb{R}^n$ where each voter has an ideological position in $\mathbb{R}^p$ where p is the number of policies. In this project we will consider only the case $p = 2$. Let $\vec{C}$ be a vector of m political candidates in $\mathbb{R}^m$ where each candidate has an ideological position in $\mathbb{R}^p$ where p is the number of policies.

So for each voter v_k the following equation represents their change in ideology over time, where a, b, γ are constants.

$$v_k' = a \cdot \frac{1}{n}\sum_{i=1}^{n} \nu_i + b \cdot f(\vec{V}, \vec{C}) + \gamma \cdot \frac{1}{m}\sum_{i=1}^{m} \xi_i + \beta$$

$\nu = \{v_i \mid ||v_k - v_i|| \leq r \ \forall i\}$ other voters within an open ball around v_k of radius r.

$f(\vec{V}, \vec{C}) =$ election results and/or the optimal ideology given the vote system.

$\xi = \{c_i \mid ||v_k - c_i|| \leq r \ \forall i\}$ candidates within an open ball around v_k of radius r.

$\beta \sim \mathcal{N}(0, r)$ a draw from the p dimensional normal distribution.

And for each political candidate c_k the following equation represents their change in ideology over time, where δ, α, ρ are constants.

$$c_k' = \delta \cdot f(\vec{V}, \vec{C}) - \alpha \cdot \frac{1}{m} \sum_{i=1}^{m} \xi_i + \rho \cdot \frac{1}{n} \sum_{i=1}^{n} \nu_i$$

$f(\vec{V}, \vec{C}) =$ election results and/or the optimal ideology given the vote system.

$\xi = \{c_i \mid ||c_k - c_i|| \leq r \ \forall i\}$ other candidates within an open ball around c_k of radius r.

$\nu = \{v_i \mid ||c_k - v_i|| \leq r \ \forall i\}$ voters within an open ball around c_k of radius r.

Since this ODE is an initial value problem, we compute the numerical solutions using the forward Euler method. Let $t_0 = 0$ and $t_f = N$ where N corresponds to the number of election cycles. Let step size $h = 1$ since a half election does not apply in this context. From the entire interval $[t_0, N]$, this defines the sub-intervals $[t_{i-1}, t_i]$ as the following:

$$t_i = t_0 + i(h), \quad i = 0, 1, ..., N.$$

Given initial values for each $v_k(t_0)$ and $c_j(t_0)$, the first step of the Forward Euler approximation is expressed as

$$v_k(t_1) = v_k(t_0) + h\left(a \cdot \frac{1}{n} \sum_{i=1}^{n} \nu_i(t_0) + b \cdot f(\vec{V}(t_0), \vec{C}(t_0) + \gamma \cdot \frac{1}{m} \sum_{i=1}^{m} \xi_i(t_0) + \beta\right),$$

$$c_j(t_1) = c_j(t_0) + h\left(\delta \cdot f(\vec{V}(t_0), \vec{C}(t_0)) - \alpha \cdot \frac{1}{m}\sum_{i=1}^{m} \xi_i(t_0) + \rho \cdot \frac{1}{n}\sum_{i=1}^{n} \nu_i(t_0)\right).$$

Knowing that the forward Euler method is iterative, this process can be generalized to

$$v_k(t_{i+1}) = v_k(t_i)+h\left(a \cdot \frac{1}{n}\sum_{l=1}^{n} \nu_l(t_i) + b \cdot f(\vec{V}(t_i), \vec{C}(t_i) + \gamma \cdot \frac{1}{m}\sum_{l=1}^{m} \xi_l(t_i) + \beta\right), \quad i = 0, 1, ..., N-1,$$

$$c_j(t_{i+1}) = c_j(t_i) + h\left(\delta \cdot f(\vec{V}(t_i), \vec{C}(t_i)) - \alpha \cdot \frac{1}{m}\sum_{l=1}^{m} \xi_l(t_i) + \rho \cdot \frac{1}{n}\sum_{l=1}^{n} \nu_l(t_i)\right), \quad i = 0, 1, ..., N-1.$$

In order to reduce the temporal complexity, this algorithm can be modified to solve for $\vec{V}(t_{i+1})$ and $\vec{C}(t_{i+1})$.

