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Advanced Financial Risk Management:

Tools and Techniques for Integrated Credit Risk, Market Risk, Liquidity Risk and Interest Rate Risk Management

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Part 4: Risk Management Applications, Instrument by Instrument

Chapter 24:

Caps and Floors

This chapter was drafted on July 2, 2012, the day that Barclays PLC chief executive officer resigned in the wake of the $451 million in fines levied on Barclays for the manipulation of Libor. Barclays Chairman Marcus Agius resigned on July 1. Given that Libor is the most common reference rate that has historically been used as the reference rate in caps and floors, big changes lie (no pun intended) ahead for the Libor market and the related market in caps and floors. Libor and the market for interest rate derivatives are so tightly tied together that a Heath Jarrow and Morton-variation called the “Libor market model” is described by some as the “industry standard model for pricing interest rate derivatives.” The Libor market model is a very useful point of reference and we recommend that serious readers review Riccardo Rebonato’s excellent book “Modern Pricing of Interest-Rate Derivatives: The LIBOR Market Model and Beyond” (John Wiley, 2012) for more on that specific implementation. In this chapter, the nature of the reference rate is irrelevant to the analysis, provided that the reference rate is determined in a competitive market with transparent price discovery and the usual assumptions of perfect competition. This description, obviously, does not apply to Libor. Therefore any results from the methods below have to be adjusted for the result of criminal activity and market manipulation. The authors recommend that readers initiate no new contracts tied to Libor or any other reference rate that does not represent actual transactions. Even in that case, sham transactions have to be eliminated from the reference rate calculation.

We now turn from the ethics of financial markets to more pleasant topics. We are making step-by-step progress in valuing the financial instruments that might be found on the balance sheet of a major financial institution. To reiterate, our ultimate objective is the full valuation of the Jarrow-Merton put option on the balance sheet of the financial institution as the best measure of integrated credit risk, market risk, liquidity risk and interest rate risk. The Jarrow-Merton put option is also an excellent basis for capital allocation, which is discussed in Chapters 36 through 41. In this chapter we turn to interest rate caps and floors using the tools from Chapter 25 extensively. We continue to employ the 3 factor Heath Jarrow and Morton model implementation from Chapter 9. Normally, in practice, one would use the parameters from Chapter 9 to value caps and floors using a Monte Carlo simulation. For ease of exposition, we use the related bushy tree from Chapter 9.

Putting aside the 2012 Libor manipulation scandal, interest rate caps and floors have become perhaps the most popular interest rate option-based instrument. They provide a valuable tool for financial managers interested in controlling their exposure to extremely high or extremely low interest rates. They also provide an excellent basis for the estimation of term structure model parameters as we discussed in Chapter 14. Common uses of caps and floors include the following examples:

- Limiting the maximum rate paid on a floating-rate loan by purchasing a cap;
- Limiting the exposure to an effective floor[1] on non-maturity consumer deposit rates paid by the bank by purchasing a floor in the open market;
- Hedging the prepayment risk on a mortgage portfolio;
- Hedging the cancellation risk on a portfolio of life insurance policies;
- Insuring that the performance of a fixed income portfolio can only improve as rates rise.

The purchaser of a cap effectively purchases an insurance policy against interest rate increases. Let us consider a hedger who has a floating-rate loan tied to a three-month U.S. dollar reference rate. By purchasing a 3% cap, the hedger insures that they will never pay more than 3% on the loan after subtracting payments from the seller of the cap. If rates rise to 5%, the hedger pays 5% on the loan, but receives 2% from the person who sold the cap, for a net expense of 3%.

Caps and floors are both traded outright and embedded in common banking and insurance

[1]For many years, U.S. bankers felt that there was an interest rate floor on consumer savings deposits, a rate below which savings deposit rates could not go. Now that we have extensive experience with extremely low interest rates in Japan, the United States and other countries, it is apparent that this floor is effectively zero. What many initially thought was a floor was more likely the lagged response of consumer savings deposit rates to changing market rates. See Chapter 30 for more on this issue.

industry assets and liabilities. A number of market conventions deserve mention here. Firstly, payment for the cap or floor typically takes place at time zero when the contract is traded outright. Embedded in a loan or deposit, the cap or floor price would be disguised as an interest rate differential relative to the identical loan or deposit with no cap or floor. Secondly, the cap or floor strike rate is stated on the same basis as the underlying interest rate. For example, the buyer of the cap above is quoted a strike rate of 3% versus a money market reference rate. Most money market interest rates are quoted on an actual/360 basis, so the cap strike rate interest amounts are calculated on an actual/360 day basis also. In other markets, actual/365 day calculations are common. Both the money market reference rate and the cap or floor rate will normally be set from the same reference point (the same 'screen' provided by a vendor of real time information, such as Reuters, Telerate, or Bloomberg) and use the same method for counting business days and the same holiday convention.

