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Advanced Financial Risk Management:

Tools and Techniques for Integrated Credit Risk, Market Risk, Liquidity Risk and Interest Rate Risk Management

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Part 4: Risk Management Applications, Instrument by Instrument

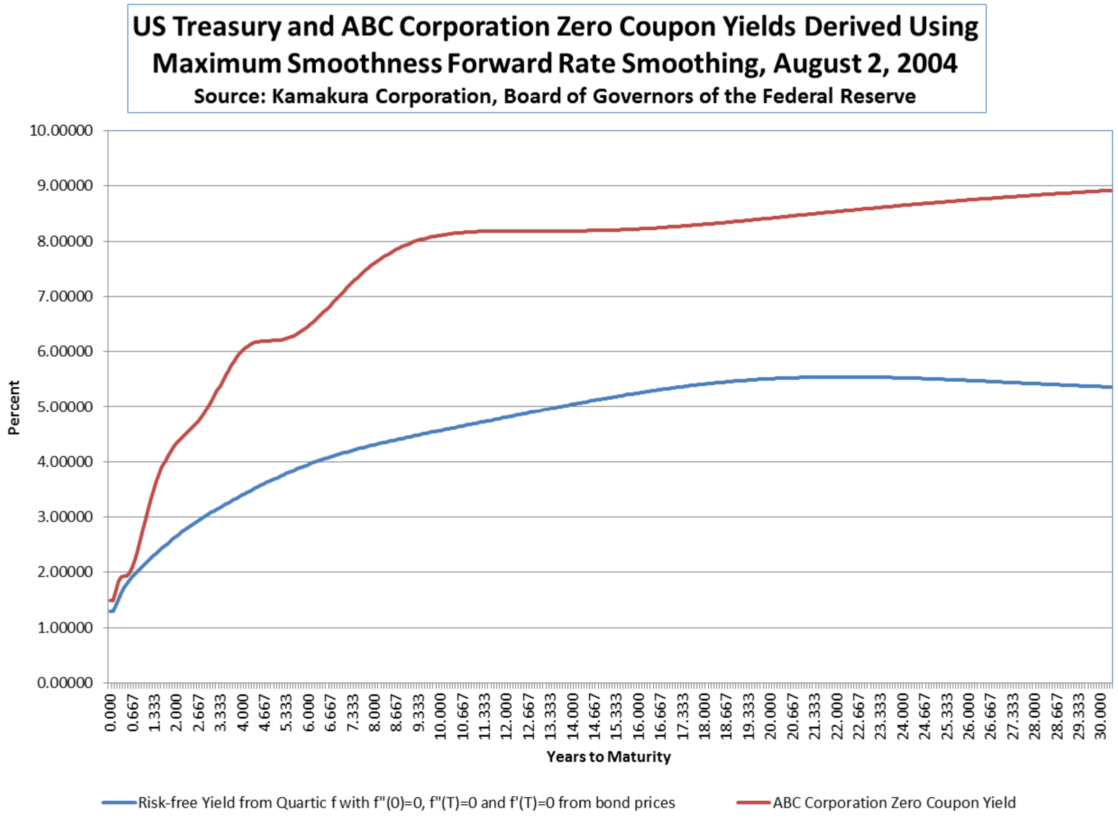
Chapter 21

European options on bonds

In this chapter, we use the three factor Heath Jarrow and Morton yield curve model to value European options on corporate bonds. As we noted in Chapter 19, Robert Jarrow ("Risky Coupon Bonds as a Portfolio of Zero-Coupon Bonds," *Finance Research Letters*, 1 (2), June 2004) has summarized the conditions under which a risky bond issuer's (we will call them ABC Corporation in this chapter) coupon-bearing bond issues can be analyzed as a portfolio of zero coupon bonds using the formulas of Chapter 16 and Chapter 19. We explained in Chapter 17 that we can create today's yield curve for ABC Corporation in one of two ways. In method one, we estimate the term structure of default risk using logistic regression for ABC and then apply the techniques of Chapter 17 to estimate what the intersection of supply and demand for credit would be for ABC corporation on that day; this would be the yield curve that we would use to price ABC securities and derivatives on them. The second method is to collect data on existing ABC bond prices and apply the yield curve smoothing techniques of Chapter 5 to extract a smooth, continuous yield curve.

As an example, we looked at the net present value ("dirty price," which is price plus accrued interest) on 77 bonds for a prestigious U.S. bond issuer on August 2, 2004. We smoothed the U.S. Treasury curve using the maximum smoothness forward rate approach of Chapter 5. We then solved for the maximum smoothness forward credit spread curve that minimized the sum of squared pricing errors on the bonds of this firm, which we call ABC Corporation. The zero coupon yield curves for the risk free curve and the ABC Corporation curve are shown in Figure 1:

Figure 1



We could then apply a Heath Jarrow and Morton interest rate model to either generate Monte Carlo simulations for both the risk free and ABC yield curves (this is the most common and most accurate technique) or to generate an HJM bushy tree. The bushy tree approach is the preferred approach for American options. We will use the 3 factor HJM bushy tree from Chapter 9 for expository purposes. We will assume in this chapter that the bushy tree in Chapter 9 is the ABC Corporation bushy tree. We could derive it in exactly the same manner we did in Chapter 9, but we leave that to the reader.

We will use the following information from Chapter 9:

1. The zero coupon bonds used as input

2. The volatility assumptions for the three factors

3. The resulting zero coupon bond prices, zero coupon bond yields, and un-compounding spot rates of interest

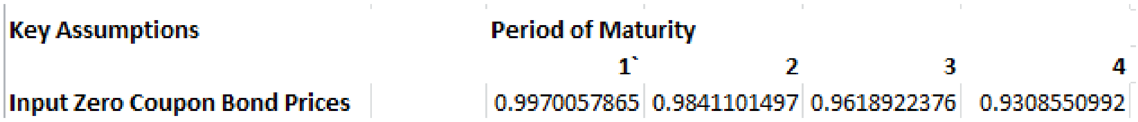
4. The probability of occurrence of the 4 nodes at time step 1, the 16 nodes at time step 2, and the 64 nodes that are relevant for time steps 3 and 4

5. The present value factors for each of those nodes

6. Most importantly, the probability weighted discount factors for each node.

For all of the examples in this chapter, we will price derivatives on an underlying reference bond of ABC Corporation. We choose as our reference instrument the 3% bond paying $3 of interest at times 1, 2, 3 and 4 and principal of $100 at time 4. We could price this bond in one of two ways. We could multiply the cash flows by the zero coupon bond prices that we assumed were visible in the market. We show them again in Figure 2:

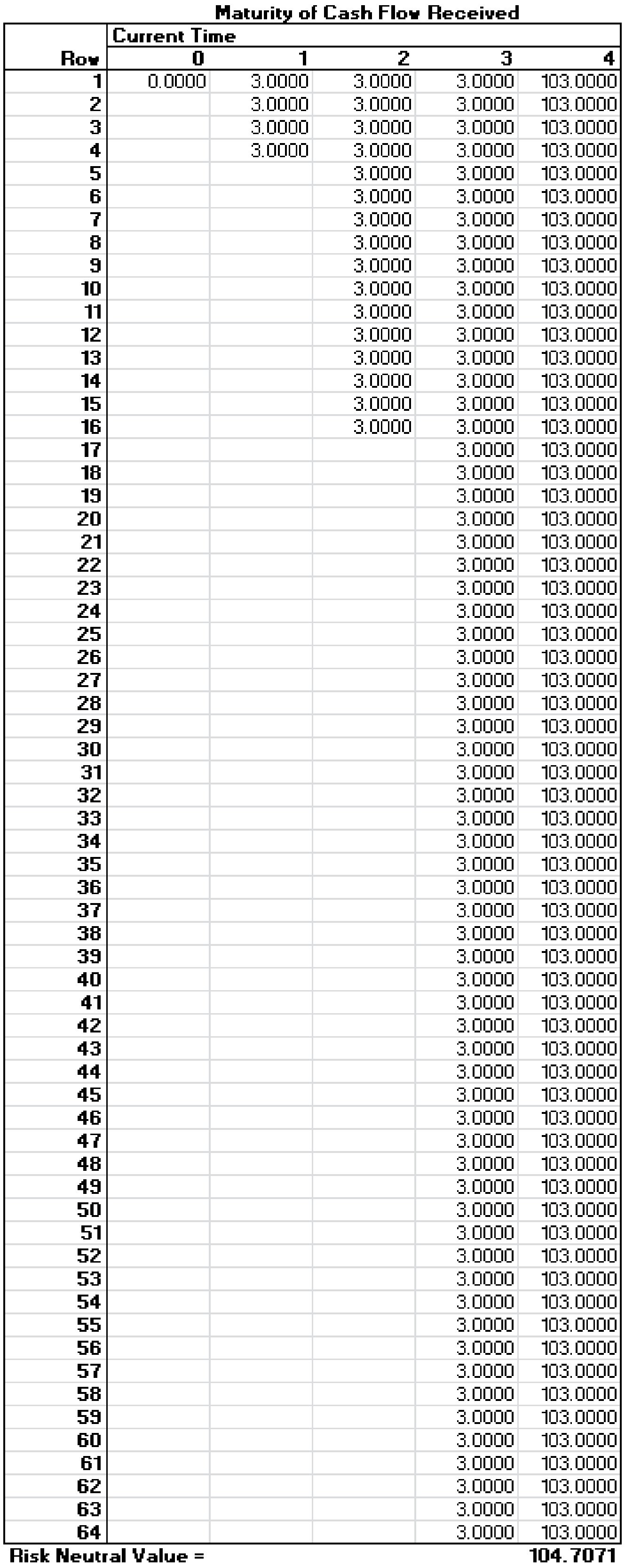
Figure 2



Key Assumptions	**Period of Maturity**			
	1`	**2**	**3**	**4**
Input Zero Coupon Bond Prices	0.9970057865	0.9841101497	0.9618922376	0.9308550992