To decide the hyper-parameters for our model, we isolated each variable affecting the voter movement and candidate movement and observed how their plot representation moved. We found that by setting a smaller influence ball around each voter, voters began clumping together in communities scattered across the policy spectrum. We set the hyper-parameter for incumbency advantage low so that the voters would not collapse onto the winner in a manner reminiscent of a regime such as ancient Egypt or modern North Korea. Candidates were set to move faster than voters and generally avoid other candidates while seeking out new voters. After running preliminary tests, we found that voters and candidates moved according to our intuition.

5 Analysis and Results

We now perform this solution method on a series of initial conditions for voters and candidates. Taking a two-dimensional policy space, we start by observing a series of randomized initial conditions for voters, with positions drawn from the standard multivariate normal distribution, and a set of 9 candidate positions with 8 in a circular pattern of radius 1.5 around a central candidate at $[0, 0]$. Second, we take sets of 1000 randomly positioned voters and 5 randomly positioned candidates, again drawing from the standard multivariate normal distribution in two-dimensions. Both of these, then, allow for voter or candidate positions to be anywhere in $\mathbb{R}^2$, but generally they will be centered within 2 unit positions of the origin, which we restrict our visuals to. Finally, we also draw from a real-world data set on voters' positions on "economic" versus "cultural" issues, setting 5 candidates at randomly selected points matching some voter's position. For each of these initial conditions, we then iterate through a series of 400 elections for a plurality, Borda count, and ranked choice voting system, updating voter and candidate positions after each to produce a series of graphs, which we have animated and posted to YouTube (https://www.youtube.com/channel/UCgJiO_pQpvsxuOL_CdhbGvA). A sample of the results of these for different election systems constitute Figures 1 through 6 at the end of this paper.

Having performed this visualization on the available data, we observe a number of distinct phenomena that result from this model, much of it matching our intuition about how voters and candidates behave in the real world. First, primarily due to the large weight that we give voters toward matching their opinions with other like-minded voters, for all initial conditions we observe immediate clumping of voters into something resembling a party system, or perhaps more accurately an ideological home space. Candidate always gravitate towards one of these clumps, suggesting the emergence of a voting base that the candidate depends on to have a chance of winning elections, and if enough voters are gathered at one of them, more than one candidate will join the clump, overcoming the distancing effect we model with candidates being negatively affected by the positions of nearby candidates. Often, voter clumps will emerge without any candidates fully aligning with them, and minute changes in candidate positions in the policy space around these "moderate" clumps seem to define which candidate wins an election. For each voting system, this tends to lead to a temporary equilibrium pattern where a set of candidates and corresponding voter clumps surround a central or set of central voter clumps, with small changes in positions except for outlying voters and candidates moving towards the major positions, and elections determined by whichever more central candidate has moved just enough to win the central votes. In the long term, with voters and candidates being drawn towards the winner of an election, these clumps converge to a single point, with all of the candidates and voters agreeing on policy, but this takes hundreds of iterations to occur, even for our simplest initial conditions, and doesn't occur within 400 iterations for non-plurality systems with random initial conditions. In the relatively short term, though, this seems to match the reality of candidates and voters building coalitions and generally consolidating their policy positions to maximize the chances of their preferred positions winning the election.

Second, the pattern of which candidates win elections has notable differences between voting systems and initial conditions, which in turn dramatically affects the behavior of candidates over time. With circular initial conditions, the central candidate wins the first dozen or so elections while moving in the direction where the slight majority of voters lie under each voting system, but at some point another candidate has drawn sufficient votes to win, at which point disparities arise. The plurality system tends towards two candidates winning every election, with the other voters and candidates drifting towards whichever of those candidates is closer, and occasionally a third candidate will attract just enough voters to outperform and replace one of the two winners, still maintaining a two-candidate domination of the election space. Borda and ranked choice provide space for short term equilibria defined by most voters and candidates remaining in a central voting space in a roughly circular pattern with outliers moving towards that space and no central candidate, but a central voter clump, and winners of elections move between three or even four candidates. This circular pattern is maintained for the plurality system as well, but again only with two winners. Figure 1 and 3 demonstrate this effect, with Figure 2 appearing similar, but with two winners until a consolidation of two voting bases around the $[0.5, 0]$ clump allows a new winner to dominate the election cycle for some time. With random initial conditions, similar patterns occur, with the same circular pattern emerging despite not initially existing, although without centralized candidates, there is more extremity in the size of these patterns and their positions in the policy space.

