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Advanced Financial Risk Management:

Tools and Techniques for Integrated Credit Risk, Market Risk, Liquidity Risk and Interest Rate Risk Management

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Part 2: Risk Management Techniques for Interest Rate Analytics

Chapter 8:
HJM interest rate modeling with two risk factors

In Chapters 6 and 7, we provided worked examples of how to use the yield curve simulation framework of Heath, Jarrow and Morton using two different assumptions about the volatility of forward rates. The first assumption was that volatility was dependent on the maturity of the forward rate and nothing else. The second assumption was that volatility of forward rates was dependent on both the level of rates and the maturity of forward rates being modeled. Both of these models were one factor models, implying that random rate shifts are either all positive, all negative, or zero. This kind of yield curve movement is not consistent with the yield curve twists that are extremely common in the U.S. Treasury market and most other fixed income markets. In this chapter, we generalize the model to include two risk factors in order to increase the realism of the simulated yield curve.

Probability of Yield Curve Twists in the U.S. Treasury Market

In this chapter, we use the same data as inputs to the yield curve simulation process that we used in Chapters 6 and 7. A critical change in this chapter is to enhance our ability to model twists in the yield curve. The table below is taken from the blog entry

> van Deventer, Donald R. "Pitfalls in Asset and Liability Management: One Factor Term Structure Models," Kamakura blog, www.kamakuraco.com, November 7, 2011. Reprinted in Bank Asset and Liability Management Newsletter, January, 2012.

It shows that, for 12,386 days of movements in U.S. Treasury forward rates, yield curve twists occurred on 94.3% of the observations.

Summary of U.S. Treasury Yield Curve Shifts
Percentage Distribution of Daily Rate Shifts
January 2, 1962 to August 22, 2011

Type of Shift	U.S. Treasury Input Data	Monthly Zero Coupon Bond Yields	Monthly Forward Rates
Negative shift	18.4%	12.0%	2.4%
Positive shift	18.6%	12.1%	2.6%
Zero shift	0.7%	0.7%	0.7%
Subtotal	37.7%	24.8%	5.7%
Twist	62.3%	75.2%	94.3%
Grand Total	100.0%	100.0%	100.0%

In order to incorporate yield curve twists in interest rate simulations under the Heath Jarrow and Morton framework, we introduce a second risk factor driving interest rates in this chapter. The single factor yield curve models of Ho and Lee, Vasicek, Hull and White, Black Derman and Toy, and Black and Karasinski are unable to model yield curve twists and therefore they do not provide a sufficient basis for interest rate risk management.

Objectives of the Example and Key Input Data

Following Jarrow (2002), we again make the same modeling assumptions for our worked example as in Chapters 6 and 7:

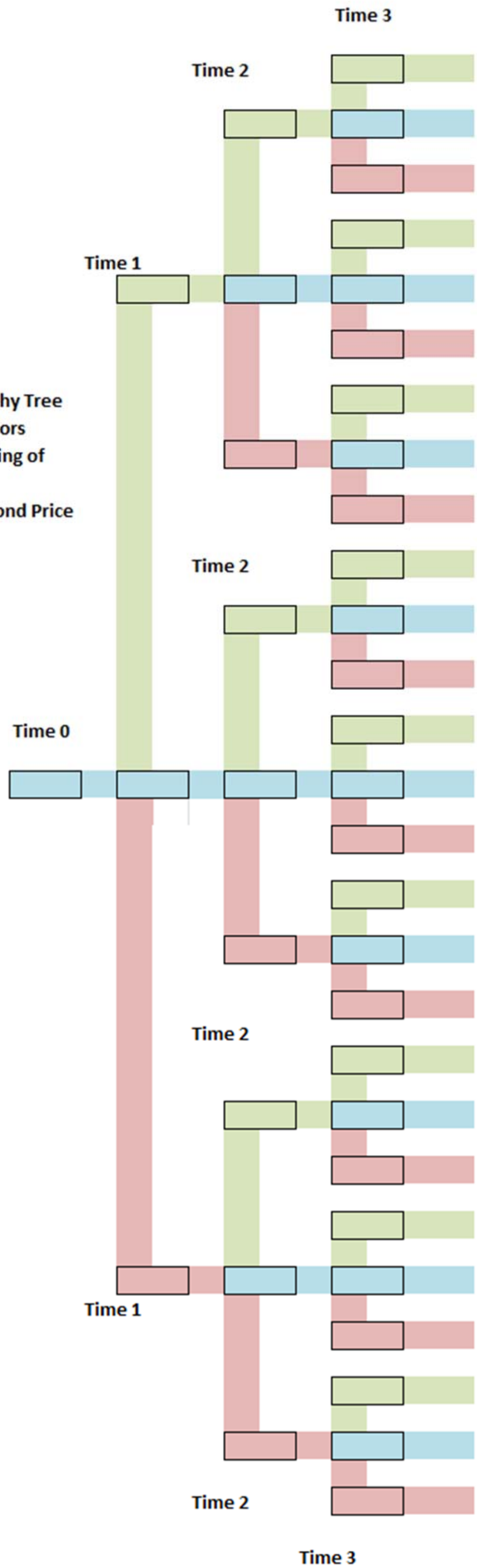
- Zero coupon bond prices for the U.S. Treasury curve on March 31, 2011 are the basic inputs.
- Interest rate volatility assumptions are based on the Dickler, Jarrow and van Deventer papers on daily U.S. Treasury yields and forward rates from 1962 to 2011. In this chapter, we retain the volatility assumptions used in Chapter 7 but expand the number of random risk factors driving interest rates to two factors.
- The modeling period is 4 equal length periods of one year each.

- The HJM implementation is that of a "bushy tree" which we describe below

In the Chapters 6 and 7, the Heath Jarrow and Morton bushy tree that we constructed consisted solely of "up shifts" and "down shifts" because we were modeling as if only one random factor was driving interest rates. In this chapter, with two random factors, we move to a bushy tree that has "up shifts," "mid shifts," and "down shifts" at each node in the tree. Please consider these terms as labels only, since forward rates and zero coupon bond prices, of course, move in opposite directions:

Figure 1

Example of Bushy Tree
with 2 Risk Factors
for HJM Modeling of
No Arbitrage
Zero Coupon Bond Price
Movements



Scenario	Scenario Description		
1	Up	Up	Up
2	Up	Up	Mid
3	Up	Up	Down
4	Up	Mid	Up
5	Up	Mid	Mid
6	Up	Mid	Down
7	Up	Down	Up
8	Up	Down	Mid
9	Up	Down	Down
10	Mid	Up	Up
11	Mid	Up	Mid
12	Mid	Up	Down
13	Mid	Mid	Up
14	Mid	Mid	Mid
15	Mid	Mid	Down
16	Mid	Down	Up
17	Mid	Down	Mid
18	Mid	Down	Down
19	Down	Up	Up
20	Down	Up	Mid
21	Down	Up	Down
22	Down	Mid	Up
23	Down	Mid	Mid
24	Down	Mid	Down
25	Down	Down	Up
26	Down	Down	Mid
27	Down	Down	Down

At each of the points in time on the lattice (time 0, 1, 2, 3 and 4) there are sets of zero coupon bond prices and forward rates. At time 0, there is one set of data. At time one, there are three sets of data, the "up set," "the mid set" and the "down set." At time two, there are nine sets of data (up up, up mid, up down, mid up, mid mid, mid down, down up, down mid, and down down), and at time three there are $27=3^3$ sets of data.

The table below shows the actual volatilities, resulting from all risk factors, for the 1 year changes in continuously compounded forward rates from 1963 to 2011. This is the same data we used in Chapter 7. We use this table later to divide total volatility between two uncorrelated risk factors.