We abstract from these market conventions in this chapter to ease exposition. For purposes of this chapter, a strike rate quoted as 3% for a money market rate cap on for a 91-day period would be converted to a decimal strike rate *K* using the following: formula:

$$K = 0.03\frac{91}{360}$$

In applying the formulas below, the strike rate should be converted to this 'K format' to allow us to abstract from day count and accrual methods in what follows.

Another important market convention is the use of Black's [1976] futures model for the quotation of caps and floors prices that we have mentioned often in previous chapters. The Black-Scholes [1973] option model was perhaps the biggest innovation in financial theory in the twentieth century, but its well-justified popularity should not obscure the reasons cited in Chapters 3 through 13 for using a term structure model approach instead to model interest rate derivatives. Hull [1993] presents a very lucid explanation of the use of the Black [1976] approximation to cap and floor valuation and the reasons why a term structure model approach works better. We ignore the Black model in what follows, consistent with our de-emphasis on historical approaches that have been passed by.

At this point, we take advantage of our work in previous chapters to illustrate cap and floor pricing under a term structure model approach using the 3 factor HJM model of Chapter 9.

Caps as European Options on Forward Rate Agreements

In Chapter 23, we analyzed the European option to buy a forward rate agreement at the FRA strike rate of *K*. Looking at the cash flows for an option on an FRA shows that a cap is really just a special case of an option on an FRA, where the exercise date of the option on the FRA is the same as the expiration date of the FRA. In a single factor Vasicek model, the bond option approach of Jamshidian [1989] from Chapter 21 can be used to derive a closed form solution for caps and floors. We now from our statistical analysis in Chapter 3 that a one factor model is not an accurate description of yield curve movements, so this chapter focuses on numerical methods using the 3 factor HJM model for expository purposes.

Forming Other Cap-Related Securities

A wide variety of cap and floor derivatives can be constructed with this basic building block as a tool:

- Floor prices are calculated in an equivalent way using the formulas for options on a zero-coupon bond in Chapter 23.
- Longer-term caps are constructed as the sum of a number of caplets.
- Longer-term floors are constructed as the sum of a number of floorlets.
- Collars represent a combination of caps and floors (typically purchasing a cap and selling a floor).
- Costless collars are created by finding the floor rate K_f, such that the value of the floor exactly equals the value of the cap with cap rate K_c (or vice versa).

We now supplement our examples from Chapter 23 with additional examples of caps, floors and related instruments.

Valuing a Cap

Recall that from Chapter 9, we know the 1 year spot rate of interest at times 0, 1, 2, and 3. There is 1 node at time 0, 4 nodes at time 1, 16 nodes at time 2, and 64 nodes at time 3. We will use the one year spot rate of interest at each of these nodes as the reference interest rate in valuing a cap. The reference rate values are given in Figure 1:

Figure 1

Spot Rate Process					
State	Row	0	1	2	3
S-1, S-1, S-1	1	0.3003%	2.1653%	4.6388%	6.1106%
S-1, S-1, S-2	2		0.6911%	1.4142%	5.5667%
S-1, S-1, S-3	3		1.0972%	2.6089%	5.1710%
S-1, S-1, S-4	4		1.3611%	4.9179%	8.0375%
S-1, S-2, S-1	5			2.2496%	3.1041%
S-1, S-2, S-2	6			0.5984%	2.5913%
S-1, S-2, S-3	7			0.8121%	2.0291%
S-1, S-2, S-4	8			1.2459%	4.0628%
S-1, S-3, S-1	9			1.9217%	5.2210%
S-1, S-3, S-2	10			1.4149%	1.9784%
S-1, S-3, S-3	11			0.8591%	3.1798%
S-1, S-3, S-4	12			2.8695%	5.5017%
S-1, S-4, S-1	13			2.2088%	5.4252%
S-1, S-4, S-2	14			1.7005%	4.8848%
S-1, S-4, S-3	15			1.1432%	4.4917%
S-1, S-4, S-4	16			3.1593%	7.3396%
S-2, S-1, S-1	17				4.1517%
S-2, S-1, S-2	18				0.9421%
S-2, S-1, S-3	19				2.1313%
S-2, S-1, S-4	20				4.4295%
S-2, S-2, S-1	21				2.2036%
S-2, S-2, S-2	22				0.5531%
S-2, S-2, S-3	23				0.7667%
S-2, S-2, S-4	24				1.2004%
S-2, S-3, S-1	25				2.6231%
S-2, S-3, S-2	26				-0.3734%
S-2, S-3, S-3	27				0.2593%
S-2, S-3, S-4	28				2.1679%
S-2, S-4, S-1	29				1.7764%
S-2, S-4, S-2	30				1.2703%
S-2, S-4, S-3	31				0.7153%
S-2, S-4, S-4	32				2.7228%
S-3, S-1, S-1	33				2.2804%
S-3, S-1, S-2	34				1.7718%
S-3, S-1, S-3	35				1.2141%
S-3, S-1, S-4	36				3.2315%
S-3, S-2, S-1	37				2.2804%
S-3, S-2, S-2	38				1.7718%
S-3, S-2, S-3	39				1.2141%
S-3, S-2, S-4	40				3.2315%
S-3, S-3, S-1	41				3.6354%
S-3, S-3, S-2	42				0.6093%
S-3, S-3, S-3	43				1.2483%
S-3, S-3, S-4	44				3.1757%
S-3, S-4, S-1	45				4.5070%
S-3, S-4, S-2	46				1.2864%
S-3, S-4, S-3	47				2.4797%
S-3, S-4, S-4	48				4.7858%
S-4, S-1, S-1	49				3.5013%
S-4, S-1, S-2	50				0.3117%
S-4, S-1, S-3	51				1.4934%
S-4, S-1, S-4	52				3.7773%
S-4, S-2, S-1	53				2.5831%
S-4, S-2, S-2	54				2.0730%
S-4, S-2, S-3	55				1.5136%
S-4, S-2, S-4	56				3.5370%
S-4, S-3, S-1	57				2.5831%
S-4, S-3, S-2	58				2.0730%
S-4, S-3, S-3	59				1.5136%
S-4, S-3, S-4	60				3.5370%
S-4, S-4, S-1	61				4.5061%
S-4, S-4, S-2	62				2.2686%
S-4, S-4, S-3	63				2.7252%
S-4, S-4, S-4	64				5.0230%

Assume we want to buy a cap on this reference rate with a 3 year maturity that pays us the difference between the ultimate value of the reference rate and our “strike price,” which we assume is 3%. We assume the reference amount is $1,000. If we are owed a payment, this payment will be received “in arrears” at time 4, the point in time at which interest is paid on the reference security. The cash flows are calculated as maximum(0,1000*(spot rate - 3%)). The cash flows associated with this cap are shown in Figure 2:

Figure 2

		Time Period			
State	**Row**	**0**	**1**	**2**	**3**
S-1, S-1, S-1	1		0	0	31.1062
S-1, S-1, S-2	2		0	0	25.6672
S-1, S-1, S-3	3		0	0	21.7103
S-1, S-1, S-4	4		0	0	50.3747
S-1, S-2, S-1	5			0	1.0407
S-1, S-2, S-2	6			0	0.0000
S-1, S-2, S-3	7			0	0.0000
S-1, S-2, S-4	8			0	10.6283
S-1, S-3, S-1	9			0	22.2100
S-1, S-3, S-2	10			0	0.0000
S-1, S-3, S-3	11			0	1.7977
S-1, S-3, S-4	12			0	25.0166
S-1, S-4, S-1	13			0	24.2520
S-1, S-4, S-2	14			0	18.8481
S-1, S-4, S-3	15			0	14.9168
S-1, S-4, S-4	16			0	43.3960
S-2, S-1, S-1	17				11.5175
S-2, S-1, S-2	18				0.0000
S-2, S-1, S-3	19				0.0000
S-2, S-1, S-4	20				14.2955
S-2, S-2, S-1	21				0.0000
S-2, S-2, S-2	22				0.0000
S-2, S-2, S-3	23				0.0000
S-2, S-2, S-4	24				0.0000
S-2, S-3, S-1	25				0.0000
S-2, S-3, S-2	26				0.0000
S-2, S-3, S-3	27				0.0000
S-2, S-3, S-4	28				0.0000
S-2, S-4, S-1	29				0.0000
S-2, S-4, S-2	30				0.0000
S-2, S-4, S-3	31				0.0000
S-2, S-4, S-4	32				0.0000
S-3, S-1, S-1	33				0.0000
S-3, S-1, S-2	34				0.0000
S-3, S-1, S-3	35				0.0000
S-3, S-1, S-4	36				2.3153
S-3, S-2, S-1	37				0.0000
S-3, S-2, S-2	38				0.0000
S-3, S-2, S-3	39				0.0000
S-3, S-2, S-4	40				2.3153
S-3, S-3, S-1	41				6.3541
S-3, S-3, S-2	42				0.0000
S-3, S-3, S-3	43				0.0000
S-3, S-3, S-4	44				1.7567
S-3, S-4, S-1	45				15.0704
S-3, S-4, S-2	46				0.0000
S-3, S-4, S-3	47				0.0000
S-3, S-4, S-4	48				17.8579
S-4, S-1, S-1	49				5.0128
S-4, S-1, S-2	50				0.0000
S-4, S-1, S-3	51				0.0000
S-4, S-1, S-4	52				7.7735
S-4, S-2, S-1	53				0.0000
S-4, S-2, S-2	54				0.0000
S-4, S-2, S-3	55				0.0000
S-4, S-2, S-4	56				5.3704
S-4, S-3, S-1	57				0.0000
S-4, S-3, S-2	58				0.0000
S-4, S-3, S-3	59				0.0000
S-4, S-3, S-4	60				5.3704
S-4, S-4, S-1	61				15.0608
S-4, S-4, S-2	62				0.0000
S-4, S-4, S-3	63				0.0000
S-4, S-4, S-4	64				20.2298