The real-world data's behavior is distinct, because voters are much more linear than the randomized initial positions, and displays how the different voting systems could have dramatically different results. Using a sample of 1000 voters, the data suggests that the majority of voters have left-wing positions on economic issues, and moderate positions on cultural issues. This causes each voting system to favor candidates in the moderate lower quadrant and a general drift to the lower-left corner by all voters and candidates, but over time the winners are distributed very differently. Plurality causes a splitting of votes in the lower-left, allowing for the minority voters in the upper-right to become a voting plurality and the remaining candidate in that area to win elections, drawing the long-term results further to the upper-right. Borda count is much more stable by contrast, with the initially central candidate drifting further to the lower-left, but still winning until a right-wing candidate moves close enough to the moderate left-leaning voters to be sufficient competition, drawing everyone to a moderate left long-term equilibrium. This long-term effect matches what happens under ranked choice, but ranked choice also allows candidates further to the lower-left to win without any vote-splitting occurring. In terms of policy, however, this suggests that in the real-world Borda and ranked choice would not result in dramatically different results, whereas plurality would have some further right policies occur. The legitimacy of applying these results to the real-world is somewhat debatable, however, since voters, who according to this data do in general hold economically and culturally left-leaning views, do repeatedly vote for candidates further to the right, but this may be due to the existence of only two legitimate candidates, and a perception that the closer candidate is in fact much further to the left than in reality they are. Regardless, further research should occur to analyze how voter's misconceptions of the likely results of the election and candidate policy positions would effect this model.

Finally, we repeatedly observed a behavior of candidates moving as a block, often incongruously with their voting base, caused by the repeated victory of a single candidate in elections. Specifically, when one candidate wins several times in a row, other candidates move closer to it, but the candidate distancing effect causes this winner to move in that same direction as the other candidates, becoming more extreme over time and distancing themselves from their voting base, allowing other candidates to pick up those former core voters. This suggests that having multiple candidates with the potential to win elections causes a larger amount of stability in our model than a single winner would have. As an example, observe in Figure 6 how all 5 candidates are initially in the lower-right corner of the graph, but over time they drift to the upper-left as a block. This is caused by the candidate furthest to the upper-left winning the vast majority of the vote for over 50 iterations, until all the candidates have moved far enough in that direction without voters immediately following them that by iteration 60 a new candidate starts winning alternately with the former winner until voters clump to the central-left candidate, where a long-term equilibrium starts to emerge.

This final effect matches our intuition that having one candidate consistently winning over time will cause them to become more extreme in their policy positions, providing more political power to their opponents, often their most distant opponents, who can draw formerly moderate voters to vote for them. Notably, extreme opposing candidates are more likely to become the competition to successive winners under the plurality system, whereas Borda and ranked choice systems prefer candidates that were already winning second or third place rankings from the majority of voters. As such, moderate candidates have a much better chance of winning under these systems, and the distancing effect is not as dramatic for candidates close to winners. This suggests that the apparent benefits of the ranked choice system can, under this model, be accomplished effectively with the Borda system, without requiring the high computational power to perform the election for large numbers of candidates.

Despite the complexities of voting behavior that consistently disrupt computational and academic analysis of these systems, we have successfully modeled realistic behavior of voters and candidates using differential equations, drawing on the positions of nearby voters and candidates, the winner of a given election, and accounting for the essentially uncountably large other potential factors with a random factor. These match the behavior we initially believed voters and candidates to perform, although when applied to real data they didn't strictly match with observation, largely, we believe, due to problems of misinformation and miscommunication between voters and candidates. Further research should account for this problem, as well as allow candidates to drop in and out of the voting space to account for more strategic behavior, but these results suggest a promising start for using differential equations to predict election results in the short-term.