Figure 2

	1 Year Rate in Data Group			Standard Deviation of Annual Changes in 1 Year Rate			
Data Group	Minimum Spot Rate	Maximum Spot Rate	Number of Observations	1 Year	2 Years	3 Years	4 Years
1	0.0000%	0.0900%	**Assumed**	**0.01000000%**	**0.02000000%**	**0.10000000%**	**0.20000000%**
2	0.0900%	0.2500%	**Assumed**	**0.04000000%**	**0.10000000%**	**0.20000000%**	**0.30000000%**
3	0.2500%	0.5000%	346	0.06363856%	0.38419072%	0.67017474%	0.86385093%
4	0.5000%	0.7500%	86	0.08298899%	0.46375394%	0.61813347%	0.80963400%
5	0.7500%	1.0000%	20	0.94371413%	1.08296308%	0.96789760%	0.74672896%
6	1.0000%	2.0000%	580	1.01658509%	0.84549720%	0.63127387%	0.66276260%
7	2.0000%	3.0000%	614	1.30986113%	1.28856305%	1.07279248%	0.80122998%
8	3.0000%	4.0000%	1492	1.22296401%	1.11359303%	1.02810902%	0.90449832%
9	4.0000%	5.0000%	1672	1.53223304%	1.21409679%	1.00407389%	0.83492621%
10	5.0000%	6.0000%	2411	1.22653783%	1.09067681%	0.97034198%	0.91785966%
11	6.0000%	7.0000%	1482	1.67496041%	1.40755286%	1.20035444%	1.04528208%
12	7.0000%	8.0000%	1218	1.70521091%	1.34675823%	1.30359062%	1.29177866%
13	8.0000%	9.0000%	765	2.16438588%	1.77497758%	1.72454046%	1.69404480%
14	9.0000%	10.0000%	500	2.28886301%	2.14441843%	2.02862693%	2.02690844%
15	10.0000%	16.6000%	959	2.86758369%	2.64068702%	2.55664943%	2.32042833%
Grand Total			12145	1.73030695%	1.49119219%	1.37014684%	1.26364341%

We will again use the zero coupon bond prices prevailing on March 31, 2011 as our other inputs:

Number of Periods:	**4**
Length of Periods (Years)	**1**
Number of Risk Factors:	**2**
Volatility Term Structure:	**Empirical, 2 Factor**

Parameter Inputs

Period Start	Period End	Zero Coupon Bond Prices
0	1	0.99700579
1	2	0.98411015
2	3	0.96189224
3	4	0.93085510

Introducing a Second Risk Factor Driving Interest Rates

In the first two worked examples of the Heath Jarrow and Morton approach in Chapters 6 and 7, the nature of the single factor shocking 1 year spot and forward rates was not specified. In this chapter, we take a cue from the popular academic models of the term structure of interest rates and postulate that one of the two factors driving changes in forward rates is the change in the short run rate of interest. In the context of our annual model, the "short rate" is the one year spot U.S. Treasury yield. For each of the 1 year U.S. Treasury forward rates $f_k(t)$, we run the regression

$$[Change\ in\ f_k(i)] = a + b[Change\ in\ 1\ year\ U.S.Treasury\ spot\ rate\ (i)] + e(i)$$

where the change in continuously compounded yields is measured over annual intervals from 1963 to 2011. The coefficients of the regressions for the 1 year forward rates maturing in years 2, 3, …, 10 are as follows:

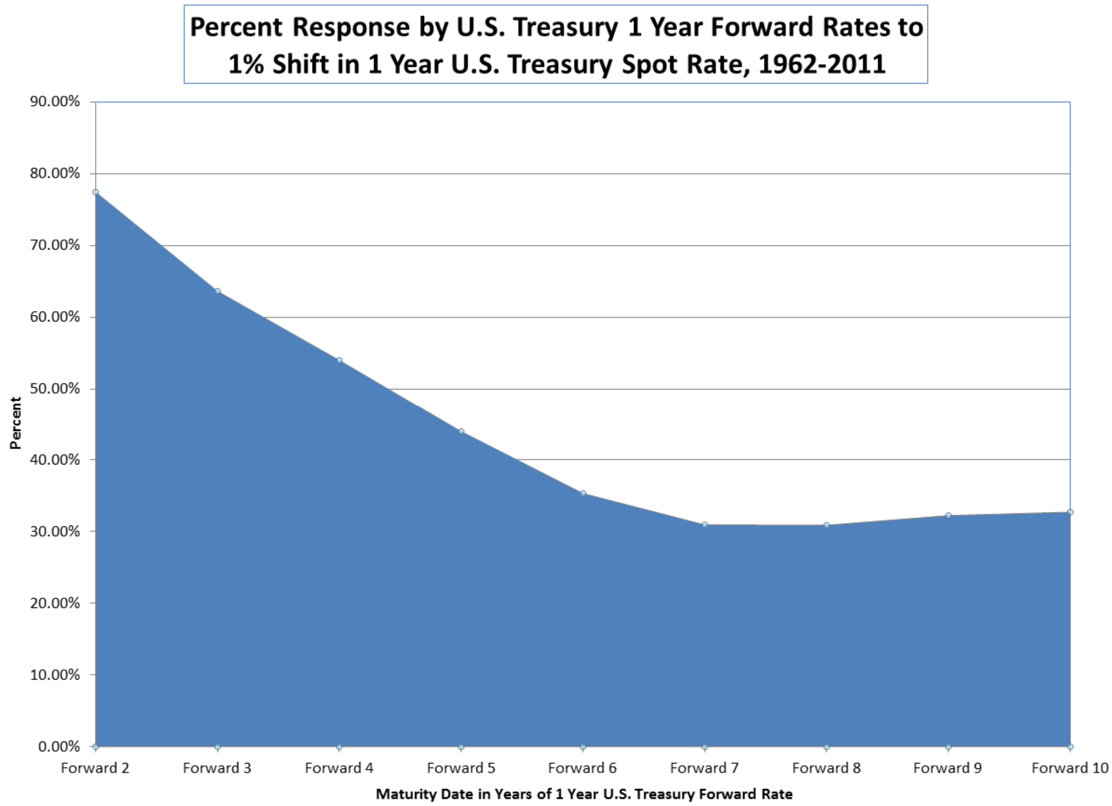
Figure 3

Regression Coefficients b for
Change in $f_k(i)$=a + b[Change in 1 Year U.S. Treasury(i)]+e(i)
1963-2011
Source: Kamakura Corporation and Federal Reserve

Dependent Variable	Coefficient b on 1 Year Change in 1 Year U.S. Treasury Rate	Standard Error of Regression
Forward 2	0.774288643	0.654783%
Forward 3	0.636076417	0.816101%
Forward 4	0.539780355	0.851190%
Forward 5	0.439882997	0.933073%
Forward 6	0.353383944	0.956797%
Forward 7	0.309579768	0.958593%
Forward 8	0.309049192	0.914843%
Forward 9	0.322513079	0.936120%
Forward 10	0.327367707	1.003651%

Graphically, it is easy to see that the historical response of forward rates to the spot rate of interest is neither constant (as assumed by Ho and Lee [1986]) nor declining, as assumed by Vasicek [1977] and Hull and White [1990]. Indeed, the response of the 1 year forwards maturing in years 8, 9, and 10 year to changes in the 1 year spot rate is larger than the response of the 1 year forward maturing in year 7:

Figure 4



We use these regression coefficients to separate the total volatility (shown in the table above) of each forward rate between two risk factors. The first risk factor is changes in the one year spot U.S. Treasury. The second factor represents all other sources of shocks to forward rates.

Because of the nature of the linear regression of changes in 1 year forward rates on the first risk factor, changes in the 1 year spot U.S. Treasury rate, we know that risk factor 1 and risk factor 2 are uncorrelated. Therefore total volatility for the forward rate maturing in T-t=k years can be divided as follows between the two risk factors:

$$\sigma_{k,total}^2 = \sigma_{k,1}^2 + \sigma_{k,2}^2$$

We also know that the risk contribution of the first risk factor, the change in the spot 1 year U.S. treasury rate, is proportional to the regression coefficient α_k of changes in forward rate maturing in year k on changes in the 1 year spot rate. We denote the total volatility of the 1 year U.S. Treasury spot rate by the subscript "1,total":

$$\sigma_{k,1}^2 = \alpha_k^2 \sigma_{1,total}^2$$

This allows us to solve for the volatility of risk factor 2 using this equation:

$$\sigma_{k,2} = \left[\sigma_{k,total}^2 - \alpha_k^2 \sigma_{1,total}^2\right]^{1/2}$$

Because the total volatility for each forward rate varies by the level of the 1 year U.S. Treasury spot rate, so will the values of $\sigma_{k,1}$ and $\sigma_{k,2}$. In one case (data group 6 for the forward rate maturing at time T=3), we set the sigma for the second risk factor to zero and ascribed all of the total volatility to risk factor one because the calculated contribution of risk factor 1 was greater than the total volatility. In the worked example, we will select the volatilities from look up tables for the appropriate risk factor volatility. Using the equations above, the look up table for risk factor 1 (changes in the 1 year spot rate) volatility is given here:

Figure 5

1 Year Rate in Data Group				Standard Deviation of Risk Factor 1, Sigma 1			
Data Group	**Minimum Spot Rate**	**Maximum Spot Rate**	**Number of Observations**	**1 Year**	**2 Years**	**3 Years**	**4 Years**
1	0.00%	0.09%	Assumed	0.01000000%	0.00774289%	0.00492507%	0.00265845%
2	0.09%	0.25%	Assumed	0.04000000%	0.03097155%	0.01970027%	0.01063382%
3	0.25%	0.50%	346	0.06363856%	0.04927462%	0.03134242%	0.01691802%
4	0.50%	0.75%	86	0.08298899%	0.06425743%	0.04087264%	0.02206225%
5	0.75%	1.00%	20	0.94371413%	0.73070713%	0.46478558%	0.25088212%
6	1.00%	2.00%	580	1.01658509%	0.78713029%	0.63127387%	0.34074924%
7	2.00%	3.00%	614	1.30986113%	1.01421060%	0.64511544%	0.34822064%
8	3.00%	4.00%	1492	1.22296401%	0.94692714%	0.60231802%	0.32511944%
9	4.00%	5.00%	1672	1.53223304%	1.18639064%	0.75463511%	0.40733721%
10	5.00%	6.00%	2411	1.22653783%	0.94969431%	0.60407816%	0.32606952%
11	6.00%	7.00%	1482	1.67496041%	1.29690282%	0.82492930%	0.44528063%
12	7.00%	8.00%	1218	1.70521091%	1.32032544%	0.83982788%	0.45332259%
13	8.00%	9.00%	765	2.16438588%	1.67585941%	1.06597465%	0.57539217%
14	9.00%	10.00%	500	2.28886301%	1.77224063%	1.12728047%	0.60848385%
15	10.00%	16.60%	959	2.86758369%	2.22033748%	1.41230431%	0.76233412%
Grand Total			12145	1.73030695%	1.33975702%	0.85218784%	0.45999426%

The look up table for the general "all other" risk factor 2 is as follows:

Figure 6

	1 Year Rate in Data Group			Standard Deviation of Risk Factor 2, Sigma 2			
Data Group	Minimum Spot Rate	Maximum Spot Rate	Number of Observations	1 Year	2 Years	3 Years	4 Years
1	0.00%	0.09%	Assumed	0.00000000%	0.01844038%	0.09987864%	0.19998233%
2	0.09%	0.25%	Assumed	0.00000000%	0.09508293%	0.19902738%	0.29981148%
3	0.25%	0.50%	346	0.00000000%	0.38101775%	0.66944144%	0.86368524%
4	0.50%	0.75%	86	0.00000000%	0.45928064%	0.61678068%	0.80933335%
5	0.75%	1.00%	20	0.00000000%	0.79929726%	0.84899949%	0.70332233%
6	1.00%	2.00%	580	0.00000000%	0.30869310%	0.00000000%	0.56845776%
7	2.00%	3.00%	614	0.00000000%	0.79484061%	0.85715213%	0.72160368%
8	3.00%	4.00%	1492	0.00000000%	0.58601914%	0.83319934%	0.84404654%
9	4.00%	5.00%	1672	0.00000000%	0.25789196%	0.66233695%	0.72881971%
10	5.00%	6.00%	2411	0.00000000%	0.53633630%	0.75937680%	0.85798894%
11	6.00%	7.00%	1482	0.00000000%	0.54703576%	0.87197617%	0.94569540%
12	7.00%	8.00%	1218	0.00000000%	0.26551545%	0.99701437%	1.20962421%
13	8.00%	9.00%	765	0.00000000%	0.58484243%	1.35563190%	1.59333350%
14	9.00%	10.00%	500	0.00000000%	1.20734981%	1.68658411%	1.93341801%
15	10.00%	16.60%	959	0.00000000%	1.42945073%	2.13116232%	2.19162824%
Grand Total			12145	0.00000000%	0.65475588%	1.07288315%	1.17694518%

Key Implications and Notation of the HJM Approach

We now re-introduce the a slightly modified version of the notation used in Chapters 6 and 7:

Δ	Length of time period, which is 1 in this example
$r(t,s_t)$	The simple 1 period risk free interest rate as of current time t in state s_t
$R(t,s_t)$	$1+r(t,s_t)$, the total return, the value of $1 dollar invested at the risk free rate for 1 period
$f(t,T,s_t)$	The simple 1 period forward rate maturing at time T in state s_t as of time t.
$F(t,T,s_t)$	$1+f(t,T,s_t)$, the return on $1 invested at the forward rate for one period.
$\sigma_1(t,T,s_t)$	Forward rate volatility due to risk factor 1 at time t for the one period forward rate maturing at T in state s_t (a sequence of ups and downs). Because σ_1 is dependent on the spot U.S. Treasury rate, the state s_t is very important in determining its level.
$\sigma_2(t,T,s_t)$	Forward rate volatility due to risk factor 2 at time t for the one period forward rate maturing at T in state s_t (a sequence of ups and downs). Because σ_2 is dependent on the spot U.S. Treasury rate, the state s_t is very important in determining its level.
$P(t,T,s_t)$	Zero coupon bond price at time t maturing at time T in state s_t (i.e. up or down)
Index(1)	An indicator of the state for risk factor 1, defined below.
Index(2)	An indicator of the state for risk factor 2, defined below.

$K(i,t,T,s_t)$ is the sum of the forward volatilities for risk factor i (either 1 or 2) as of time t for the 1 period forward rates from t+Δ to T, as shown here:

$$K(i,t,T,s_t) = \sum_{j=t+\Delta}^{T} \sigma_i(t,j,s_t)\sqrt{\Delta}$$

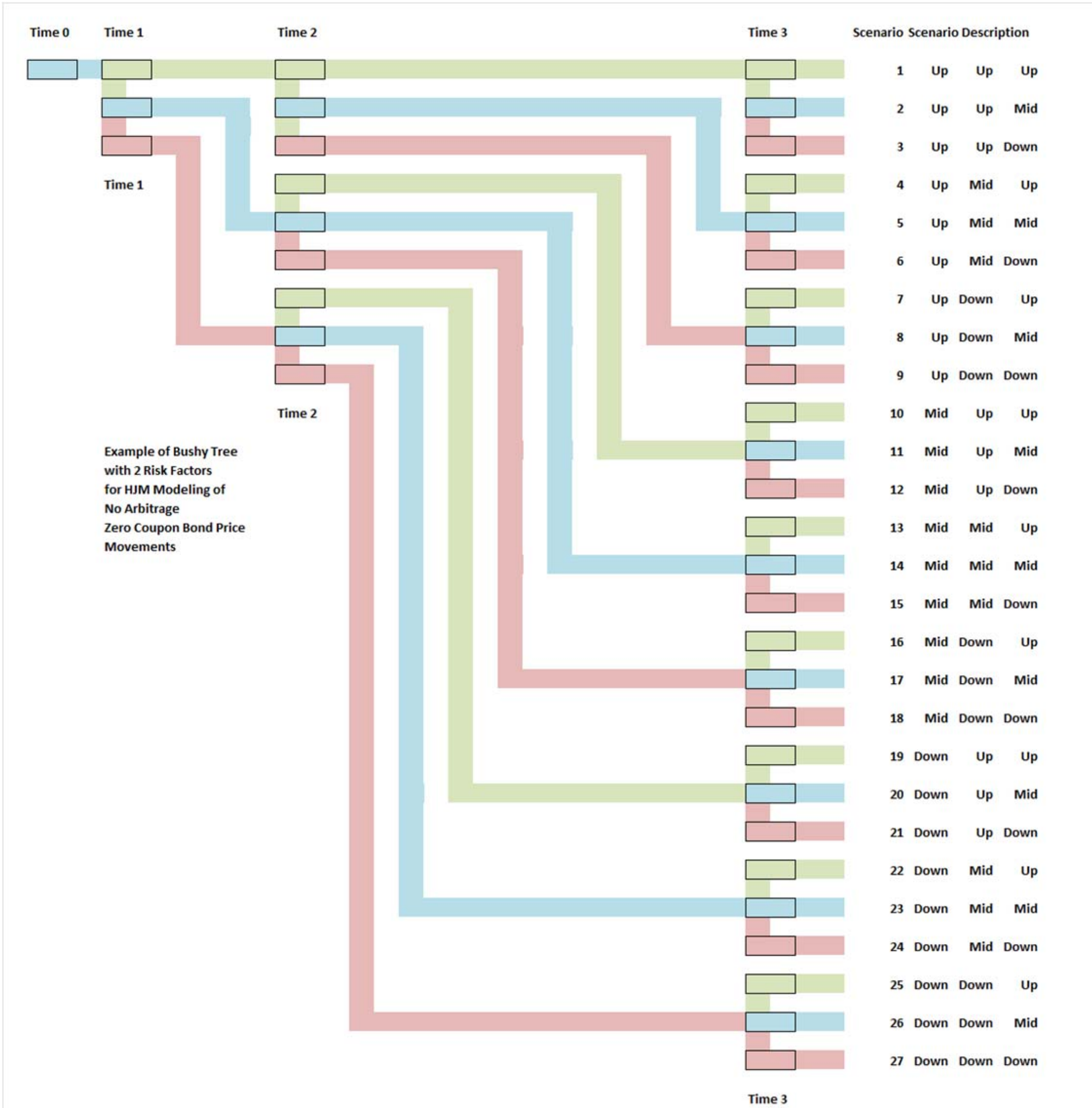
Note that this is a slightly different definition of K than we used in Chapters 6 and 7.