Alternatively, we could populate the cash flow table cash flows received at each point of time in each state, as shown in Figure 3:

Figure 3

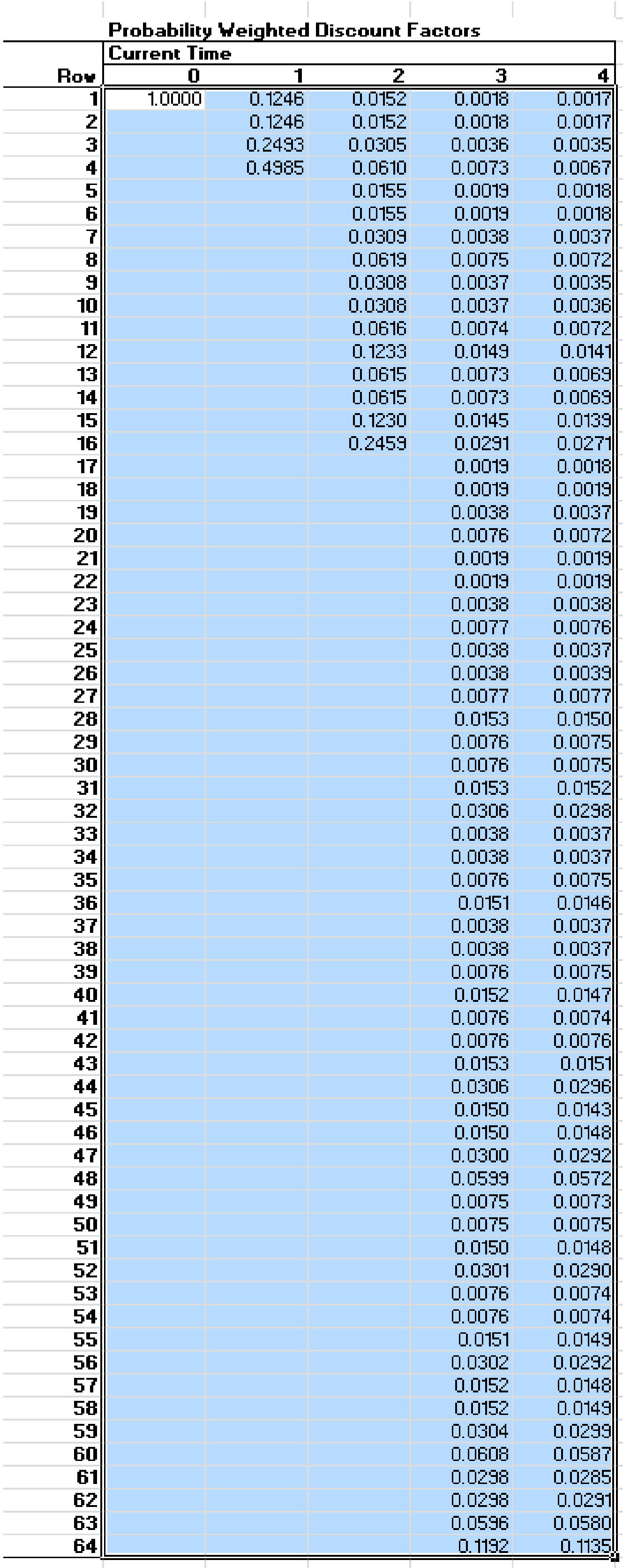


Maturity of Cash Flow Received

	Current Time				
Row	**0**	**1**	**2**	**3**	**4**
1	0.0000	3.0000	3.0000	3.0000	103.0000
2		3.0000	3.0000	3.0000	103.0000
3		3.0000	3.0000	3.0000	103.0000
4		3.0000	3.0000	3.0000	103.0000
5			3.0000	3.0000	103.0000
6			3.0000	3.0000	103.0000
7			3.0000	3.0000	103.0000
8			3.0000	3.0000	103.0000
9			3.0000	3.0000	103.0000
10			3.0000	3.0000	103.0000
11			3.0000	3.0000	103.0000
12			3.0000	3.0000	103.0000
13			3.0000	3.0000	103.0000
14			3.0000	3.0000	103.0000
15			3.0000	3.0000	103.0000
16			3.0000	3.0000	103.0000
17				3.0000	103.0000
18				3.0000	103.0000
19				3.0000	103.0000
20				3.0000	103.0000
21				3.0000	103.0000
22				3.0000	103.0000
23				3.0000	103.0000
24				3.0000	103.0000
25				3.0000	103.0000
26				3.0000	103.0000
27				3.0000	103.0000
28				3.0000	103.0000
29				3.0000	103.0000
30				3.0000	103.0000
31				3.0000	103.0000
32				3.0000	103.0000
33				3.0000	103.0000
34				3.0000	103.0000
35				3.0000	103.0000
36				3.0000	103.0000
37				3.0000	103.0000
38				3.0000	103.0000
39				3.0000	103.0000
40				3.0000	103.0000
41				3.0000	103.0000
42				3.0000	103.0000
43				3.0000	103.0000
44				3.0000	103.0000
45				3.0000	103.0000
46				3.0000	103.0000
47				3.0000	103.0000
48				3.0000	103.0000
49				3.0000	103.0000
50				3.0000	103.0000
51				3.0000	103.0000
52				3.0000	103.0000
53				3.0000	103.0000
54				3.0000	103.0000
55				3.0000	103.0000
56				3.0000	103.0000
57				3.0000	103.0000
58				3.0000	103.0000
59				3.0000	103.0000
60				3.0000	103.0000
61				3.0000	103.0000
62				3.0000	103.0000
63				3.0000	103.0000
64				3.0000	103.0000
Risk Neutral Value =					**104.7071**

We then multiply these cash flows by the matching probability weighted discount factor from the table in Figure 4:

Figure 4

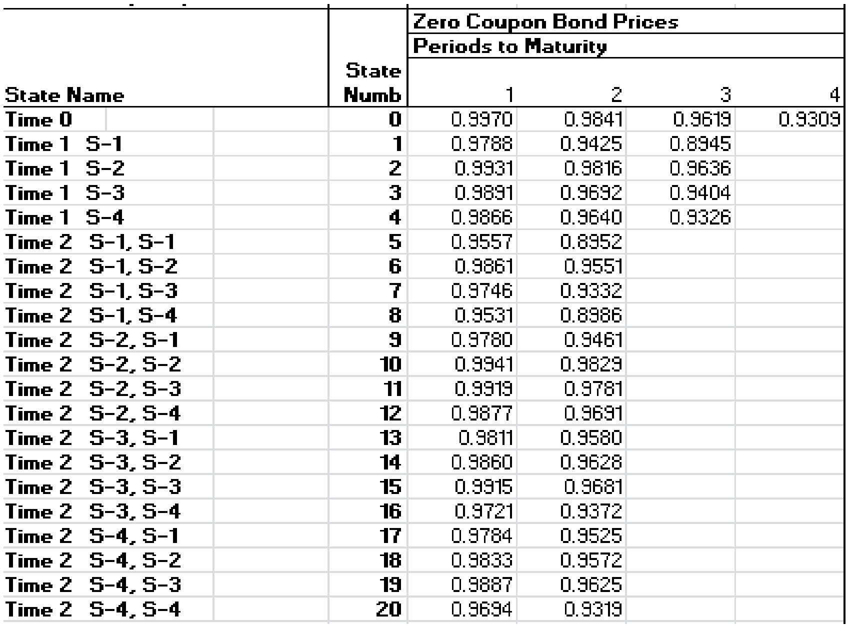


Probability Weighted Discount Factors

	Current Time				
Row	0	1	2	3	4
1	1.0000	0.1246	0.0152	0.0018	0.0017
2		0.1246	0.0152	0.0018	0.0017
3		0.2493	0.0305	0.0036	0.0035
4		0.4985	0.0610	0.0073	0.0067
5			0.0155	0.0019	0.0018
6			0.0155	0.0019	0.0018
7			0.0309	0.0038	0.0037
8			0.0619	0.0075	0.0072
9			0.0308	0.0037	0.0035
10			0.0308	0.0037	0.0036
11			0.0616	0.0074	0.0072
12			0.1233	0.0149	0.0141
13			0.0615	0.0073	0.0069
14			0.0615	0.0073	0.0069
15			0.1230	0.0145	0.0139
16			0.2459	0.0291	0.0271
17				0.0019	0.0018
18				0.0019	0.0019
19				0.0038	0.0037
20				0.0076	0.0072
21				0.0019	0.0019
22				0.0019	0.0019
23				0.0038	0.0038
24				0.0077	0.0076
25				0.0038	0.0037
26				0.0038	0.0039
27				0.0077	0.0077
28				0.0153	0.0150
29				0.0076	0.0075
30				0.0076	0.0075
31				0.0153	0.0152
32				0.0306	0.0298
33				0.0038	0.0037
34				0.0038	0.0037
35				0.0076	0.0075
36				0.0151	0.0146
37				0.0038	0.0037
38				0.0038	0.0037
39				0.0076	0.0075
40				0.0152	0.0147
41				0.0076	0.0074
42				0.0076	0.0076
43				0.0153	0.0151
44				0.0306	0.0296
45				0.0150	0.0143
46				0.0150	0.0148
47				0.0300	0.0292
48				0.0599	0.0572
49				0.0075	0.0073
50				0.0075	0.0075
51				0.0150	0.0148
52				0.0301	0.0290
53				0.0076	0.0074
54				0.0076	0.0074
55				0.0151	0.0149
56				0.0302	0.0292
57				0.0152	0.0148
58				0.0152	0.0149
59				0.0304	0.0299
60				0.0608	0.0587
61				0.0298	0.0285
62				0.0298	0.0291
63				0.0596	0.0580
64				0.1192	0.1135

Regardless of the method chosen, we get the same values because we have been careful to construct the bushy tree so that the input zero coupon bond prices are correctly valued using the bushy tree: 104.7071 is the value for the 3% coupon-bearing bond. The table we will use the most in this chapter is the term structure of zero coupon bond prices that will prevail at time 0 (1 state), time 1 (4 states), and especially time 2 (16 states), because we will be analyzing European options with an exercise time of 2 years. We see in Figure 5 that there are 16 combinations of zero coupon bonds that mature at time T=3 and T=4:

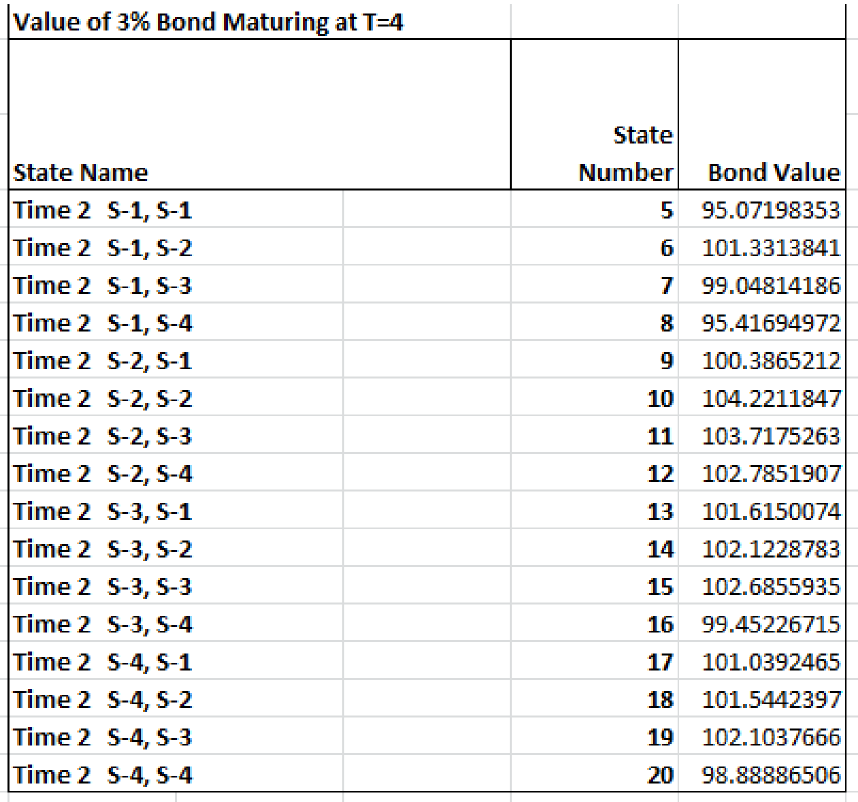
Figure 5



		Zero Coupon Bond Prices			
		Periods to Maturity			
State Name	State Numb	1	2	3	4
Time 0	0	0.9970	0.9841	0.9619	0.9309
Time 1 S-1	1	0.9788	0.9425	0.8945	
Time 1 S-2	2	0.9931	0.9816	0.9636	
Time 1 S-3	3	0.9891	0.9692	0.9404	
Time 1 S-4	4	0.9866	0.9640	0.9326	
Time 2 S-1, S-1	5	0.9557	0.8952		
Time 2 S-1, S-2	6	0.9861	0.9551		
Time 2 S-1, S-3	7	0.9746	0.9332		
Time 2 S-1, S-4	8	0.9531	0.8986		
Time 2 S-2, S-1	9	0.9780	0.9461		
Time 2 S-2, S-2	10	0.9941	0.9829		
Time 2 S-2, S-3	11	0.9919	0.9781		
Time 2 S-2, S-4	12	0.9877	0.9691		
Time 2 S-3, S-1	13	0.9811	0.9580		
Time 2 S-3, S-2	14	0.9860	0.9628		
Time 2 S-3, S-3	15	0.9915	0.9681		
Time 2 S-3, S-4	16	0.9721	0.9372		
Time 2 S-4, S-1	17	0.9784	0.9525		
Time 2 S-4, S-2	18	0.9833	0.9572		
Time 2 S-4, S-3	19	0.9887	0.9625		
Time 2 S-4, S-4	20	0.9694	0.9319		

To get the value of the bond as of time T=2, we multiply the \$3 received at time T=3 and the \$103 received at time T=4 by the appropriate zero coupon bond price in each state (we assume we are doing the analysis 1 second after the \$3 coupon due at time T=2 has been paid). We get a different value for the bond in each of the 16 states:

Figure 6



Value of 3% Bond Maturing at T=4

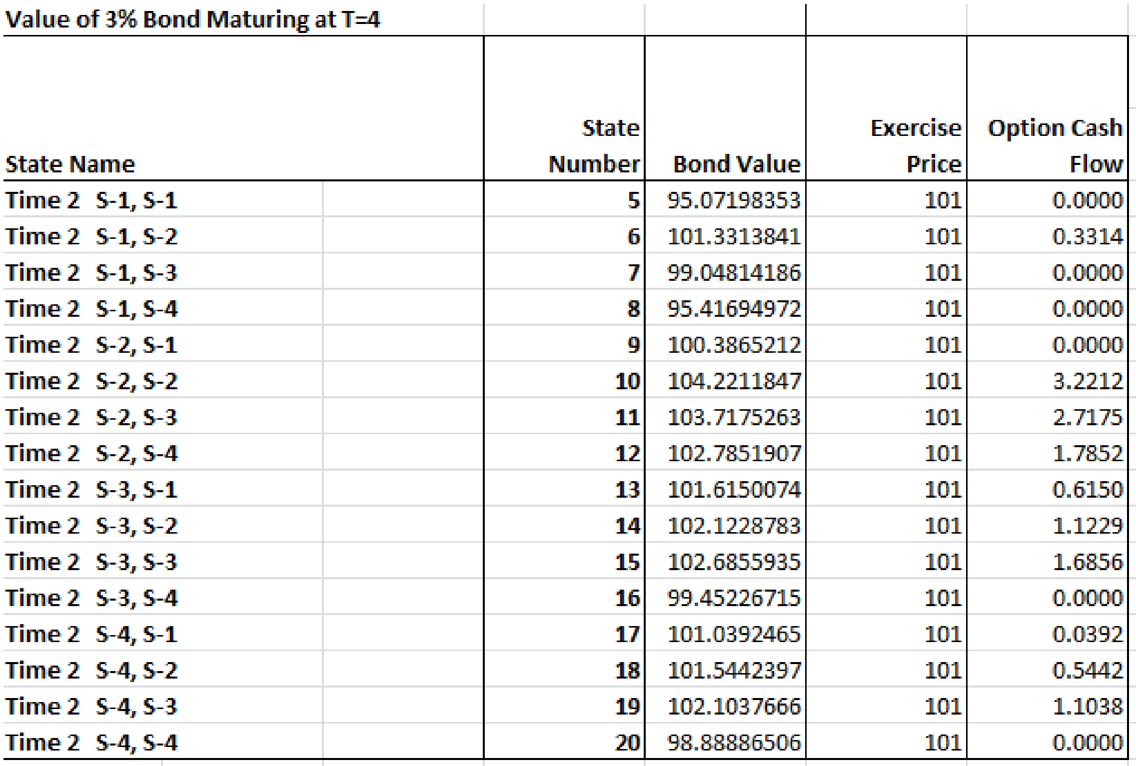
State Name	State Number	Bond Value
Time 2 S-1, S-1	5	95.07198353
Time 2 S-1, S-2	6	101.3313841
Time 2 S-1, S-3	7	99.04814186
Time 2 S-1, S-4	8	95.41694972
Time 2 S-2, S-1	9	100.3865212
Time 2 S-2, S-2	10	104.2211847
Time 2 S-2, S-3	11	103.7175263
Time 2 S-2, S-4	12	102.7851907
Time 2 S-3, S-1	13	101.6150074
Time 2 S-3, S-2	14	102.1228783
Time 2 S-3, S-3	15	102.6855935
Time 2 S-3, S-4	16	99.45226715
Time 2 S-4, S-1	17	101.0392465
Time 2 S-4, S-2	18	101.5442397
Time 2 S-4, S-3	19	102.1037666
Time 2 S-4, S-4	20	98.88886506

Now that we know all of the relevant time T=2 values for the reference instrument, we can value European options on this instrument.