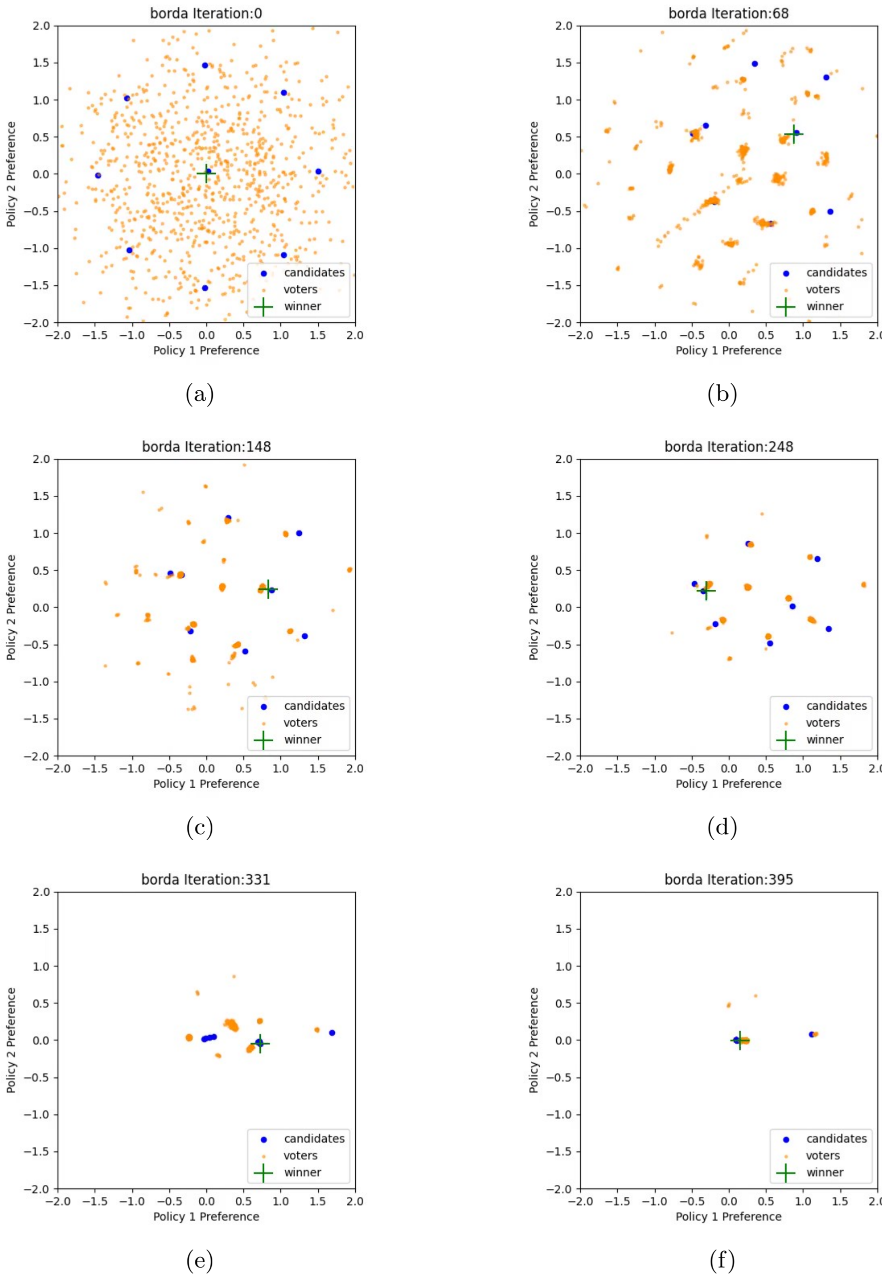
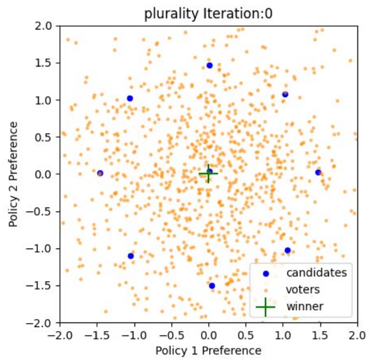


Figure 1: Borda with fixed initial candidates and random voters.