We again use the trigonometric function cosh(x):

$$Cosh(x) = \frac{1}{2}[e^x + e^{-x}]$$

We'll again arrange the tree to look like a table by stretching the bushy tree as follows:

Figure 7



A completely populated zero coupon bond price tree would then be summarized like this; prices shown are for the zero coupon bond price maturing at time T=4 at times 0, 1, 2, and 3:

Figure 8

	Four Year Zero Coupon Bond Price			
	Current Time			
Row Number	0	1	2	3
1	0.93085510	0.96002942	0.98271534	0.99567640
2		0.90943520	0.95325638	0.98282580
3		0.93256899	0.96584087	0.98795974
4			0.94311642	0.98343465
5			0.93491776	0.97488551
6			0.91274456	0.96385706
7			0.96285825	0.99123966
8			0.95448798	0.98262266
9			0.93185064	0.97150669
10				0.98272135
11				0.96087480
12				0.95222434
13				0.98272135
14				0.96087480
15				0.95222434
16				0.96472003
17				0.95770868
18				0.93866906
19				0.98545473
20				0.97688802
21				0.96583693
22				0.98545473
23				0.97688802
24				0.96583693
25				0.97845156
26				0.96236727
27				0.95217151

The mapping of the sequence of up and down states is shown here, consistent with the stretched tree above:

Figure 9

	Map of Sequence of States			
	Current Time			
Row Number	0	1	2	3
1	Time 0	Up	Up Up	Up Up Up
2		Mid	Up Mid	Up Up Mid
3		Down	Up Down	Up Up Down
4			Mid Up	Up Mid Up
5			Mid Mid	Up Mid Mid
6			Mid Down	Up Mid Down
7			Down Up	Up Down Up
8			Down Mid	Up Down Mid
9			Down Down	Up Down Down
10				Mid Up Up
11				Mid Up Mid
12				Mid Up Down
13				Mid Mid Up
14				Mid Mid Mid
15				Mid Mid Down
16				Mid Down Up
17				Mid Down Mid
18				Mid Down Down
19				Down Up Up
20				Down Up Mid
21				Down Up Down
22				Down Mid Up
23				Down Mid Mid
24				Down Mid Down
25				Down Down Up
26				Down Down Mid
27				Down Down Down

In order to populate the trees with zero coupon bond prices and forward rates, we again need to select the relevant pseudo probabilities.

Pseudo Probabilities

Consistent with Chapter 7 of Jarrow (2002), and without loss of generality, we set the probability of an up shift to ¼, the probability of a mid shift to ¼, and the probability of a down shift to ½. The no arbitrage restrictions that stem from this set of pseudo probabilities are given below.

The Formula for Zero Coupon Bond Price Shifts with Two Risk Factors

In Chapters 6 and 7, we used equation 15.17 in Jarrow (2002, page 286) to calculate the no arbitrage shifts in zero coupon bond prices. Alternatively, when there is one risk factor, we could have used equation 15.19 in Jarrow (2002, page 287) to shift forward rates and derive zero coupon bond prices from the forward rates. Now that we have two risk factors, it is convenient to calculate the forward rates first. We do this using equations 15.32, 15.39a, and 15.39b in Jarrow (2002, pages 293 and 296). We use this equation for the shift in forward rates:

(1)

$$F(t+\Delta,T,s_{t+\Delta}) = F(t,T,s_t)e^{[\mu(t,T,s_t)\Delta]\Delta}e^{[Index(1)\sigma_1(t,T,s_t)\sqrt{\Delta}+Index(2)\sqrt{2}\sigma_2(t,T,s_t)\sqrt{\Delta}]\Delta}$$

Where when T=t+Δ, the expression μ becomes

$$[\mu(t,T)\Delta]\Delta = log\left[\frac{1}{2}e^{[K(1,t,t+\Delta)]\Delta}\cosh(\Delta K(2,t,t+\Delta)\sqrt{2}) + \frac{1}{2}e^{-[K(1,t,t+\Delta)]\Delta}\right]$$

We use this expression to evaluate formula (1) when T>t+Δ:

$$[\mu(t,T)\Delta]\Delta = log\left[\frac{1}{2}e^{[K(1,t,T)]\Delta}\cosh(\Delta K(2,t,T)\sqrt{2}) + \frac{1}{2}e^{-[K(1,t,T)]\Delta}\right] - \sum_{j=t+\Delta}^{T-\Delta}[\mu(t,j)\Delta]\,\Delta$$

The values for the pseudo probabilities and Index (1) and Index (2) are set as follows:

Type of Shift	Pseudo Probability	Index(1)	Index (2)
Up shift	1/4	-1	-1
Mid shift	1/4	-1	1
Down shift	1/2	1	0

Building the Bushy Tree for Zero Coupon Bonds Maturing at Time T=2

We now populate the bushy tree for the 2 year zero coupon bond. We calculate each element of equation (1). When t=0 and T=2, we know Δ=1 and

$P(0,2,s_t) = 0.98411015$.

The one period risk free return (1 plus the risk free rate) is again

$R(0,s_t)=1/P(0,1,s_t)=1/0.997005786=1.003003206$

The 1 period spot rate for U.S. Treasuries is r(0, st) =R(0,st)-1=0.3003206%. At this level of the spot rate for U.S. Treasuries, volatilities for risk factor 1 are selected from data group 3 in the look up table above. The volatilities for risk factor 1 for the 1 year forward rates maturing in years 2, 3 and 4 are 0.000492746, 0.000313424, and 0.00016918. The volatilities for risk factor 2 for the 1 year forward rates maturing in years 2, 3 and 4 are 0.003810177, 0.006694414, and 0.008636852.

The scaled sum of sigmas $K(1,t,T,s_t)$ for t=0 and T=2 becomes

$$K(1,t,T,s_t) = \sum_{j=t+\Delta}^{T-\Delta} \sigma_1(t,j,s_t)\sqrt{\Delta} = \sum_{j=1}^{1} \sigma_1(0,j,s_t)\sqrt{\Delta}$$

and therefore $K(1,0,T,s_t) = (\sqrt{1})(0.000492746) = 0.000492746$. Similarly, $K(2,0,T,s_t) = 0.003810177$. We also can calculate that $[\mu(t,t+\Delta)\Delta]\Delta = 0.00000738$.

Using formula 1 with these inputs and the fact that the variable Index(1)=-1 and Index(2)=-1 for an up shift gives us the forward returns for an up shift, mid shift and down shift as follows: 1.007170561, 1.018083343, and 1.013610665. From these we calculate the zero coupon bond prices:

$P(1,2,s_t = up) = 0.992880 = 1/F(1,2,s_t = up)$

For a mid shift, we set Index(1)=-1 and Index(2)=+1 and calculate

$P(1,2,s_t = \text{mid}) = 0.982238 = 1/F(1,2,s_t = \text{mid})$

For a down shift we set Index(1) = 1 and Index(2)= 0 and recalculate formula 1 to get

$P(1,2,st = \text{down}) = 0.986572 = 1/F(1,2,s_t = \text{down})$

We have fully populated the bushy tree for the zero coupon bond maturing at T=2 (note values have been rounded to six decimal places above for display only), since all of the up, mid and down states at time t=2 result in a riskless pay-off of the zero coupon bond at its face value of 1.