Example: European call option on coupon-bearing bond

We first value a European call option that gives us the right, but not the obligation, to buy the 3% coupon bond at a price of 101 at time T=2, 1 second after the time T=2 coupon of $3 has been paid. We assume in this chapter that we will exercise this call option rationally. In later chapters we discuss retail options where "irrational exercise" is a possibility. We will exercise our option and buy the bond whenever the time T=2 price is more than 101. The cash flow in this case will be the price of the bond minus the 101 exercise price. If exercise of the option is not rational, the cash flow is zero. The cash flows attributable to the option are shown in Figure 7:

Figure 7

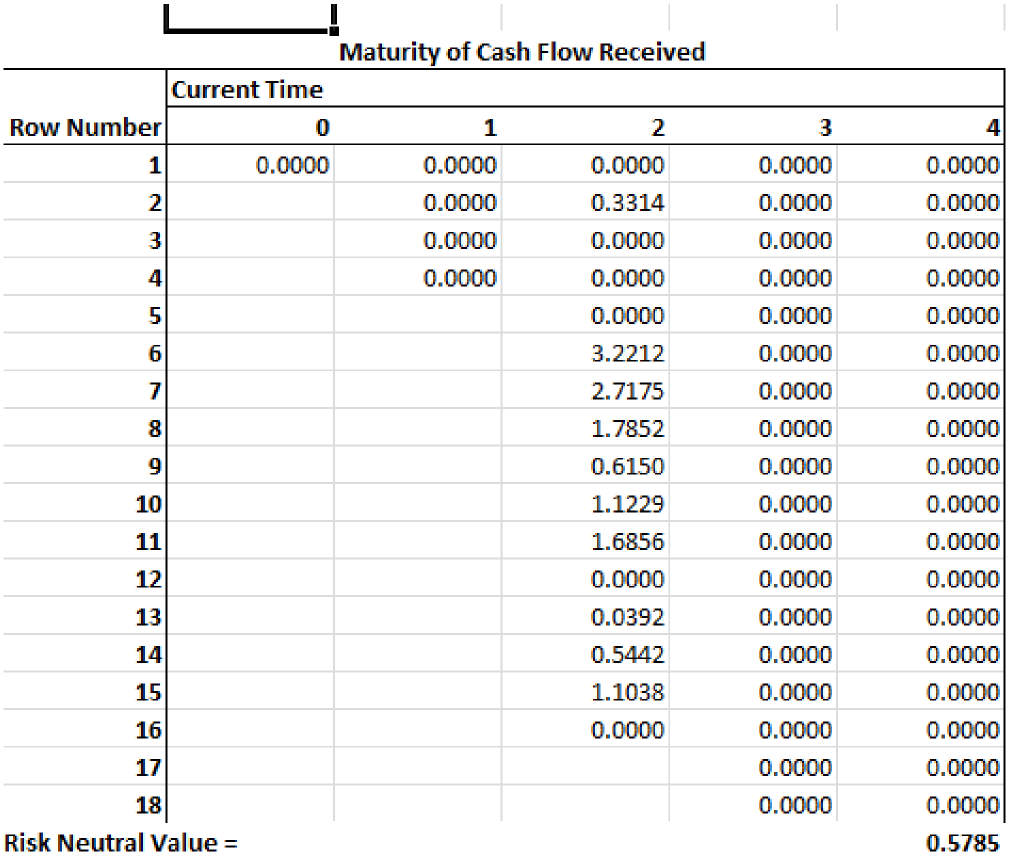


Value of 3% Bond Maturing at T=4

State Name	State Number	Bond Value	Exercise Price	Option Cash Flow
Time 2 S-1, S-1	5	95.07198353	101	0.0000
Time 2 S-1, S-2	6	101.3313841	101	0.3314
Time 2 S-1, S-3	7	99.04814186	101	0.0000
Time 2 S-1, S-4	8	95.41694972	101	0.0000
Time 2 S-2, S-1	9	100.3865212	101	0.0000
Time 2 S-2, S-2	10	104.2211847	101	3.2212
Time 2 S-2, S-3	11	103.7175263	101	2.7175
Time 2 S-2, S-4	12	102.7851907	101	1.7852
Time 2 S-3, S-1	13	101.6150074	101	0.6150
Time 2 S-3, S-2	14	102.1228783	101	1.1229
Time 2 S-3, S-3	15	102.6855935	101	1.6856
Time 2 S-3, S-4	16	99.45226715	101	0.0000
Time 2 S-4, S-1	17	101.0392465	101	0.0392
Time 2 S-4, S-2	18	101.5442397	101	0.5442
Time 2 S-4, S-3	19	102.1037666	101	1.1038
Time 2 S-4, S-4	20	98.88886506	101	0.0000

We then drop these option values into the cash flow table and multiply them by the respective probability-weighted discount factor. The cash flow table is shown in Figure 8, with the time T=3 and T=4 states partially hidden because they are all zero. The value of the European option to buy the bond at time T=2 at a price of 101 is 0.5785:

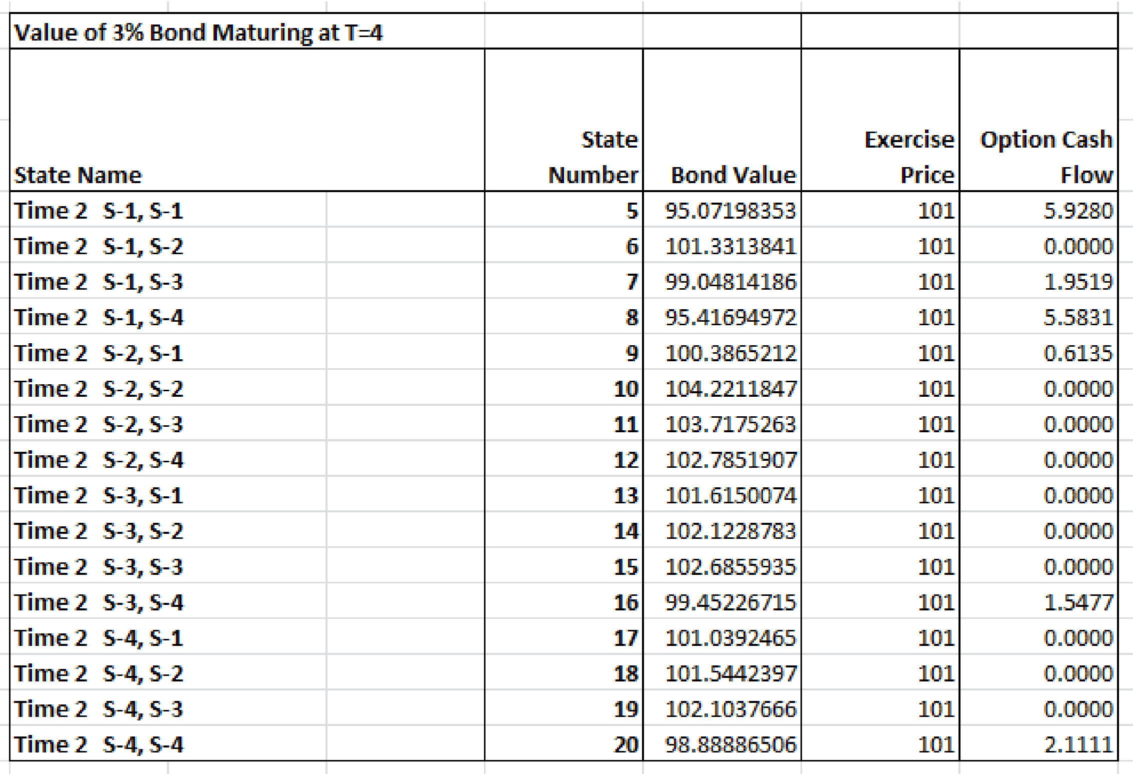
Figure 8



	Maturity of Cash Flow Received				
	Current Time				
Row Number	0	1	2	3	4
1	0.0000	0.0000	0.0000	0.0000	0.0000
2		0.0000	0.3314	0.0000	0.0000
3		0.0000	0.0000	0.0000	0.0000
4		0.0000	0.0000	0.0000	0.0000
5			0.0000	0.0000	0.0000
6			3.2212	0.0000	0.0000
7			2.7175	0.0000	0.0000
8			1.7852	0.0000	0.0000
9			0.6150	0.0000	0.0000
10			1.1229	0.0000	0.0000
11			1.6856	0.0000	0.0000
12			0.0000	0.0000	0.0000
13			0.0392	0.0000	0.0000
14			0.5442	0.0000	0.0000
15			1.1038	0.0000	0.0000
16			0.0000	0.0000	0.0000
17				0.0000	0.0000
18				0.0000	0.0000
Risk Neutral Value =					**0.5785**