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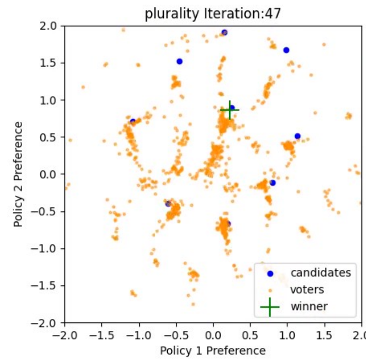

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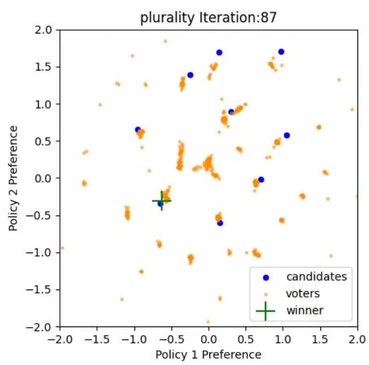

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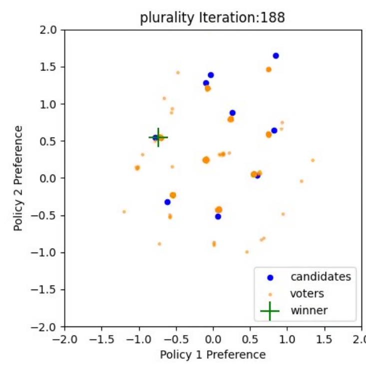

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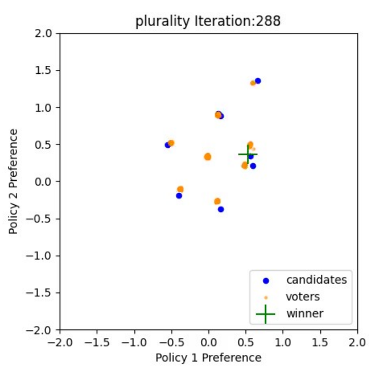

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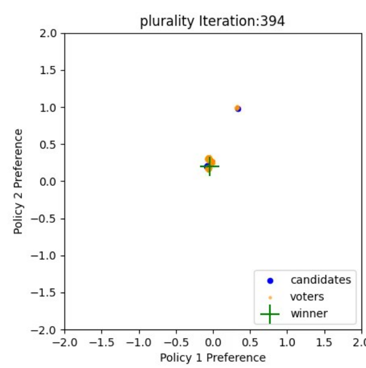

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Figure 2: Plurality with fixed initial candidates and random voters.