	Two Year Zero Coupon Bond Price			
	Current Time			
Row Number	0	1	2	3
1	0.98411015	0.99288049	1	
2		0.98223786	1	
3		0.98657210	1	
4			1	
5			1	
6			1	
7			1	
8			1	
9			1	

Building the Bushy Tree for Zero Coupon Bonds Maturing at Time T=3

For the zero coupon bonds and 1 period forward returns (= 1 + forward rate) maturing at time T=3, we use the same volatilities listed above for risk factors 1 and 2 to calculate

$K(1,0,3,s_t) = 0.00080617$

$K(2,0,3,s_t) = 0.010504592$

$[\mu(t,T)\Delta]\Delta = 0.00004816$

The resulting forward returns for an up shift, mid shift and down shift are 1.013189027, 1.032556197, and 1.023468132. Zero coupon bond prices are calculated from the 1 period forward returns, so

$P(1,3,s_t = up) = 1/[F(1,2,s_t = up)F(1,3,s_t = up)]$

The zero coupon bond prices for an up shift, mid shift, and down shift are 0.979956, 0.951268, and 0.963950. To eight decimal places, we have populated the second column of the zero coupon bond price table for the zero coupon bond maturing at T=3 as follows:

Figure 10

	Three Year Zero Coupon Bond Price			
	Current Time			
Row Number	0	1	2	3
1	0.96189224	0.97995583	0.99404203	1
2		0.95126818	0.98121252	1
3		0.96394999	0.98633804	1
4			0.98035908	1
5			0.97183666	1
6			0.96084271	1
7			0.98906435	1
8			0.98046626	1
9			0.96937468	1
10				1
11				1
12				1
13				1
14				1
15				1
16				1
17				1
18				1
19				1
20				1
21				1
22				1
23				1
24				1
25				1
26				1
27				1

Building the Bushy Tree for Zero Coupon Bonds Maturing at Time T = 4

We now populate the bushy tree for the zero coupon bond maturing at T=4. Using the same volatilities as before for both risk factors 1 and 2, we find that

$K(1,0,4,s_t) = 0.000975351$

$K(2,0,4,s_t) = 0.019141444$

$[\mu(t,T)\Delta]\Delta = 0.00012830$

Using formula (1) with the correct values for Index (1) and Index (2) leads to the following forward returns for an up shift, mid shift and down shift: 1.020756042, 1.045998862, and 1.033650059. The zero coupon bond price is calculated as follows:

$P(1,4,s_t = \text{up}) = 1/[F(1,2,s_t = \text{up})F(1,3,s_t = \text{up})\, F(1,4,s_t = \text{up})]$

This gives us the three zero coupon bond prices of the column labeled 1 in this table for up, mid and down shifts: 0.96002942, 0.90943520, and 0.93256899.

Figure 11

	Four Year Zero Coupon Bond Price			
	Current Time			
Row Number	0	1	2	3
1	0.93085510	0.96002942	0.98271534	0.99567640
2		0.90943520	0.95325638	0.98282580
3		0.93256899	0.96584087	0.98795974
4			0.94311642	0.98343465
5			0.93491776	0.97488551
6			0.91274456	0.96385706
7			0.96285825	0.99123966
8			0.95448798	0.98262266
9			0.93185064	0.97150669
10				0.98272135
11				0.96087480
12				0.95222434
13				0.98272135
14				0.96087480
15				0.95222434
16				0.96472003
17				0.95770868
18				0.93866906
19				0.98545473
20				0.97688802
21				0.96583693
22				0.98545473
23				0.97688802
24				0.96583693
25				0.97845156
26				0.96236727
27				0.95217151

Now we move to the third column, which displays the outcome of the T=4 zero coupon bond price after 9 scenarios: up-up, up-mid, up-down, mid-up, mid-mid, mid-down, down-up, down-mid, and down-down. We calculate P(2,4,s_t = up), P(2,4,s_t=mid) and P(2,4,s_t = down) after the initial "down" state as follows. When t=1, T=4, and Δ=1 then the volatilities for the two remaining 1 period forward rates that are relevant are taken from the lookup table for data group 6 for risk factor 1: 0.007871303, and 0.006312739. For risk factor 2, the volatilities for the two remaining 1 period forward rates are also chosen from data group 6: 0.003086931, and 0. The zero value was described above; the implied volatility for risk factor 1 was greater than measured total volatility for the forward rate maturing at T=3 in data group 6, so the risk factor 1 volatility was set to total volatility and risk factor 2 volatility was set to zero.

At time 1 in the down state, the zero coupon bond prices for maturities at T=2, 3 and 4 are 0.986572, 0.963950, and 0.932569. We make the intermediate calculations as above for the zero coupon bond maturing at T=4:

$K(1,1,4,s_t) = 0.014184042$

$K(2,1,4,s_t) = 0.003086931$

$[\mu(t,T)\Delta]\Delta = 0.00006964$

We can calculate the values of the 1 period forward return maturing at time T=4 in the up state, mid state, and down state as follows: 1.027216983, 1.027216983, and 1.040268304. Similarly, using the appropriate intermediate calculations, we can calculate the forward returns for maturity at T=3: 1.011056564, 1.01992291, and 1.031592858. Since

$P(2,4,s_t = \text{down up}) = 1/[F(1,3,s_t = \text{down up})\ F(1,4,s_t = \text{down up})]$

the zero coupon bond prices for maturity at T=4 in the down up, down mid, and down down states are as follows:

0.962858

0.954488

0.931851

We have correctly populated the seventh, eighth and ninth rows of column 3 (t=2) of the bushy tree above for the zero coupon bond maturing at T=4 (note values have been rounded to six decimal places for display only). The remaining calculations are left to the reader.

If we combine all of these tables, we can create a table of the term structure of zero coupon bond prices in each scenario as in examples one and two. The shading highlights two nodes of the bushy tree where values are identical because of the occurrence of $\sigma_2 = 0$ at one point on the bushy tree:

Figure 12

State Name	State Number	Zero Coupon Bond Prices Periods to Maturity 1	2	3	4
Time 0	0	0.99700579	0.98411015	0.96189224	0.93085510
Time 1 Up	1	0.99288049	0.97995583	0.96002942	
Time 1 Mid	2	0.98223786	0.95126818	0.90943520	
Time 1 Down	3	0.98657210	0.96394999	0.93256899	
Time 2 Up Up	4	0.99404203	0.98271534		
Time 2 Up Mid	5	0.98121252	0.95325638		
Time 2 Up Down	6	0.98633804	0.96584087		
Time 2 Mid Up	7	0.98035908	0.94311642		
Time 2 Mid Mid	8	0.97183666	0.93491776		
Time 2 Mid Down	9	0.96084271	0.91274456		
Time 2 Down Up	10	0.98906435	0.96285825		
Time 2 Down Mid	11	0.98046626	0.95448798		
Time 2 Down Down	12	0.96937468	0.93185064		
Time 3 Up Up Up	13	0.99567640			
Time 3 Up Up Mid	14	0.98282580			
Time 3 Up Up Down	15	0.98795974			
Time 3 Up Mid Up	16	0.98343465			
Time 3 Up Mid Mid	17	0.97488551			
Time 3 Up Mid Down	18	0.96385706			
Time 3 Up Down Up	19	0.99123966			
Time 3 Up Down Mid	20	0.98262266			
Time 3 Up Down Down	21	0.97150669			
Time 3 Mid Up Up	22	0.98272135			
Time 3 Mid Up Mid	23	0.96087480			
Time 3 Mid Up Down	24	0.95222434			
Time 3 Mid Mid Up	25	0.98272135			
Time 3 Mid Mid Mid	26	0.96087480			
Time 3 Mid Mid Down	27	0.95222434			
Time 3 Mid Down Up	28	0.96472003			
Time 3 Mid Down Mid	29	0.95770868			
Time 3 Mid Down Down	30	0.93866906			
Time 3 Down Up Up	31	0.98545473			
Time 3 Down Up Mid	32	0.97688802			
Time 3 Down Up Down	33	0.96583693			
Time 3 Down Mid Up	34	0.98545473			
Time 3 Down Mid Mid	35	0.97688802			
Time 3 Down Mid Down	36	0.96583693			
Time 3 Down Down Up	37	0.97845156			
Time 3 Down Down Mid	38	0.96236727			
Time 3 Down Down Down	39	0.95217151			