What if the European option were a PUT option, not a CALL option? That gives us the right to sell the bond at a price of 101 at time T=2. The cash flows from rational exercise of this put option are shown in Figure 9:

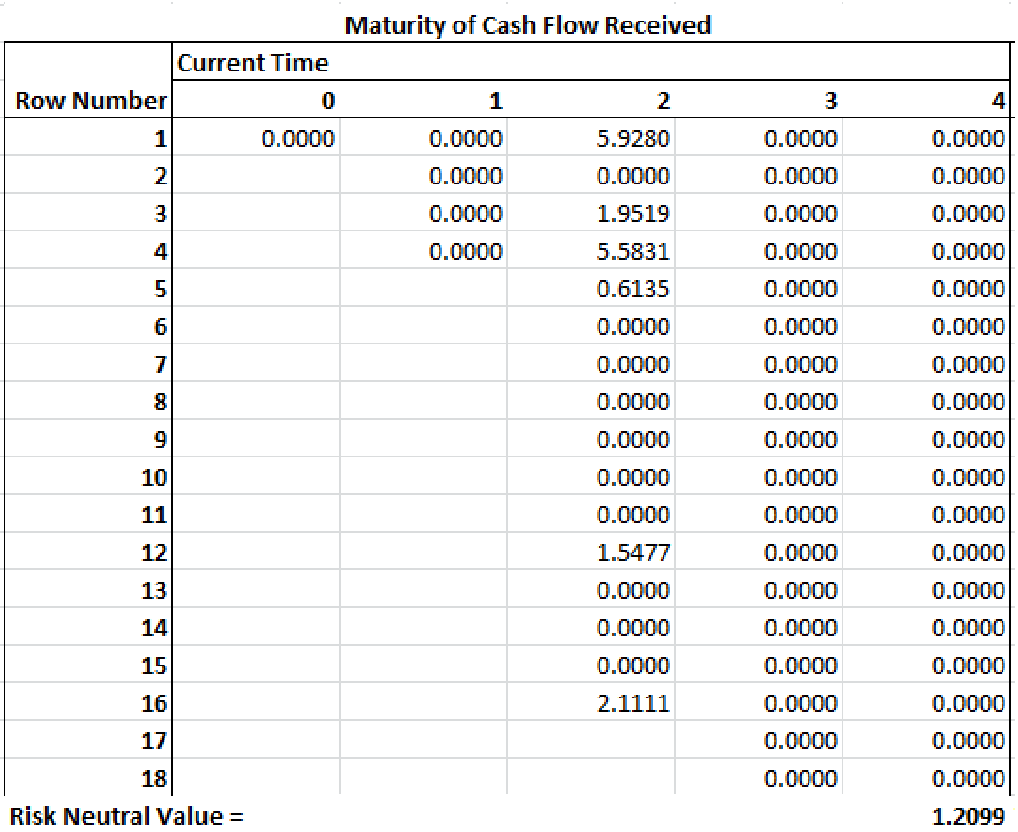
Figure 9



Value of 3% Bond Maturing at T=4				
State Name	State Number	Bond Value	Exercise Price	Option Cash Flow
Time 2 S-1, S-1	5	95.07198353	101	5.9280
Time 2 S-1, S-2	6	101.3313841	101	0.0000
Time 2 S-1, S-3	7	99.04814186	101	1.9519
Time 2 S-1, S-4	8	95.41694972	101	5.5831
Time 2 S-2, S-1	9	100.3865212	101	0.6135
Time 2 S-2, S-2	10	104.2211847	101	0.0000
Time 2 S-2, S-3	11	103.7175263	101	0.0000
Time 2 S-2, S-4	12	102.7851907	101	0.0000
Time 2 S-3, S-1	13	101.6150074	101	0.0000
Time 2 S-3, S-2	14	102.1228783	101	0.0000
Time 2 S-3, S-3	15	102.6855935	101	0.0000
Time 2 S-3, S-4	16	99.45226715	101	1.5477
Time 2 S-4, S-1	17	101.0392465	101	0.0000
Time 2 S-4, S-2	18	101.5442397	101	0.0000
Time 2 S-4, S-3	19	102.1037666	101	0.0000
Time 2 S-4, S-4	20	98.88886506	101	2.1111

We again drop these put option cash flows into the cash flow table and multiply them by the respective probability weighted discount factor. We show the cash flow table in Figure 10, along with the put option value of 1.2099.

Figure 10



Maturity of Cash Flow Received

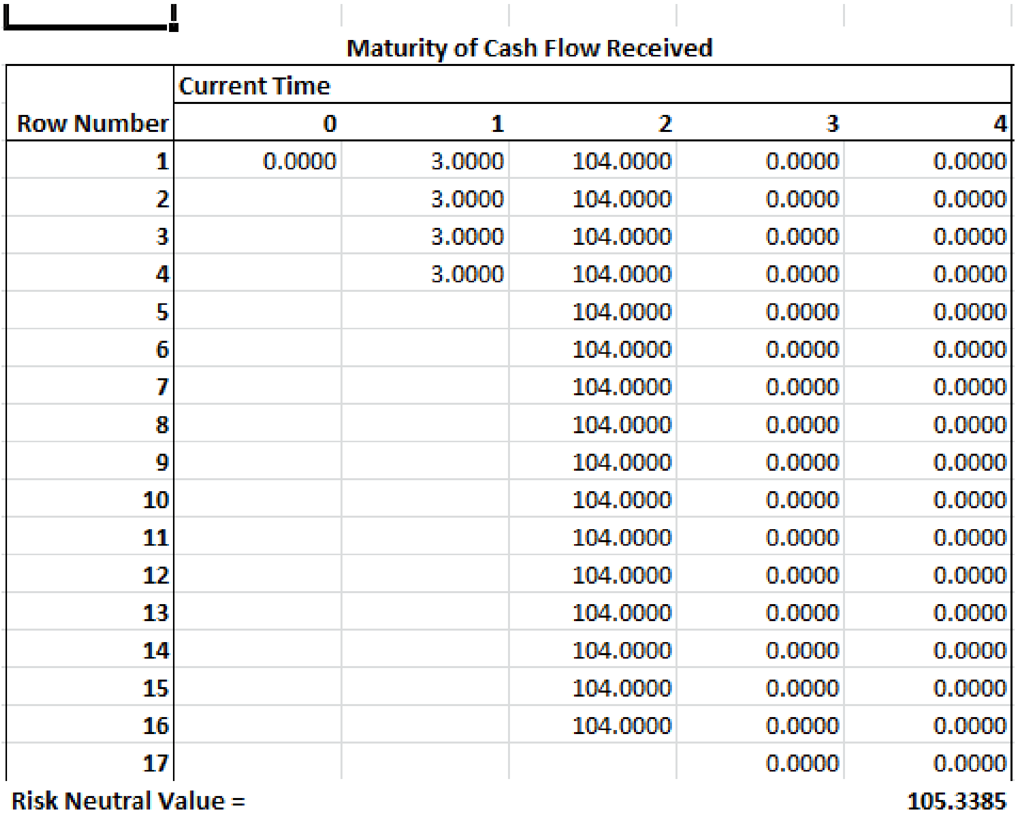
Row Number	Current Time				
	0	1	2	3	4
1	0.0000	0.0000	5.9280	0.0000	0.0000
2		0.0000	0.0000	0.0000	0.0000
3		0.0000	1.9519	0.0000	0.0000
4		0.0000	5.5831	0.0000	0.0000
5			0.6135	0.0000	0.0000
6			0.0000	0.0000	0.0000
7			0.0000	0.0000	0.0000
8			0.0000	0.0000	0.0000
9			0.0000	0.0000	0.0000
10			0.0000	0.0000	0.0000
11			0.0000	0.0000	0.0000
12			1.5477	0.0000	0.0000
13			0.0000	0.0000	0.0000
14			0.0000	0.0000	0.0000
15			0.0000	0.0000	0.0000
16			2.1111	0.0000	0.0000
17				0.0000	0.0000
18				0.0000	0.0000
Risk Neutral Value =					1.2099

We can check that these options values are consistent with no arbitrage in a very simple way. Consider this trade:

1. Buy the 3% bond maturing in T=4.
2. Sell the call option, allowing the option holder to buy the bond from us at 101.
3. Buy the put option, allowing us to sell the bond to the put option holder at 101.

The cost of this strategy is 104.7071 - 0.5785 + 1.2099 = 105.3385. Another way to receive these cash flows would be to buy a bond that pays $3 coupons at time T=1 and T=2 and principal of 101 at T=2. We drop these cash flows into the cash flow table and verify that the value/cost of this bond is the same number: 105.3385.