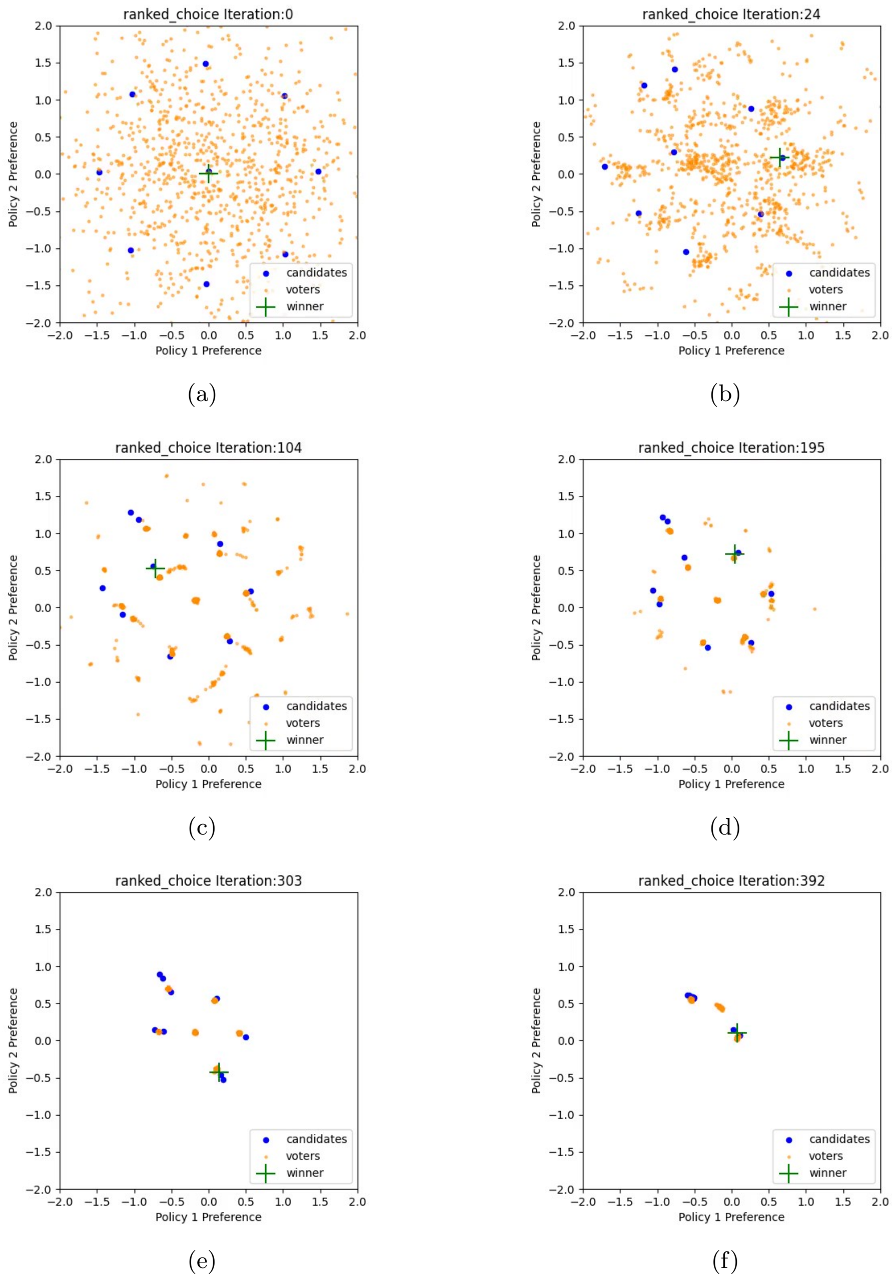


Figure 3: Ranked Choice with fixed initial candidates and random voters.

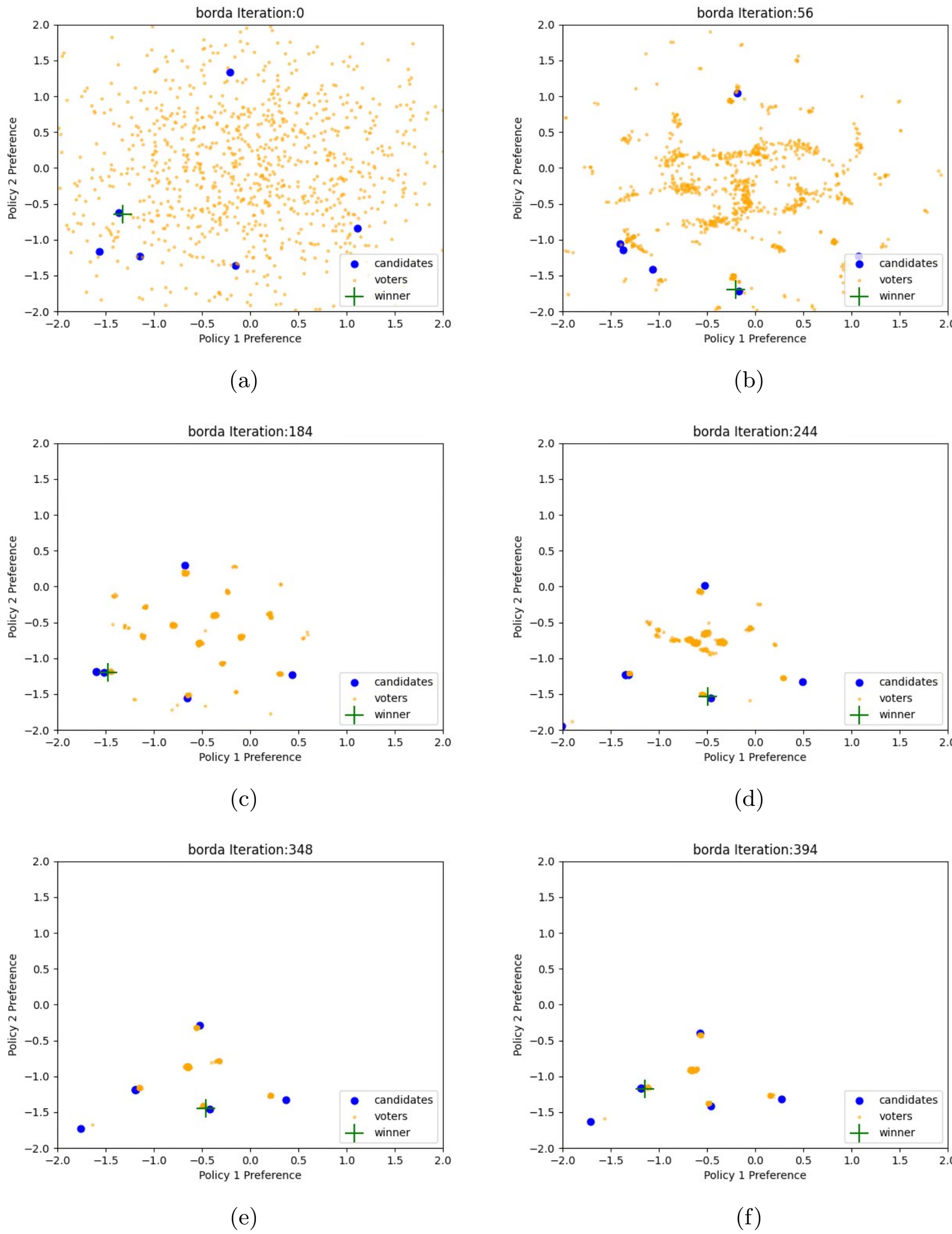
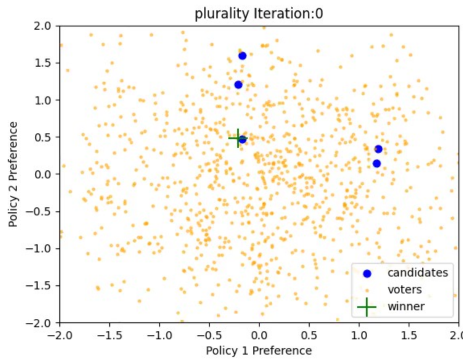