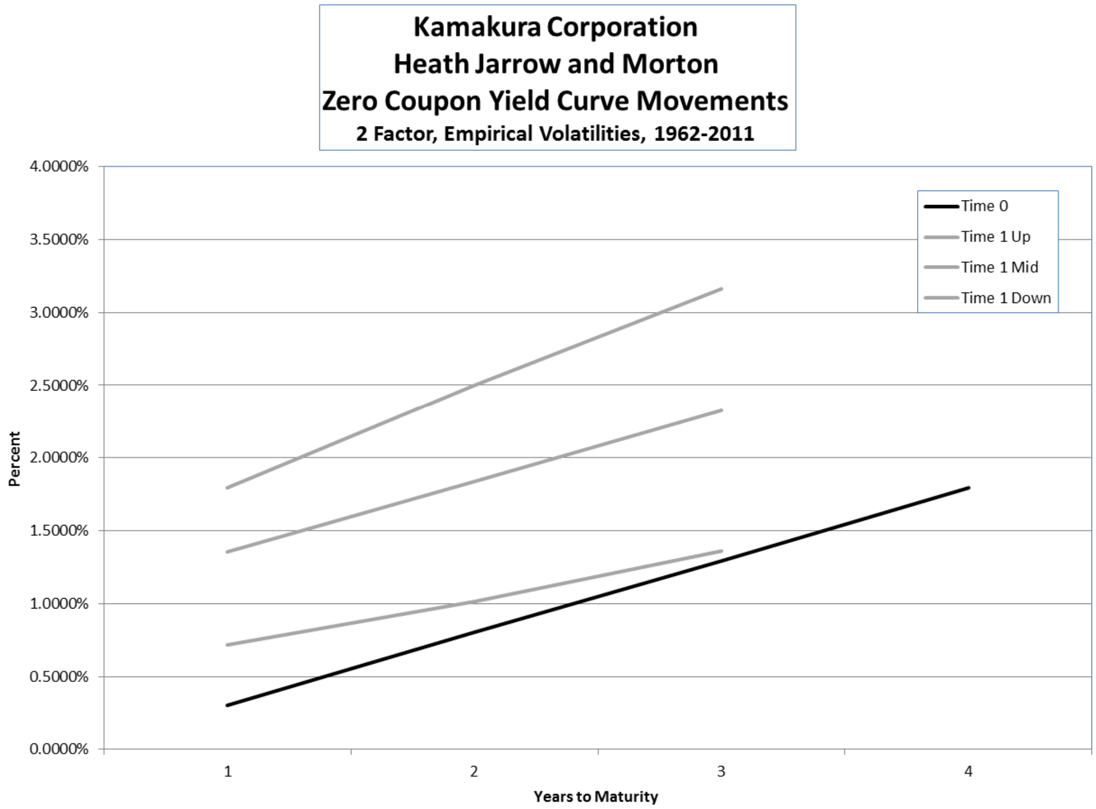
At any point in time t, the continuously compounded yield to maturity at time T can be calculated as y(T-t)=-ln[P(t,T)]/(T-t). Note that we have no negative rates on this bushy tree and that yield curve shifts are much more complex than in the two prior examples using one risk factor:

Figure 13

State Name	State Number	Continuously Compounded Zero Yields			
		Periods to Maturity			
		1	2	3	4
Time 0	0	0.2999%	0.8009%	1.2951%	1.7913%
Time 1 Up	1	0.7145%	1.0124%	1.3597%	
Time 1 Mid	2	1.7922%	2.4980%	3.1644%	
Time 1 Down	3	1.3519%	1.8358%	2.3271%	
Time 2 Up Up	4	0.5976%	0.8718%		
Time 2 Up Mid	5	1.8966%	2.3936%		
Time 2 Up Down	6	1.3756%	1.7378%		
Time 2 Mid Up	7	1.9836%	2.9283%		
Time 2 Mid Mid	8	2.8568%	3.3648%		
Time 2 Mid Down	9	3.9945%	4.5650%		
Time 2 Down Up	10	1.0996%	1.8925%		
Time 2 Down Mid	11	1.9727%	2.3290%		
Time 2 Down Down	12	3.1104%	3.5291%		
Time 3 Up Up Up	13	0.4333%			
Time 3 Up Up Mid	14	1.7323%			
Time 3 Up Up Down	15	1.2113%			
Time 3 Up Mid Up	16	1.6704%			
Time 3 Up Mid Mid	17	2.5435%			
Time 3 Up Mid Down	18	3.6812%			
Time 3 Up Down Up	19	0.8799%			
Time 3 Up Down Mid	20	1.7530%			
Time 3 Up Down Down	21	2.8907%			
Time 3 Mid Up Up	22	1.7430%			
Time 3 Mid Up Mid	23	3.9911%			
Time 3 Mid Up Down	24	4.8955%			
Time 3 Mid Mid Up	25	1.7430%			
Time 3 Mid Mid Mid	26	3.9911%			
Time 3 Mid Mid Down	27	4.8955%			
Time 3 Mid Down Up	28	3.5917%			
Time 3 Mid Down Mid	29	4.3212%			
Time 3 Mid Down Down	30	6.3292%			
Time 3 Down Up Up	31	1.4652%			
Time 3 Down Up Mid	32	2.3383%			
Time 3 Down Up Down	33	3.4760%			
Time 3 Down Mid Up	34	1.4652%			
Time 3 Down Mid Mid	35	2.3383%			
Time 3 Down Mid Down	36	3.4760%			
Time 3 Down Down Up	37	2.1784%			
Time 3 Down Down Mid	38	3.8359%			
Time 3 Down Down Down	39	4.9010%			

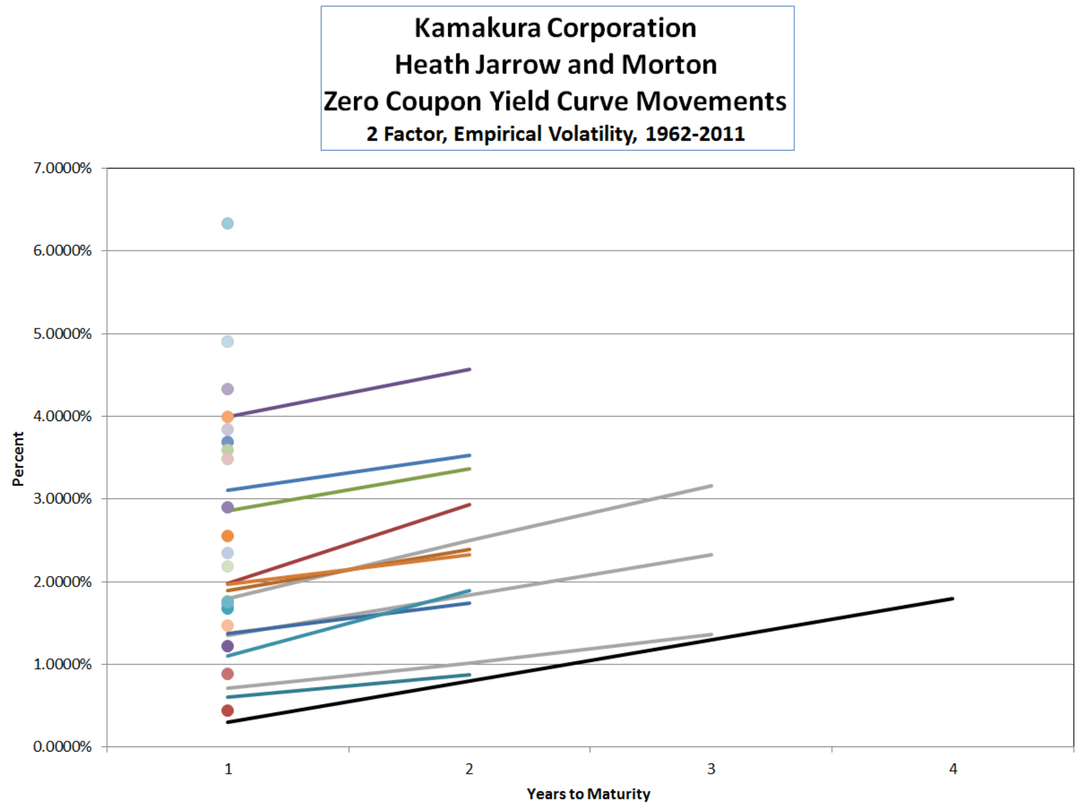
We can graph yield curve movements as shown below at time t=1. We plot yield curves for the up, mid and down shifts. These shifts are relative to the 1 period forward rates prevailing at time zero for maturity at time T=2 and T=3. Because these 1 period forward rates were much higher than yields as of time t=0, all three shifts produce yields higher than time zero yields.

Figure 14



When we add 9 yield curves prevailing at time t=3 and 27 "single point" yield curves prevailing at time t=4, two things are very clear. First, yield curve movements in a two factor model are much more complex and much more realistic than what we saw in the two one-factor examples. Second, in the low yield environment prevailing as of March 31, 2011, no arbitrage yield curve simulation shows "there is nowhere to go but up" from a yield curve perspective.