We drop these cash flows into the cash flow table and multiply times the probability weighted discount factors from Chapter 9. Figure 3 shows these cash flows and produces the value of the cap: $7.7028. Note that the cash flows occur at time 4 even though the reference interest rate is known at time 3 because interest is paid in arrears, the normal money market practice.

Figure 3

	Maturity of Cash Flow Received				
	Current Time				
Row	0	1	2	3	4
1	0.0000	0.0000	0.0000	0.0000	31.1062
2		0.0000	0.0000	0.0000	25.6672
3		0.0000	0.0000	0.0000	21.7103
4		0.0000	0.0000	0.0000	50.3747
5			0.0000	0.0000	1.0407
6			0.0000	0.0000	0.0000
7			0.0000	0.0000	0.0000
8			0.0000	0.0000	10.6283
9			0.0000	0.0000	22.2100
10			0.0000	0.0000	0.0000
11			0.0000	0.0000	1.7977
12			0.0000	0.0000	25.0166
13			0.0000	0.0000	24.2520
14			0.0000	0.0000	18.8481
15			0.0000	0.0000	14.9168
16			0.0000	0.0000	43.3960
17				0.0000	11.5175
18				0.0000	0.0000
19				0.0000	0.0000
20				0.0000	14.2955
21				0.0000	0.0000
22				0.0000	0.0000
23				0.0000	0.0000
24				0.0000	0.0000
25				0.0000	0.0000
26				0.0000	0.0000
27				0.0000	0.0000
28				0.0000	0.0000
29				0.0000	0.0000
30				0.0000	0.0000
31				0.0000	0.0000
32				0.0000	0.0000
33				0.0000	0.0000
34				0.0000	0.0000
35				0.0000	0.0000
36				0.0000	2.3153
37				0.0000	0.0000
38				0.0000	0.0000
39				0.0000	0.0000
40				0.0000	2.3153
41				0.0000	6.3541
42				0.0000	0.0000
43				0.0000	0.0000
44				0.0000	1.7567
45				0.0000	15.0704
46				0.0000	0.0000
47				0.0000	0.0000
48				0.0000	17.8579
49				0.0000	5.0128
50				0.0000	0.0000
51				0.0000	0.0000
52				0.0000	7.7735
53				0.0000	0.0000
54				0.0000	0.0000
55				0.0000	0.0000
56				0.0000	5.3704
57				0.0000	0.0000
58				0.0000	0.0000
59				0.0000	0.0000
60				0.0000	5.3704
61				0.0000	15.0608
62				0.0000	0.0000
63				0.0000	0.0000
64				0.0000	20.2298
Risk Neutral Value =					7.7028

Valuing a Floor

What if we had agreed to pay our counterparty whenever the reference interest rate was below 1.5% on the same terms as before? Those terms were $1,000 in the reference money market instrument to be issued at time T=3 for maturity at time T=4. The cash flows on our promise to pay can be written Minimum (0, 1000*(spot rate - 1.5%)). Those cash flows are given in Figure 4:

Figure 4

		Time Period			
State	**Row**	**0**	**1**	**2**	**3**
S-1, S-1, S-1	1		0	0	0.0000
S-1, S-1, S-2	2		0	0	0.0000
S-1, S-1, S-3	3		0	0	0.0000
S-1, S-1, S-4	4		0	0	0.0000
S-1, S-2, S-1	5			0	0.0000
S-1, S-2, S-2	6			0	0.0000
S-1, S-2, S-3	7			0	0.0000
S-1, S-2, S-4	8			0	0.0000
S-1, S-3, S-1	9			0	0.0000
S-1, S-3, S-2	10			0	0.0000
S-1, S-3, S-3	11			0	0.0000
S-1, S-3, S-4	12			0	0.0000
S-1, S-4, S-1	13			0	0.0000
S-1, S-4, S-2	14			0	0.0000
S-1, S-4, S-3	15			0	0.0000
S-1, S-4, S-4	16			0	0.0000
S-2, S-1, S-1	17				0.0000
S-2, S-1, S-2	18				-5.5789
S-2, S-1, S-3	19				0.0000
S-2, S-1, S-4	20				0.0000
S-2, S-2, S-1	21				0.0000
S-2, S-2, S-2	22				-9.4690
S-2, S-2, S-3	23				-7.3327
S-2, S-2, S-4	24				-2.9963
S-2, S-3, S-1	25				0.0000
S-2, S-3, S-2	26				-18.7340
S-2, S-3, S-3	27				-12.4068
S-2, S-3, S-4	28				0.0000
S-2, S-4, S-1	29				0.0000
S-2, S-4, S-2	30				-2.2973
S-2, S-4, S-3	31				-7.8469
S-2, S-4, S-4	32				0.0000
S-3, S-1, S-1	33				0.0000
S-3, S-1, S-2	34				0.0000
S-3, S-1, S-3	35				-2.8592
S-3, S-1, S-4	36				0.0000
S-3, S-2, S-1	37				0.0000
S-3, S-2, S-2	38				0.0000
S-3, S-2, S-3	39				-2.8592
S-3, S-2, S-4	40				0.0000
S-3, S-3, S-1	41				0.0000
S-3, S-3, S-2	42				-8.9066
S-3, S-3, S-3	43				-2.5170
S-3, S-3, S-4	44				0.0000
S-3, S-4, S-1	45				0.0000
S-3, S-4, S-2	46				-2.1355
S-3, S-4, S-3	47				0.0000
S-3, S-4, S-4	48				0.0000
S-4, S-1, S-1	49				0.0000
S-4, S-1, S-2	50				-11.8831
S-4, S-1, S-3	51				-0.0659
S-4, S-1, S-4	52				0.0000
S-4, S-2, S-1	53				0.0000
S-4, S-2, S-2	54				0.0000
S-4, S-2, S-3	55				0.0000
S-4, S-2, S-4	56				0.0000
S-4, S-3, S-1	57				0.0000
S-4, S-3, S-2	58				0.0000
S-4, S-3, S-3	59				0.0000
S-4, S-3, S-4	60				0.0000
S-4, S-4, S-1	61				0.0000
S-4, S-4, S-2	62				0.0000
S-4, S-4, S-3	63				0.0000
S-4, S-4, S-4	64				0.0000

If we drop these cash flows into the cash flow table shown in Figure 5, we can calculate the value of our promise to pay on this floor contract to be -0.6528:

Figure 5

	Maturity of Cash Flow Received				
	Current Time				
Row	**0**	**1**	**2**	**3**	**4**
1	0.0000	0.0000	0.0000	0.0000	0.0000
2		0.0000	0.0000	0.0000	0.0000
3		0.0000	0.0000	0.0000	0.0000
4		0.0000	0.0000	0.0000	0.0000
5			0.0000	0.0000	0.0000
6			0.0000	0.0000	0.0000
7			0.0000	0.0000	0.0000
8			0.0000	0.0000	0.0000
9			0.0000	0.0000	0.0000
10			0.0000	0.0000	0.0000
11			0.0000	0.0000	0.0000
12			0.0000	0.0000	0.0000
13			0.0000	0.0000	0.0000
14			0.0000	0.0000	0.0000
15			0.0000	0.0000	0.0000
16			0.0000	0.0000	0.0000
17				0.0000	0.0000
18				0.0000	-5.5789
19				0.0000	0.0000
20				0.0000	0.0000
21				0.0000	0.0000
22				0.0000	-9.4690
23				0.0000	-7.3327
24				0.0000	-2.9963
25				0.0000	0.0000
26				0.0000	-18.7340
27				0.0000	-12.4068
28				0.0000	0.0000
29				0.0000	0.0000
30				0.0000	-2.2973
31				0.0000	-7.8469
32				0.0000	0.0000
33				0.0000	0.0000
34				0.0000	0.0000
35				0.0000	-2.8592
36				0.0000	0.0000
37				0.0000	0.0000
38				0.0000	0.0000
39				0.0000	-2.8592
40				0.0000	0.0000
41				0.0000	0.0000
42				0.0000	-8.9066
43				0.0000	-2.5170
44				0.0000	0.0000
45				0.0000	0.0000
46				0.0000	-2.1355
47				0.0000	0.0000
48				0.0000	0.0000
49				0.0000	0.0000
50				0.0000	-11.8831
51				0.0000	-0.0659
52				0.0000	0.0000
53				0.0000	0.0000
54				0.0000	0.0000
55				0.0000	0.0000
56				0.0000	0.0000
57				0.0000	0.0000
58				0.0000	0.0000
59				0.0000	0.0000
60				0.0000	0.0000
61				0.0000	0.0000
62				0.0000	0.0000
63				0.0000	0.0000
64				0.0000	0.0000
Risk Neutral Value =					**-0.6528**