Figure 4: Borda with random candidates and random voters.



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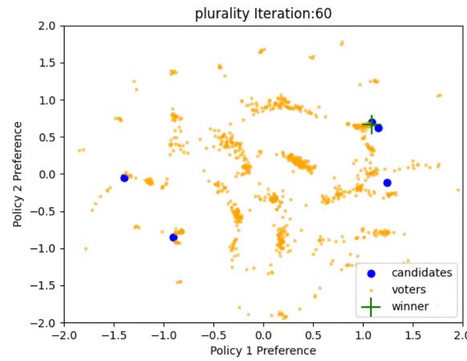

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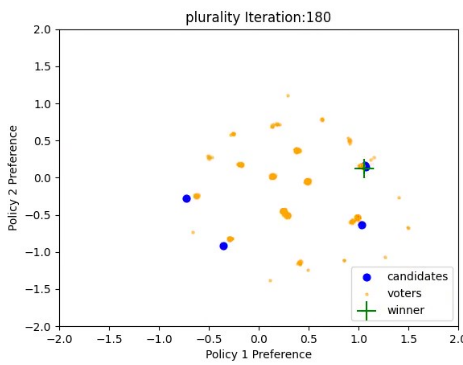

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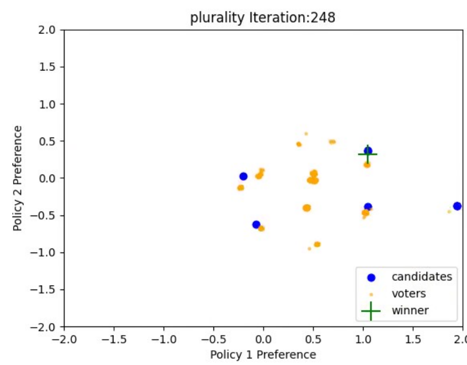

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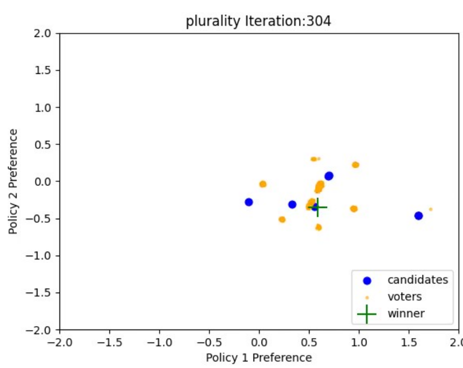

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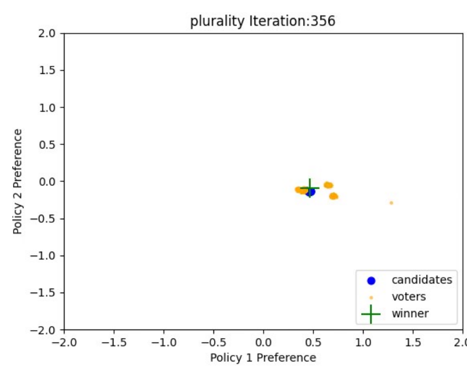

(f)

Figure 5: Plurality with random candidates and random voters.