Figure 15



Finally, we can display the 1 year U.S. Treasury spot rates and the associated term structure of 1 year forward rates in each scenario.

Figure 16

State Name	State Number	1 Year Forward Rates, Simple Interest Periods to Maturity 1	2	3	4
Time 0	0	0.3003%	1.3104%	2.3098%	3.3343%
Time 1 Up	1	0.7171%	1.3189%	2.0756%	
Time 1 Mid	2	1.8083%	3.2556%	4.5999%	
Time 1 Down	3	1.3611%	2.3468%	3.3650%	
Time 2 Up Up	4	0.5994%	1.1526%		
Time 2 Up Mid	5	1.9147%	2.9327%		
Time 2 Up Down	6	1.3851%	2.1222%		
Time 2 Mid Up	7	2.0034%	3.9489%		
Time 2 Mid Mid	8	2.8979%	3.9489%		
Time 2 Mid Down	9	4.0753%	5.2696%		
Time 2 Down Up	10	1.1057%	2.7217%		
Time 2 Down Mid	11	1.9923%	2.7217%		
Time 2 Down Down	12	3.1593%	4.0268%		
Time 3 Up Up Up	13	0.4342%			
Time 3 Up Up Mid	14	1.7474%			
Time 3 Up Up Down	15	1.2187%			
Time 3 Up Mid Up	16	1.6844%			
Time 3 Up Mid Mid	17	2.5761%			
Time 3 Up Mid Down	18	3.7498%			
Time 3 Up Down Up	19	0.8838%			
Time 3 Up Down Mid	20	1.7685%			
Time 3 Up Down Down	21	2.9329%			
Time 3 Mid Up Up	22	1.7582%			
Time 3 Mid Up Mid	23	4.0718%			
Time 3 Mid Up Down	24	5.0173%			
Time 3 Mid Mid Up	25	1.7582%			
Time 3 Mid Mid Mid	26	4.0718%			
Time 3 Mid Mid Down	27	5.0173%			
Time 3 Mid Down Up	28	3.6570%			
Time 3 Mid Down Mid	29	4.4159%			
Time 3 Mid Down Down	30	6.5338%			
Time 3 Down Up Up	31	1.4760%			
Time 3 Down Up Mid	32	2.3659%			
Time 3 Down Up Down	33	3.5371%			
Time 3 Down Mid Up	34	1.4760%			
Time 3 Down Mid Mid	35	2.3659%			
Time 3 Down Mid Down	36	3.5371%			
Time 3 Down Down Up	37	2.2023%			
Time 3 Down Down Mid	38	3.9104%			
Time 3 Down Down Down	39	5.0231%			

Valuation in the Heath Jarrow and Morton Framework

Prof. Jarrow in a quote in Chapter 6 described valuation as the expected value of cash flows using the risk neutral probabilities. Note that column 1 denotes the riskless 1 period interest rate in each scenario. For the scenario number 39 (three consecutive down shifts in zero coupon bond prices), cash flows at time T=4 would be discounted by the one year spot rates at time t=0, by the one year spot rate at time t=1 in scenario 3 ("down"), by the one year spot rate in scenario 12 ("down down") at time t=2, and by the one year spot rate at time t=3 in scenario 39 ("down down down"). The discount factor is

Discount Factor (0,4, down down down) =1/(1.003003)(1.013611)(1.031593)(1.050231)

These discount factors are displayed here for each potential cash flow date:

Figure 17

	Discount Factors				
	Current Time				
Row Number	0	1	2	3	4
1	1	0.997005786	0.989907593	0.984009757	0.979755294
2		0.997005786	0.989907593	0.984009757	0.967110174
3		0.997005786	0.989907593	0.984009757	0.972162025
4			0.979296826	0.971309727	0.955219645
5			0.979296826	0.971309727	0.946915774
6			0.979296826	0.971309727	0.93620374
7			0.98361809	0.976383515	0.967830059
8			0.98361809	0.976383515	0.959416563
9			0.98361809	0.976383515	0.948563113
10				0.960062532	0.943473945
11				0.960062532	0.922499895
12				0.960062532	0.914194906
13				0.951716561	0.935272181
14				0.951716561	0.914480461
15				0.951716561	0.906247669
16				0.940950217	0.907753518
17				0.940950217	0.901156195
18				0.940950217	0.883240853
19				0.972861584	0.958711053
20				0.972861584	0.950376831
21				0.972861584	0.939625643
22				0.964404349	0.950376831
23				0.964404349	0.942115059
24				0.964404349	0.931457333
25				0.953494474	0.932948155
26				0.953494474	0.917611875
27				0.953494474	0.907890273

When taking expected values, we can calculate the probability of each scenario coming about since the probabilities of an up shift, mid shift, and down shift are ¼, ¼ and 1/2:

Figure 18

	Probability of Each State				
	Current Time				
Row Number	0	1	2	3	4
1	100.0000%	25.0000%	6.2500%	1.5625%	1.5625%
2		25.0000%	6.2500%	1.5625%	1.5625%
3		50.0000%	12.5000%	3.1250%	3.1250%
4			6.2500%	1.5625%	1.5625%
5			6.2500%	1.5625%	1.5625%
6			12.5000%	3.1250%	3.1250%
7			12.5000%	3.1250%	3.1250%
8			12.5000%	3.1250%	3.1250%
9			25.0000%	6.2500%	6.2500%
10				1.5625%	1.5625%
11				1.5625%	1.5625%
12				3.1250%	3.1250%
13				1.5625%	1.5625%
14				1.5625%	1.5625%
15				3.1250%	3.1250%
16				3.1250%	3.1250%
17				3.1250%	3.1250%
18				6.2500%	6.2500%
19				3.1250%	3.1250%
20				3.1250%	3.1250%
21				6.2500%	6.2500%
22				3.1250%	3.1250%
23				3.1250%	3.1250%
24				6.2500%	6.2500%
25				6.2500%	6.2500%
26				6.2500%	6.2500%
27				12.5000%	12.5000%
Total	100.0000%	100.0000%	100.0000%	100.0000%	100.0000%

It is again convenient to calculate the “probability weighted discount factors” for use in calculating the expected present value of cash flows:

Figure 19

	Probability Weighted Discount Factors				
	Current Time				
Row Number	0	1	2	3	4
1	1	0.249251447	0.061869225	0.015375152	0.015308676
2		0.249251447	0.061869225	0.015375152	0.015111096
3		0.498502893	0.123738449	0.030750305	0.030380063
4			0.061206052	0.015176714	0.014925307
5			0.061206052	0.015176714	0.014795559
6			0.122412103	0.030353429	0.029256367
7			0.122952261	0.030511985	0.030244689
8			0.122952261	0.030511985	0.029981768
9			0.245904522	0.06102397	0.059285195
10				0.015000977	0.01474178
11				0.015000977	0.014414061
12				0.030001954	0.028568591
13				0.014870571	0.014613628
14				0.014870571	0.014288757
15				0.029741143	0.02832024
16				0.029404694	0.028367297
17				0.029404694	0.028161131
18				0.058809389	0.055202553
19				0.030401924	0.02995972
20				0.030401924	0.029699276
21				0.060803849	0.058726603
22				0.030137636	0.029699276
23				0.030137636	0.029441096
24				0.060275272	0.058216083
25				0.059593405	0.05830926
26				0.059593405	0.057350742
27				0.119186809	0.113486284

We now use the HJM bushy trees we have generated to value representative securities.

Valuation of a Zero Coupon Bond Maturing at Time T=4

A riskless zero coupon bond pays $1 in each of the 8 nodes of the bushy tree that prevail at time T=4:

Figure 20

	Maturity of Cash Flow Received				
	Current Time				
Row Number	0	1	2	3	4
1	0.00	0.00	0.00	0.00	1.00
2		0.00	0.00	0.00	1.00
3		0.00	0.00	0.00	1.00
4			0.00	0.00	1.00
5			0.00	0.00	1.00
6			0.00	0.00	1.00
7			0.00	0.00	1.00
8			0.00	0.00	1.00
9			0.00	0.00	1.00
10				0.00	1.00
11				0.00	1.00
12				0.00	1.00
13				0.00	1.00
14				0.00	1.00
15				0.00	1.00
16				0.00	1.00
17				0.00	1.00
18				0.00	1.00
19				0.00	1.00
20				0.00	1.00
21				0.00	1.00
22				0.00	1.00
23				0.00	1.00
24				0.00	1.00
25				0.00	1.00
26				0.00	1.00
27				0.00	1.00
Risk Neutral Value =					0.93085510

When we multiply this vector of 1s times the probability weighted discount factors in the time T=4 column in the previous table and add them, we get a zero coupon bond price of 0.93085510, which is the value we should get in a no-arbitrage economy, the value observable in the market and used as an input to create the tree.