Valuing a Floating Rate Loan with a Cap

We now embed the caps and floors in a loan contract. First, we revert to Chapter 19 and value the cash flows of a loan that pays the spot rate of interest in every period (see Figure 1) in arrears and pays the principal of $1000 at time T=4. As we showed in Chapter 19, the value of that loan is the par value of $1000 because we are using a valuation yield curve for that borrower, ABC Company, derived from observable zero coupon bonds that correctly reflect ABC's default risk and related market risk aversion. Figure 6 shows those cash flows and the resulting valuation, exactly par value of $1,000.00, which results from multiplying each cash flow times the relevant probability-weighted discount factor from Chapter 9. Note that, as in Chapter 9, there is one node with a negative interest rate in row 26 so the period 4 payment is less than par value:

Figure 6

	Maturity of Cash Flow Received				
	Current Time				
Row	0	1	2	3	4
1	0.0000	3.0032	21.6527	46.3884	1,061.1062
2		3.0032	21.6527	46.3884	1,055.6672
3		3.0032	21.6527	46.3884	1,051.7103
4		3.0032	21.6527	46.3884	1,080.3747
5			6.9105	14.1419	1,031.0407
6			6.9105	14.1419	1,025.9132
7			6.9105	14.1419	1,020.2913
8			6.9105	14.1419	1,040.6283
9			10.9723	26.0890	1,052.2100
10			10.9723	26.0890	1,019.7841
11			10.9723	26.0890	1,031.7977
12			10.9723	26.0890	1,055.0166
13			13.6107	49.1794	1,054.2520
14			13.6107	49.1794	1,048.8481
15			13.6107	49.1794	1,044.9168
16			13.6107	49.1794	1,073.3960
17				22.4959	1,041.5175
18				22.4959	1,009.4211
19				22.4959	1,021.3126
20				22.4959	1,044.2955
21				5.9835	1,022.0359
22				5.9835	1,005.5310
23				5.9835	1,007.6673
24				5.9835	1,012.0037
25				8.1208	1,026.2311
26				8.1208	996.2660
27				8.1208	1,002.5932
28				8.1208	1,021.6787
29				12.4592	1,017.7642
30				12.4592	1,012.7027
31				12.4592	1,007.1531
32				12.4592	1,027.2283
33				19.2173	1,022.8044
34				19.2173	1,017.7178
35				19.2173	1,012.1408
36				19.2173	1,032.3153
37				14.1486	1,022.8044
38				14.1486	1,017.7178
39				14.1486	1,012.1408
40				14.1486	1,032.3153
41				8.5911	1,036.3541
42				8.5911	1,006.0934
43				8.5911	1,012.4830
44				8.5911	1,031.7567
45				28.6949	1,045.0704
46				28.6949	1,012.8645
47				28.6949	1,024.7966
48				28.6949	1,047.8579
49				22.0883	1,035.0128
50				22.0883	1,003.1169
51				22.0883	1,014.9341
52				22.0883	1,037.7735
53				17.0053	1,025.8313
54				17.0053	1,020.7297
55				17.0053	1,015.1361
56				17.0053	1,035.3704
57				11.4322	1,025.8313
58				11.4322	1,020.7297
59				11.4322	1,015.1361
60				11.4322	1,035.3704
61				31.5926	1,045.0608
62				31.5926	1,022.6860
63				31.5926	1,027.2519
64				31.5926	1,050.2298
Risk Neutral Value =					1,000.0000

Now let's embed a cap on the interest rate in the loan of 3.00%, applicable only to the interest payment made at time T=4 based on the reference rate at time T=3. This changes the cash flows to those shown in Figure 7 and produces a valuation of $992.2972:

Figure 7

	Maturity of Cash Flow Received				
	Current Time				
Row	**0**	**1**	**2**	**3**	**4**
1	0.0000	3.0032	21.6527	46.3884	1,030.0000
2		3.0032	21.6527	46.3884	1,030.0000
3		3.0032	21.6527	46.3884	1,030.0000
4		3.0032	21.6527	46.3884	1,030.0000
5			6.9105	14.1419	1,030.0000
6			6.9105	14.1419	1,025.9132
7			6.9105	14.1419	1,020.2913
8			6.9105	14.1419	1,030.0000
9			10.9723	26.0890	1,030.0000
10			10.9723	26.0890	1,019.7841
11			10.9723	26.0890	1,030.0000
12			10.9723	26.0890	1,030.0000
13			13.6107	49.1794	1,030.0000
14			13.6107	49.1794	1,030.0000
15			13.6107	49.1794	1,030.0000
16			13.6107	49.1794	1,030.0000
17				22.4959	1,030.0000
18				22.4959	1,009.4211
19				22.4959	1,021.3126
20				22.4959	1,030.0000
21				5.9835	1,022.0359
22				5.9835	1,005.5310
23				5.9835	1,007.6673
24				5.9835	1,012.0037
25				8.1208	1,026.2311
26				8.1208	996.2660
27				8.1208	1,002.5932
28				8.1208	1,021.6787
29				12.4592	1,017.7642
30				12.4592	1,012.7027
31				12.4592	1,007.1531
32				12.4592	1,027.2283
33				19.2173	1,022.8044
34				19.2173	1,017.7178
35				19.2173	1,012.1408
36				19.2173	1,030.0000
37				14.1486	1,022.8044
38				14.1486	1,017.7178
39				14.1486	1,012.1408
40				14.1486	1,030.0000
41				8.5911	1,030.0000
42				8.5911	1,006.0934
43				8.5911	1,012.4830
44				8.5911	1,030.0000
45				28.6949	1,030.0000
46				28.6949	1,012.8645
47				28.6949	1,024.7966
48				28.6949	1,030.0000
49				22.0883	1,030.0000
50				22.0883	1,003.1169
51				22.0883	1,014.9341
52				22.0883	1,030.0000
53				17.0053	1,025.8313
54				17.0053	1,020.7297
55				17.0053	1,015.1361
56				17.0053	1,030.0000
57				11.4322	1,025.8313
58				11.4322	1,020.7297
59				11.4322	1,015.1361
60				11.4322	1,030.0000
61				31.5926	1,030.0000
62				31.5926	1,022.6860
63				31.5926	1,027.2519
64				31.5926	1,030.0000
Risk Neutral Value =					**992.2972**

To check that there is no arbitrage between the loan market and cap market, the value of the loan with no cap ($1000.000) less the value of the cap that the lender has given to the borrower ($7.7028) should equal the value of the loan. We can confirm that the loan value of $992.2972 matches exactly.

Value of a Loan with a Cap and a Floor

Now we modify the terms of the loan one more time. We assume that the borrower agrees to a floor of 1.5% on the interest payment to be made at time T=4 based on the reference rate levels prevailing at time T=3. In that case, the cash flows are as shown in Figure 8 and the resulting value of the loan is $992.9501.

Figure 8

	Maturity of Cash Flow Received				
	Current Time				
Row	0	1	2	3	4
1	0.0000	3.0032	21.6527	46.3884	1,030.0000
2		3.0032	21.6527	46.3884	1,030.0000
3		3.0032	21.6527	46.3884	1,030.0000
4		3.0032	21.6527	46.3884	1,030.0000
5			6.9105	14.1419	1,030.0000
6			6.9105	14.1419	1,025.9132
7			6.9105	14.1419	1,020.2913
8			6.9105	14.1419	1,030.0000
9			10.9723	26.0890	1,030.0000
10			10.9723	26.0890	1,019.7841
11			10.9723	26.0890	1,030.0000
12			10.9723	26.0890	1,030.0000
13			13.6107	49.1794	1,030.0000
14			13.6107	49.1794	1,030.0000
15			13.6107	49.1794	1,030.0000
16			13.6107	49.1794	1,030.0000
17				22.4959	1,030.0000
18				22.4959	1,015.0000
19				22.4959	1,021.3126
20				22.4959	1,030.0000
21				5.9835	1,022.0359
22				5.9835	1,015.0000
23				5.9835	1,015.0000
24				5.9835	1,015.0000
25				8.1208	1,026.2311
26				8.1208	1,015.0000
27				8.1208	1,015.0000
28				8.1208	1,021.6787
29				12.4592	1,017.7642
30				12.4592	1,015.0000
31				12.4592	1,015.0000
32				12.4592	1,027.2283
33				19.2173	1,022.8044
34				19.2173	1,017.7178
35				19.2173	1,015.0000
36				19.2173	1,030.0000
37				14.1486	1,022.8044
38				14.1486	1,017.7178
39				14.1486	1,015.0000
40				14.1486	1,030.0000
41				8.5911	1,030.0000
42				8.5911	1,015.0000
43				8.5911	1,015.0000
44				8.5911	1,030.0000
45				28.6949	1,030.0000
46				28.6949	1,015.0000
47				28.6949	1,024.7966
48				28.6949	1,030.0000
49				22.0883	1,030.0000
50				22.0883	1,015.0000
51				22.0883	1,015.0000
52				22.0883	1,030.0000
53				17.0053	1,025.8313
54				17.0053	1,020.7297
55				17.0053	1,015.1361
56				17.0053	1,030.0000
57				11.4322	1,025.8313
58				11.4322	1,020.7297
59				11.4322	1,015.1361
60				11.4322	1,030.0000
61				31.5926	1,030.0000
62				31.5926	1,022.6860
63				31.5926	1,027.2519
64				31.5926	1,030.0000
Risk Neutral Value =					992.9501