Figure 11



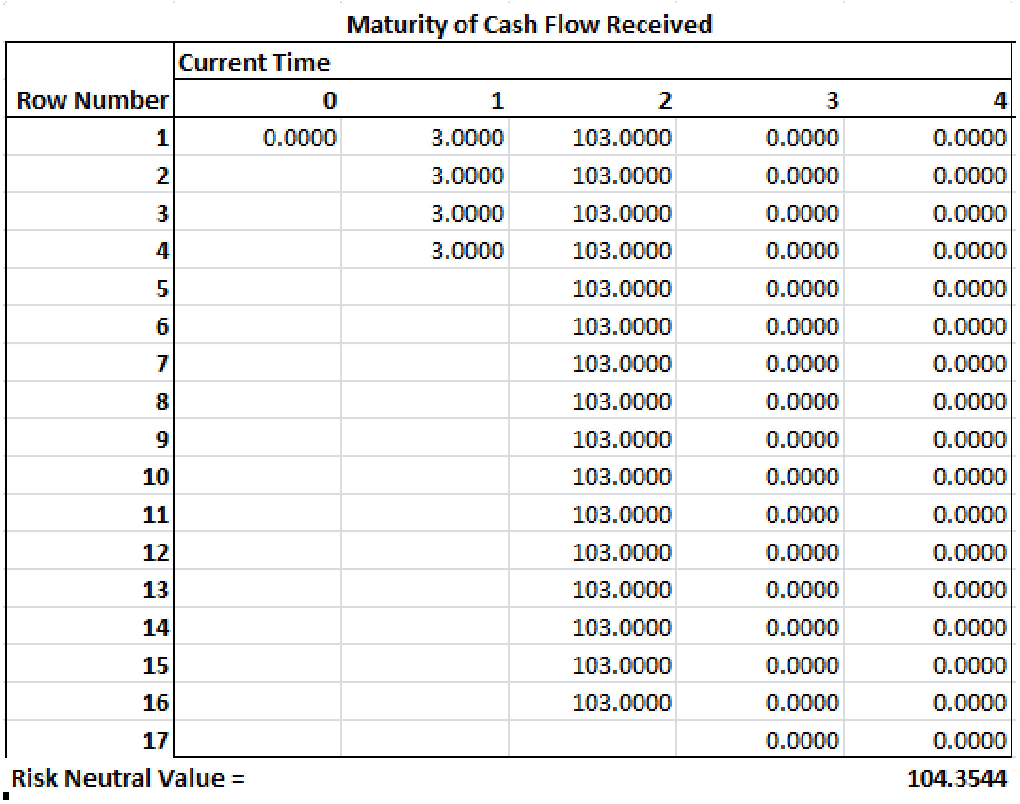
Maturity of Cash Flow Received

Row Number	Current Time				
	0	1	2	3	4
1	0.0000	3.0000	104.0000	0.0000	0.0000
2		3.0000	104.0000	0.0000	0.0000
3		3.0000	104.0000	0.0000	0.0000
4		3.0000	104.0000	0.0000	0.0000
5			104.0000	0.0000	0.0000
6			104.0000	0.0000	0.0000
7			104.0000	0.0000	0.0000
8			104.0000	0.0000	0.0000
9			104.0000	0.0000	0.0000
10			104.0000	0.0000	0.0000
11			104.0000	0.0000	0.0000
12			104.0000	0.0000	0.0000
13			104.0000	0.0000	0.0000
14			104.0000	0.0000	0.0000
15			104.0000	0.0000	0.0000
16			104.0000	0.0000	0.0000
17				0.0000	0.0000
Risk Neutral Value =					105.3385

Example: Coupon-bearing bond with embedded European call option

We now turn to another common structure for retail borrowers, corporate borrowers and sovereign borrows. That is, a structure that allows the borrower to prepay the borrowing. Let's continue with the case of the 3% bond due at time T=4 with principal of 100. What if we grant the issuer of the bond the one time only right to prepay at 100 immediately after the time T=2 coupon of $3 is paid. If the issuer does not exercise the option, the bond is repaid as scheduled at time T=4. What is the true nature of this arrangement? This structure is labeled with the misleading name of a "callable bond" but it in fact involves a put option held by the bond issuer. The issuer sells a bond maturing at time T=2 with a 3% coupon and principal of 100 and buys the "put option" that allows the issuer to compel the holder to buy another 2 year 3% bond maturing at time T=4 at the issuer's option at time T=2 at a price of 100. We can model this structure as the sum of the two year bond less the value of put option or we can model the entire structure together. The value of the two year bond is easily calculated either using our input zero coupon bonds or via an input to the cash flow table. We show the latter and the value of 104.3544 in Figure 12:

Figure 12

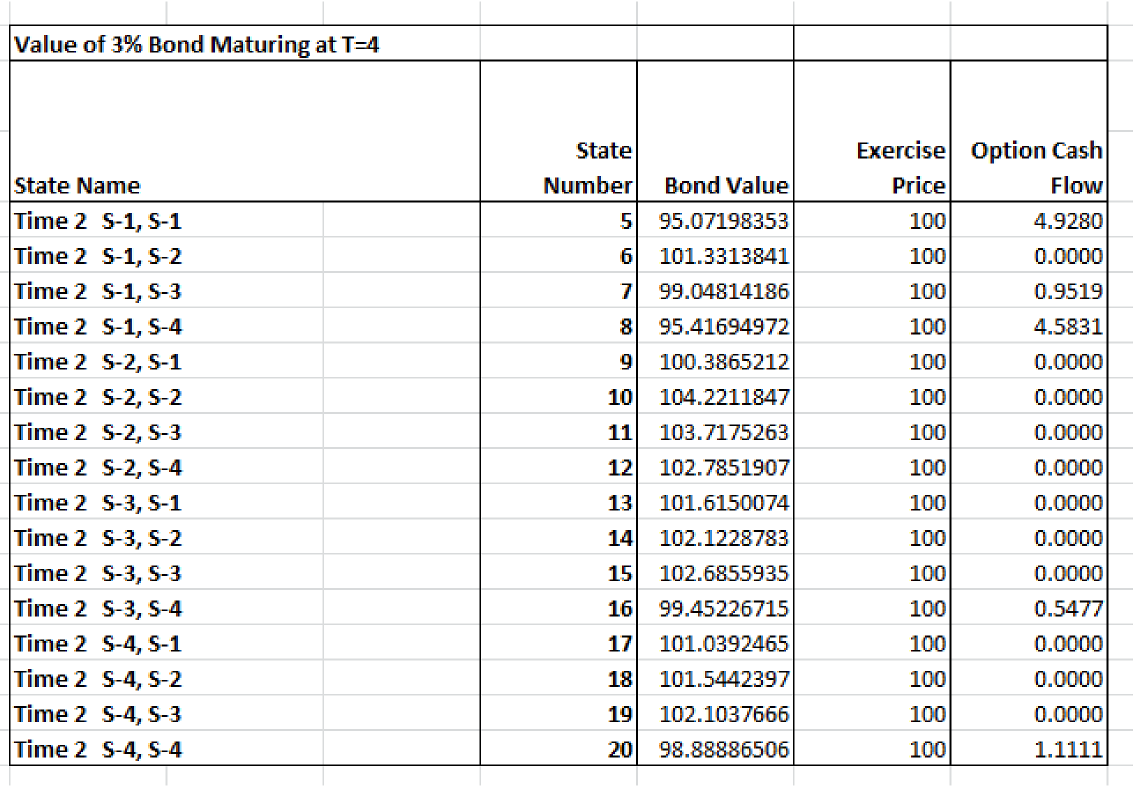


Maturity of Cash Flow Received

	Current Time				
Row Number	**0**	**1**	**2**	**3**	**4**
1	0.0000	3.0000	103.0000	0.0000	0.0000
2		3.0000	103.0000	0.0000	0.0000
3		3.0000	103.0000	0.0000	0.0000
4		3.0000	103.0000	0.0000	0.0000
5			103.0000	0.0000	0.0000
6			103.0000	0.0000	0.0000
7			103.0000	0.0000	0.0000
8			103.0000	0.0000	0.0000
9			103.0000	0.0000	0.0000
10			103.0000	0.0000	0.0000
11			103.0000	0.0000	0.0000
12			103.0000	0.0000	0.0000
13			103.0000	0.0000	0.0000
14			103.0000	0.0000	0.0000
15			103.0000	0.0000	0.0000
16			103.0000	0.0000	0.0000
17				0.0000	0.0000
Risk Neutral Value =					**104.3544**

The value of the put option on bond that pays a $3 coupon at time T=3 and T=4 and principal of 100 at T=4 is just the value of the cash flow gains that accrue to the owner of the put option (the issuer). Those cash flows are shown in Figure 13. The "put option" is exercised in 5 of 16 scenarios, so the bond continues to time T=4. In the other 10 scenarios, the bond is "called" or repaid at time T=2. The cash flows from the put are given here:

Figure 13

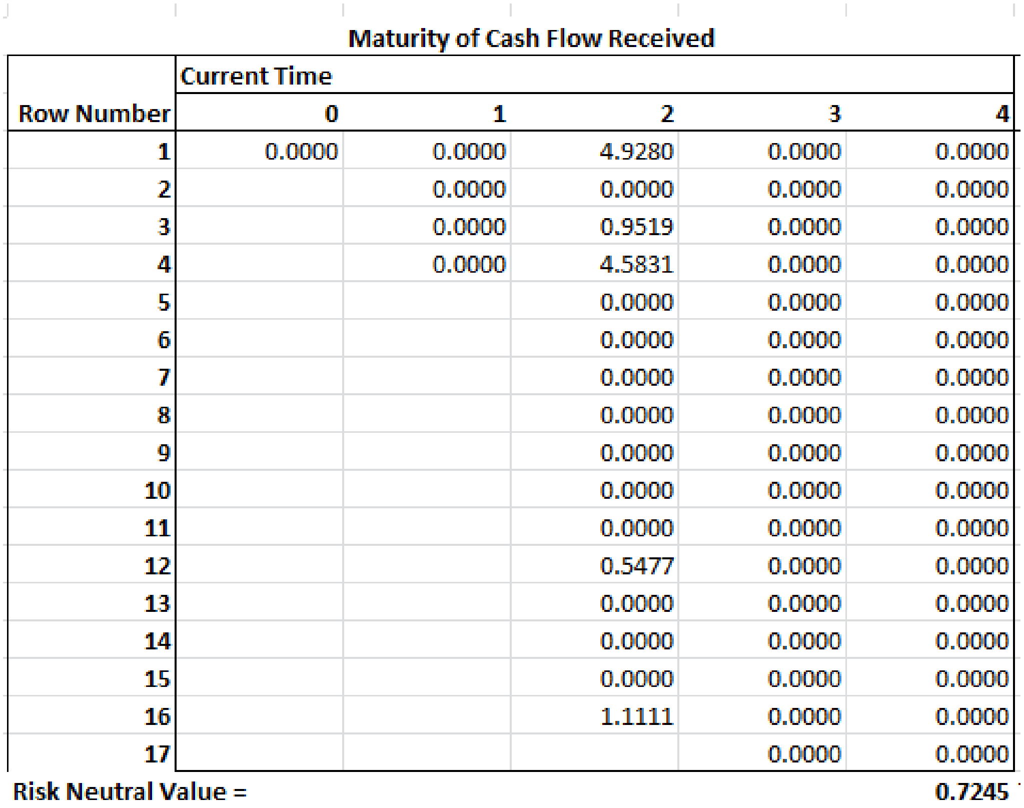


Value of 3% Bond Maturing at T=4

State Name	State Number	Bond Value	Exercise Price	Option Cash Flow
Time 2 S-1, S-1	5	95.07198353	100	4.9280
Time 2 S-1, S-2	6	101.3313841	100	0.0000
Time 2 S-1, S-3	7	99.04814186	100	0.9519
Time 2 S-1, S-4	8	95.41694972	100	4.5831
Time 2 S-2, S-1	9	100.3865212	100	0.0000
Time 2 S-2, S-2	10	104.2211847	100	0.0000
Time 2 S-2, S-3	11	103.7175263	100	0.0000
Time 2 S-2, S-4	12	102.7851907	100	0.0000
Time 2 S-3, S-1	13	101.6150074	100	0.0000
Time 2 S-3, S-2	14	102.1228783	100	0.0000
Time 2 S-3, S-3	15	102.6855935	100	0.0000
Time 2 S-3, S-4	16	99.45226715	100	0.5477
Time 2 S-4, S-1	17	101.0392465	100	0.0000
Time 2 S-4, S-2	18	101.5442397	100	0.0000
Time 2 S-4, S-3	19	102.1037666	100	0.0000
Time 2 S-4, S-4	20	98.88886506	100	1.1111

The put cash flows are put into the cash flow table as usual, giving us a value of 0.7245 in Figure 14:

Figure 14

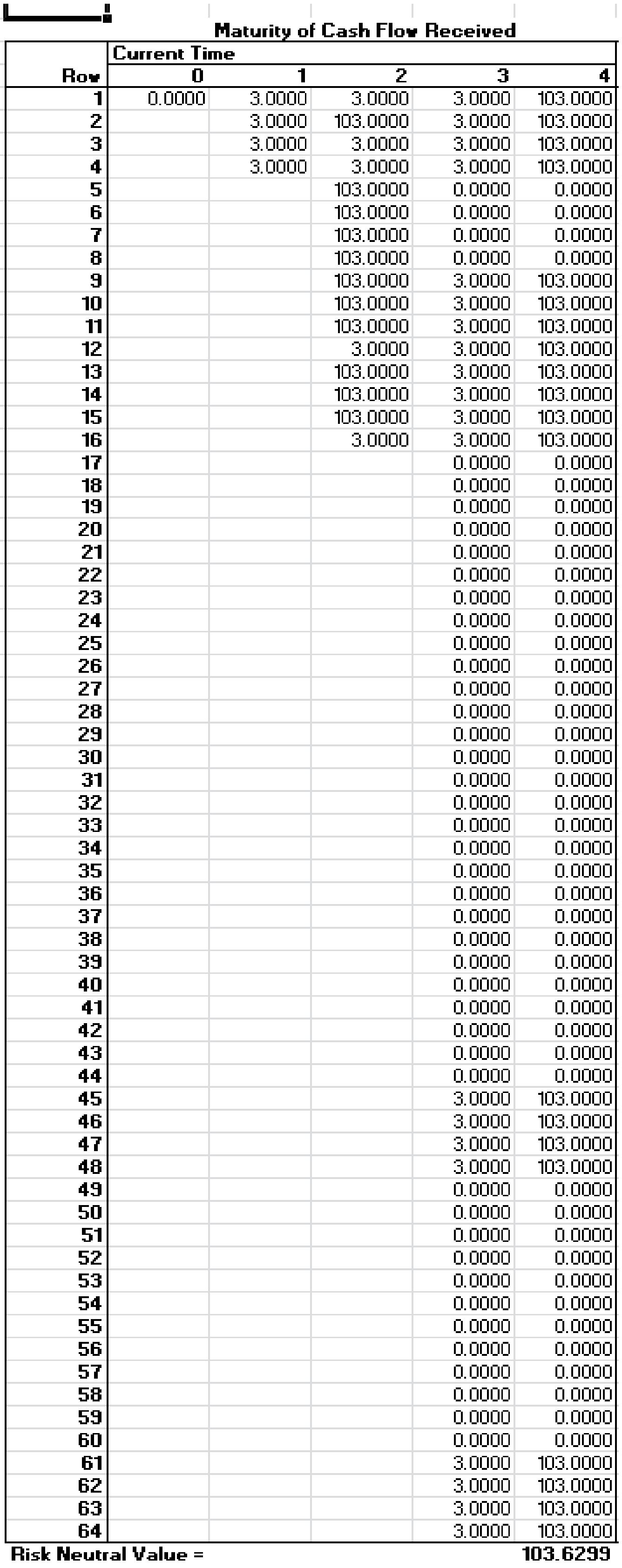


Maturity of Cash Flow Received

Row Number	Current Time				
	0	1	2	3	4
1	0.0000	0.0000	4.9280	0.0000	0.0000
2		0.0000	0.0000	0.0000	0.0000
3		0.0000	0.9519	0.0000	0.0000
4		0.0000	4.5831	0.0000	0.0000
5			0.0000	0.0000	0.0000
6			0.0000	0.0000	0.0000
7			0.0000	0.0000	0.0000
8			0.0000	0.0000	0.0000
9			0.0000	0.0000	0.0000
10			0.0000	0.0000	0.0000
11			0.0000	0.0000	0.0000
12			0.5477	0.0000	0.0000
13			0.0000	0.0000	0.0000
14			0.0000	0.0000	0.0000
15			0.0000	0.0000	0.0000
16			1.1111	0.0000	0.0000
17				0.0000	0.0000
Risk Neutral Value =					**0.7245**

Alternatively, we can model the full table of cash flows over all 4 periods. In the 4 nodes at time T=1, cash flow is $3. The put option, extending the bonds to year 4, is exercised in 5 scenarios at time T=2. The time 2 cash flows for these five scenarios are $3. For the 4 x 5 = 20 nodes at time T=3 associated with the decision to extend the bonds, cash flow is also $3. For the associated 20 nodes at time T=4, cash flow is 103. Going back to time T=2, for the 10 scenarios in which the put is not exercised, the bonds are repaid at a cash flow of 103. All other nodes at times 3 and 4 are set to zero because the bonds have been retired. The value of this “callable” bond is 103.6299 as shown in Figure 15:

Figure 15



Maturity of Cash Flow Received

Row	Current Time 0	1	2	3	4
1	0.0000	3.0000	3.0000	3.0000	103.0000
2		3.0000	103.0000	3.0000	103.0000
3		3.0000	3.0000	3.0000	103.0000
4		3.0000	3.0000	3.0000	103.0000
5			103.0000	0.0000	0.0000
6			103.0000	0.0000	0.0000
7			103.0000	0.0000	0.0000
8			103.0000	0.0000	0.0000
9			103.0000	3.0000	103.0000
10			103.0000	3.0000	103.0000
11			103.0000	3.0000	103.0000
12			3.0000	3.0000	103.0000
13			103.0000	3.0000	103.0000
14			103.0000	3.0000	103.0000
15			103.0000	3.0000	103.0000
16			3.0000	3.0000	103.0000
17				0.0000	0.0000
18				0.0000	0.0000
19				0.0000	0.0000
20				0.0000	0.0000
21				0.0000	0.0000
22				0.0000	0.0000
23				0.0000	0.0000
24				0.0000	0.0000
25				0.0000	0.0000
26				0.0000	0.0000
27				0.0000	0.0000
28				0.0000	0.0000
29				0.0000	0.0000
30				0.0000	0.0000
31				0.0000	0.0000
32				0.0000	0.0000
33				0.0000	0.0000
34				0.0000	0.0000
35				0.0000	0.0000
36				0.0000	0.0000
37				0.0000	0.0000
38				0.0000	0.0000
39				0.0000	0.0000
40				0.0000	0.0000
41				0.0000	0.0000
42				0.0000	0.0000
43				0.0000	0.0000
44				0.0000	0.0000
45				3.0000	103.0000
46				3.0000	103.0000
47				3.0000	103.0000
48				3.0000	103.0000
49				0.0000	0.0000
50				0.0000	0.0000
51				0.0000	0.0000
52				0.0000	0.0000
53				0.0000	0.0000
54				0.0000	0.0000
55				0.0000	0.0000
56				0.0000	0.0000
57				0.0000	0.0000
58				0.0000	0.0000
59				0.0000	0.0000
60				0.0000	0.0000
61				3.0000	103.0000
62				3.0000	103.0000
63				3.0000	103.0000
64				3.0000	103.0000

Risk Neutral Value = **103.6299**

We can verify there is no arbitrage by showing that the value of the callable bond (103.6299) equals the value of the non-call 2 year bond (104.3544) less the cost of the put option (0.7245). One can easily use the solver function in common spreadsheet software to answer this question: what coupon level causes a new issue bond that matures in 4 years with a European call option (i.e. option to prepay or put option to extend) to be valued at 100% of par value?

European options on defaultable bonds

This review of European options on bonds emphasizes the power and ease of use of the HJM framework. Once the bushy tree is set up or the Monte Carlo simulations have been generated, valuation is quick and easy. We have discussed ABC Corporation as a risky issuer and taken advantage of the Jarrow insight that the risky zero coupon bonds of ABC can be used to value coupon bearing bonds in the same manner as if ABC had no credit risk. In analyzing the callable bonds in the prior example, the issuer holds the option so there is no counterparty risk to be analyzed by the issuer of the bonds.

In the case of discrete call or put options (i.e. point in time credit default swaps) on the bonds, there is indeed the risk that one's counter-party on the options may default. We have ignored that risk here. In Chapter 20, we reviewed the Jarrow formula for the impact of counterparty risk. In general, we strongly recommend a simulation approach to counterparty risk as the only realistic way to accurately capture the joint impact of macro-economic factors on the default risk and credit spreads of both the bond issuer and those who hold call or put options on those securities.

Note that this chapter provides worked examples of how to value the Jarrow-Merton put option on the entire organization or portfolio as a measure of comprehensive risk. The only difference in methodology is that cash flows are generated from a collection of instruments valued on a collection of yield curves. Modern computers are expert at adding numbers together. The process is no more complex than the examples above. Repetition does not make the task more complex. It merely takes longer.

We close with a brief review of how simpler and less realistic term structure models can be used to obtain closed form solutions for fixed income options values. We present them in this chapter for exposition purposes. MODEL RISK ALERT: We do not recommend their use due to the inaccuracy of one factor term structure models.

HJM Special Case: European Options in the One Factor Vasicek Model

MODEL RISK ALERT: The one factor option pricing model in this section is presented for exposition purposes only and we do not recommend its use. The lack of accuracy of one factor models was reviewed in great detail in Chapters 3 through 10. We recommend that anyone tempted to use what follows in practice review Chapters 3 through 10 again. We present this section as an aid to understanding for those who are mathematically inclined. We remind you that those who prefer mathematics based on unrealistic assumptions over accuracy have been labeled "F9 Model Monkeys" by the always entertaining Felix Salmon (2012) of Reuters.

In this section, we focus on the Vasicek model rather than the Heath Jarrow and Morton 3 factor model used in the rest of the chapter and the rest of the book. We do so to illustrate that closed form solutions are possible for options valuation if one makes enough simplifying (and realism destroying) assumptions.

In Chapter 13 on legacy interest rate models, we introduced the Vasicek model in which short term interest rates move randomly according to the stochastic process

$$dr = \alpha(\ \mu - r)dt + \sigma dz$$

Using the hedging argument from Chapter 13, riskless arbitrage requires that zero coupon bonds for all maturities are related to the market price of risk lambda by the following equation:

$$\frac{P_r\,\alpha(\ \mu - r) + \frac{1}{2}\sigma^2 P_{rr} + P_t - rP}{\sigma P_r} = -\lambda$$

Jamshidian [1989] derived the value of a European call option on a zero coupon bond. We assume that the call option is to be valued at current time t. It is an option to purchase a risk-free zero coupon bond with maturity at time T_2 that is exercisable at a purchase price K at time T_1.

The value of the call option as of time t given the observable short rate r is

$$V(r,t,T_1,T_2,K) = P(r,t,T_2)N(h) - P(r,t,T_1)KN(h - \sigma_P)$$

where N is the standard cumulative normal distribution. We use the following definitions:

$$h = \frac{1}{\sigma_P}\ln\left[\frac{P(r,t,T_2)}{KP(r,t,T_1)}\right] + \frac{\sigma_P}{2}$$

$$\sigma_P = v\,F_1$$

$$F_i(T - t) = \frac{1}{\alpha}\left(1 - e^{-\alpha(T-t)}\right)$$

The latter expression is the standard deviation of the price of the T_2 maturity zero coupon bond's price.

Readers familiar with the Black-Scholes options model will notice a very strong similarity to the pricing formula in that model. The similarities occur because the zero coupon bond's price is lognormally distributed since its yield is linear in the short rate of interest and the yield is therefore normally distributed. There are very important differences in this formulation compared to the Black-Scholes model that are worth noting:

- The volatility of the bond's price declines over time and reaches zero at maturity, which is fully captured by the Jamshidian formulation. It is not captured by the Black-Scholes model if applied to bond options, since the bond price volatility is assumed to be constant in the Black-Scholes model. As we noted earlier, many traders in caps, floors and swaptions erroneously live with the Black model's assumption of constant bond price volatility, which is a partial explanation for the volatility "smile" observed when using the Black model for caps, floors and swaptions.

- The Jamshidian formulation looks through bond price fluctuations to the economic source of the fluctuations, random interest rates, and provides an internally consistent methodology for bond option valuation. Using the Black-Scholes model for bond options pricing relies on the dangerous inconsistency that bond options are being valued using a model that assumes interest rates are constant.

- The Jamshidian formula, in combination with the Vasicek or Extended Vasicek term structure model, provides a theoretically consistent approach that allows cross-hedging of a one-year option on a three-year zero coupon bond with the appropriate hedging amount of two year bonds. MODEL RISK ALERT: That being said, the one factor model is too simple but relative to Black-Scholes, it's an improvement of considerable magnitude. The Black-Scholes model does not provide an explanation of the relationship between price changes on bonds with different maturities.

Options on Coupon-Bearing Bonds

Jamshidian notes that the price of any security with a positive amount of contractual cash flows a_i at n points of time in the future is a monotonically decreasing function of the short rate of interest r. Therefore, the zero coupon bond formulas for European options can be applied directly to the problem of European options on coupon-bearing bonds. For each cash flow date, the strike price K_i is set to equal the zero coupon bond price as of the exercise date T such that the present value of the remaining payments on the security as of time T exactly equal the true strike price on the entire security K. The interest rate r* is the short term interest rate at which this is true:

$$\sum_{i=j}^{n} a_i P(r^*, T, T_i) = K$$

and

$$K_i = P(r^*, T, T_i)$$

Jamshidian shows that the value of a European option on the entire security is the weighted sum of options on each cash flow:

$$Option(r,t,T,K) = \sum_{i=j}^{n} a_i Option(r,t,T,T_i,K_i)$$

The Jarrow-Merton Put Option

The worked examples using the 3 factor Heath Jarrow and Morton framework and the single factor Jamshidian formula for bond options in the legacy Vasicek model both serve an important purpose: to show that the Jarrow-Merton put option calculation for measurement of total risk is not complicated or mysterious. It is a calculation that modern enterprise risk management software should be able to accomplish with ease. As we shall see in later chapters, the Jarrow-Merton put option as a total risk measure is far superior to the varied uses of "value at risk" that have been called into question in the wake of financial institutions failures around the world in the 2007-2010 credit crisis and in more recent times. Before demonstrating that concretely, we continue to work through other security types to illustrate the power of the HJM approach.