Valuation of a Coupon-Bearing Bond Paying Annual Interest

Next we value a bond with no credit risk that pays $3 in interest at every scenario at times T=1, 2, 3, and 4 plus principal of 100 at time T=4. The valuation is calculated by multiplying each cash flow by the matching probability weighted discount factor, to get a value of 104.70709974. It will surprise many that this is the same value that we arrived at in Chapters 6 and 7, even though the volatilities used and number of risk factors used are different. The values are the same because, by construction, our valuations for the zero coupon bond prices at time zero for maturities at T = 1, 2, 3, and 4 continue to match the inputs. Multiplying these zero coupon bond prices times 3, 3, 3, and 103 also leads to a value of 104.70709974 as it should.

Figure 21

	Maturity of Cash Flow Received				
	Current Time				
Row Number	0	1	2	3	4
1	0.00	3.00	3.00	3.00	103.00
2		3.00	3.00	3.00	103.00
3		3.00	3.00	3.00	103.00
4			3.00	3.00	103.00
5			3.00	3.00	103.00
6			3.00	3.00	103.00
7			3.00	3.00	103.00
8			3.00	3.00	103.00
9			3.00	3.00	103.00
10				3.00	103.00
11				3.00	103.00
12				3.00	103.00
13				3.00	103.00
14				3.00	103.00
15				3.00	103.00
16				3.00	103.00
17				3.00	103.00
18				3.00	103.00
19				3.00	103.00
20				3.00	103.00
21				3.00	103.00
22				3.00	103.00
23				3.00	103.00
24				3.00	103.00
25				3.00	103.00
26				3.00	103.00
27				3.00	103.00
Risk Neutral Value =					104.70709974

Valuation of a Digital Option on the 1 Year U.S. Treasury Rate

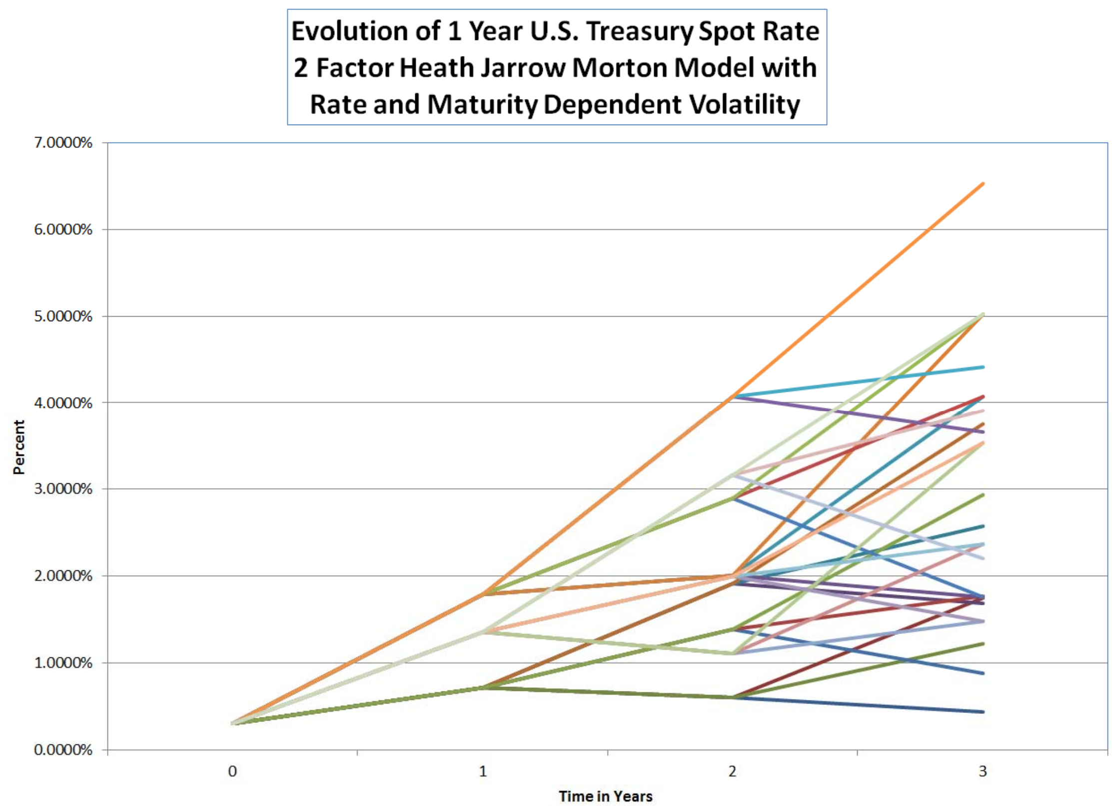
Now we value a digital option that pays $1 at time T=3 if (at that time) the one year U.S. Treasury rate (for maturity at T=4) is over 4%. If we look at the table of the term structure of one year spot rates over time, this happens at the seven shaded scenarios out of 27 possibilities at time t=3.

Figure 22

Spot Rate Process					
State	Row	0	1	2	3
Up Up Up	1	0.3003%	0.7145%	0.5994%	0.4342%
Up Up Mid	2		1.7922%	1.9147%	1.7474%
Up Up Down	3		1.3519%	1.3851%	1.2187%
Up Mid Up	4			2.0034%	1.6844%
Up Mid Mid	5			2.8979%	2.5761%
Up Mid Down	6			4.0753%	3.7498%
Up Down Up	7			1.1057%	0.8838%
Up Down Mid	8			1.9923%	1.7685%
Up Down Down	9			3.1593%	2.9329%
Mid Up Up	10				1.7582%
Mid Up Mid	11				4.0718%
Mid Up Down	12				5.0173%
Mid Mid Up	13				1.7582%
Mid Mid Mid	14				4.0718%
Mid Mid Down	15				5.0173%
Mid Down Up	16				3.6570%
Mid Down Mid	17				4.4159%
Mid Down Down	18				6.5338%
Down Up Up	19				1.4760%
Down Up Mid	20				2.3659%
Down Up Down	21				3.5371%
Down Mid Up	22				1.4760%
Down Mid Mid	23				2.3659%
Down Mid Down	24				3.5371%
Down Down Up	25				2.2023%
Down Down Mid	26				3.9104%
Down Down Down	27				5.0231%

The evolution of the spot rate can be displayed graphically as follows:

Figure 23



The cash flow payoffs in the 7 relevant scenarios can be input in the table below and multiplied by the probability weighted discount factors to find that this option has a value of 0.29701554:

Figure 24

Maturity of Cash Flow Received					
	Current Time				
Row Number	0	1	2	3	4
1	0.00	0.00	0.00	0.00	0.00
2		0.00	0.00	0.00	0.00
3		0.00	0.00	0.00	0.00
4			0.00	0.00	0.00
5			0.00	0.00	0.00
6			0.00	0.00	0.00
7			0.00	0.00	0.00
8			0.00	0.00	0.00
9			0.00	0.00	0.00
10				0.00	0.00
11				1.00	0.00
12				1.00	0.00
13				0.00	0.00
14				1.00	0.00
15				1.00	0.00
16				0.00	0.00
17				1.00	0.00
18				1.00	0.00
19				0.00	0.00
20				0.00	0.00
21				0.00	0.00
22				0.00	0.00
23				0.00	0.00
24				0.00	0.00
25				0.00	0.00
26				0.00	0.00
27				1.00	0.00
Risk Neutral Value =					0.29701554

Replication of HJM Example 3 in Common Spreadsheet Software

The readers of this book are almost certain to be proficient in the use of common spreadsheet software. In replicating the examples in this and other chapters, any discrepancy in values within the first 10 decimal places is most likely an error by the analyst or, heaven forbid, the authors of this volume. In the construction of the examples in this book, every discrepancy in the first 10 decimal places has been an error, not the normal computer science rounding error that the authors would typically blame first as the cause.

Conclusion

This chapter, continuing the examples of Chapters 6 and 7, shows how to simulate zero coupon bond prices, forward rates and zero coupon bond yields in an HJM framework with two risk factors and rate-dependent and maturity-dependent interest rate volatility. The results show a rich twist in simulated yield curves and a pull of rates upward from a very low rate environment. Monte Carlo simulation, an alternative to the bushy tree framework, can be done in a fully consistent way. We discuss that in detail in Chapter 10.

In the next chapter, we introduce a third risk factor to further advance the realism of the model.