In the absence of arbitrage, the loan value should increase by 0.6528 (which is the value of the floor agreed by the borrower to the lender's favor) and we confirm that $992.2972+0.6528 = 992.9501 (after rounding).

Variations on Caps and Floors

We saw in Chapters 6 through 9 that it is easy to value a digital option on interest rates in the HJM framework. Indeed, there are an infinite variety of "derivatives" that can be priced in the HJM framework. Any function of interest rates can be dropped into the cash flow table and valued on a risk neutral basis. Extending the coverage of the caps and floors to the first 3 years of the loan is very simple. This means that the interest rate on the loan will vary only between 1.5% and 3.00% for the life of the loan. The variety of structures that can be priced (and hedged) is truly mind-boggling. That being said, only a few of the structures one can imagine have real value to both borrowers and lenders so there is a practical limit to the complexity that one sees in the market place. As we saw in Chapter 20, in the case of the CDO market, the level of complexity exceeded the level of complexity that could be quickly and accurate priced, and the CDO structure has gone into hibernation, perhaps permanently.

In valuing and hedging complex interest rate derivatives, the analyst must be sure that the HJM framework is properly adjusted. For example, the simple 4 step bushy tree from the 3 factor HJM model in Chapter 9 has only 4 nodes at time T=1. If one is pricing instruments with a short term maturity, the granularity of the analysis has to be much greater than 4 outcomes in a bushy tree or Monte Carlo simulation. The degree of granularity necessary can be calculated explicitly once an analyst can state the accuracy of valuation that is necessary. One such definition of accuracy is the sentence "I need the model valuation to be within the bid-offered spread for this instrument 95% of the time." This defines the maximum sampling error that can be tolerated, and that in turn defines the number of nodes on a bushy tree or the number of Monte Carl scenarios that are needed. See Fishman [1996] for an example of this calculation.

In the reminder of this book, we cover the most practical variations on interest rate structures.

Measuring the Credit Risk of Counterparties on Caps and Floors

All of our comments in Chapters 24-26 regarding the valuation of 'vulnerable' derivatives apply here as well. Since the default probability of our counterparty on the cap or floor is very likely to be driven by interest rate levels, the credit risk-adjusted valuation of caps and floors will involve either a complex closed form solution in a reduced form model or (more accurately) the equivalent Monte Carlo simulation with explicit interest rate drivers of our counterparties' default probabilities. This is a standard capability of a best practice enterprise risk management system. Clearly, a 10-year cap on interest rates purchased in the early 1980s from a soon-to-be bankrupt savings and loan association (driven into bankruptcy by high interest rates in the U.S.) is much less valuable than the same cap on U.S. interest rates provided by a non-U.S. financial institution since their exposure to U.S. interest rate risk is so much less than a savings and loan association in the early 1980s, just prior to the worst spike in interest rates in U.S. history.

Best practice requires that we take this into account in order to avoid the $1 trillion in losses that U.S. tax-payers suffered from failed financial institutions from the savings and loan crisis. Similarly, a systematic risk factor like home prices drove much of the second $1 trillion bail-out in the United States in the 2007 to 2010 period. The FDIC Loss Distribution Model announced on December 10 2003 (and discussed frequently in this book) does exactly this. Given that the 2007 to 2010 credit crisis firmly validated the FDIC modeling approach (which unfortunately came too late to stave off the crisis), one would hope that the probability of the next crisis can be calculated precisely and early. Perhaps this will lead to preventative action by bankers and regulators. Then again, perhaps it will not. The readers of this book will have much to say about that.