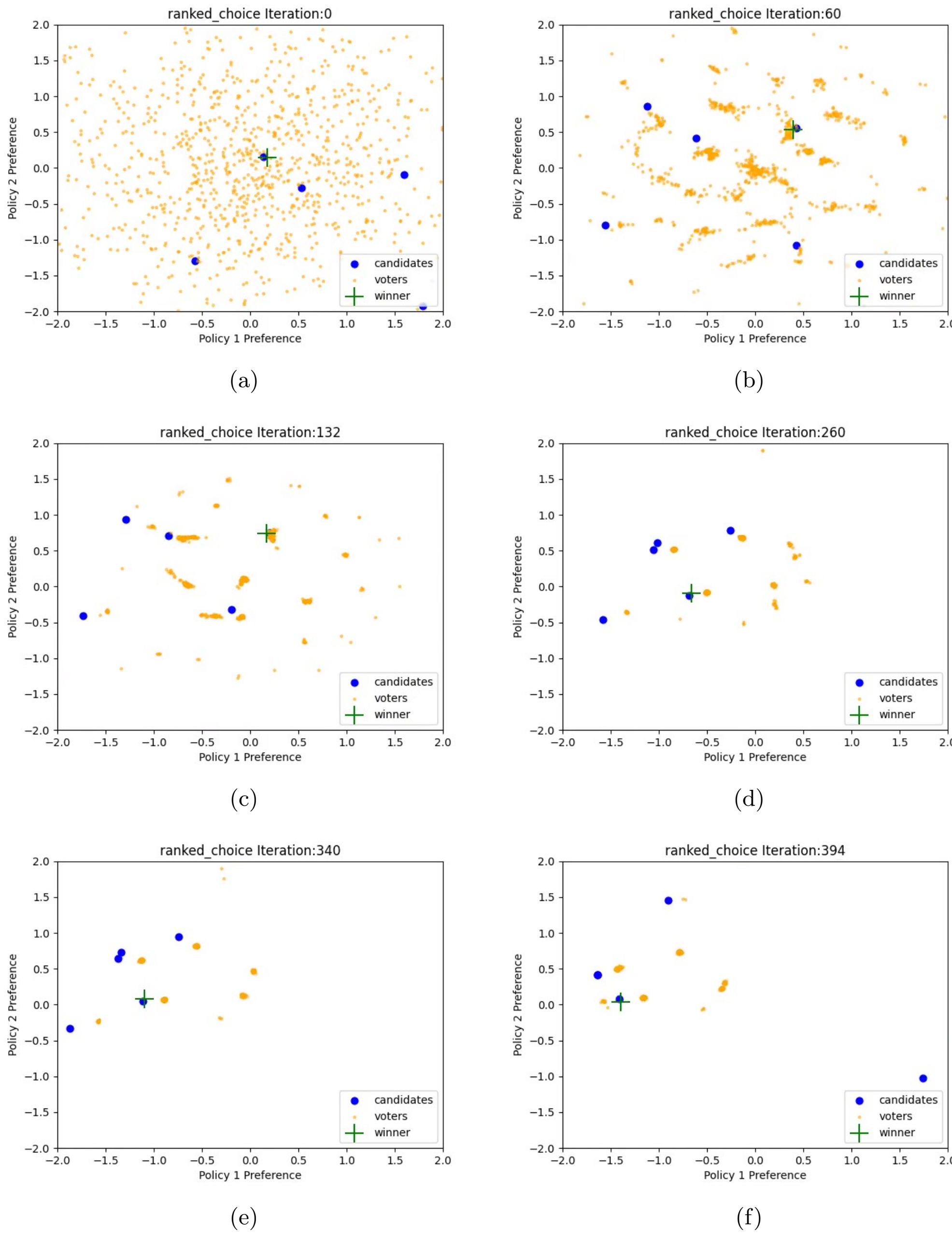

Figure 6: Ranked Choice with random candidates and random voters.