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Advanced Financial Risk Management:

Tools and Techniques for Integrated Credit Risk, Market Risk, Liquidity Risk and Interest Rate Risk Management

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Part 2: Risk Management Techniques for Interest Rate Analytics

Chapter 9:
HJM interest rate modeling with three risk factors

This chapter incorporates some important revisions in previously published work by Robert A. Jarrow (Jarrow, 2002, Chapter 15). We would like to add a special thanks to Professor Jarrow for his contributions to this revision.

In Chapters 6 through 8, we provided worked examples of how to use the yield curve simulation framework of Heath, Jarrow and Morton using two different assumptions about the volatility of forward rates and one or two independent risk factors. The first volatility assumption was that volatility was dependent on the maturity of the forward rate and nothing else. The second volatility assumption was that the volatility of forward rates was dependent on both the level of rates and the maturity of the forward rates being modeled. The first two models were one factor models, implying that random rate shifts are either all positive, all negative, or zero. Our third example postulated that the change in 1 year spot rates, along with an unspecified second risk factor, was in fact driving interest rate movements. In this chapter, we generalize the model to include three risk factors in order to increase further the realism of the simulated yield curve. Consistent with the research of Dickler, Jarrow and van Deventer, we explain why even three risk factors understate the number of risk factors driving the U.S. Treasury yield curve.

Probability of Yield Curve Twists in the U.S. Treasury Market

In Chapter 8, we explained that one factor models imply no twists in the yield curve. Random yield shocks drive yields either all up or all down together. Yield curve twists, where some rates rise and others fall, are assumed never to occur. Yet during 12,386 days of movements in U.S. Treasury forward rates, yield curve twists occurred on 94.3% of the observations (see Dickler, Jarrow and van Deventer, 2011). We now ask a simple question. To what extent did the two factor model of Chapter 8 succeed in explaining the correlations between changes in forward rates? We answer that question by analyzing the correlation between the residuals of the regression we used to determine the impact of our first risk factor, the annual change in the one year spot U.S. Treasury rate, on changes in forward rates at various maturities:

$$[Change\ in\ f_k(i)] = a + b[Change\ in\ 1\ year\ U.S.Treasury\ spot\ rate\ (i)] + e_k(i)$$

If the two factor model is correct, all forward rate movements would be explained by our two factors: the 1 year change in the one year spot rate and the unnamed second factor. What would the correlations remaining after use of the first risk factor show if our hypothesis was correct? The correlation of the residuals at any two maturities would show a 100% correlation, with all residuals either moving up (+100% correlation) together or down (-100% correlation) together after eliminating the common impact of changes in the 1 year spot rate. Unfortunately the correlation of the residuals shows that the 2 factor model is not rich enough to achieve that objective:

Figure 1

Correlation of Residuals from Regression 1 on U.S. Treasury Spot and Forward Rates, 1962-2011
Source: Kamakura Corporation and Federal Reserve

	1 Year U.S. Treasury Forward Rate Maturing in Year:							
Maturity in Year:	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
2	1							
3	0.920156	1						
4	0.840913	0.936176	1					
5	0.807619	0.823928	0.942743	1				
6	0.790989	0.808105	0.875853	0.950392	1			
7	0.753103	0.803682	0.820229	0.859572	0.967012	1		
8	0.737788	0.806994	0.833124	0.838943	0.910462	0.961099	1	
9	0.677499	0.741696	0.808673	0.796858	0.786531	0.816912	0.93905	1

The correlations are positive and far from 100%, which leads to an immediate conclusion. We need at least one more risk factor to realistically mimic real yield curve movements. We incorporate a third factor in this chapter for that reason.

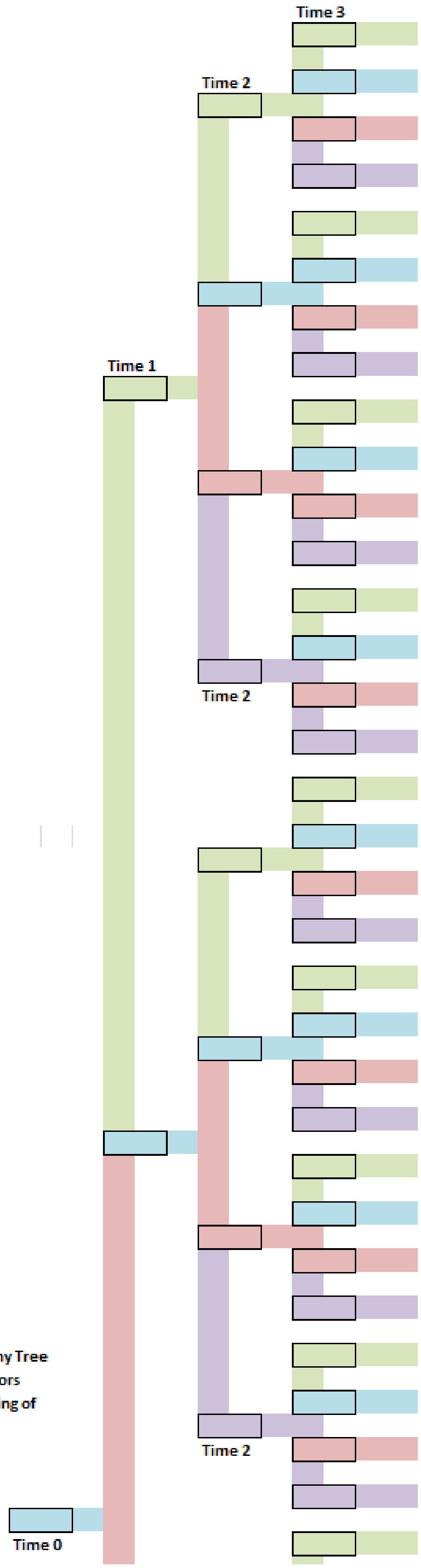
Objectives of the Example and Key Input Data

Following Jarrow (2002), we make the same modeling assumptions for our worked example as in Chapters 6 through 8:

- Zero coupon bond prices for the U.S. Treasury curve on March 31, 2011 are the basic inputs.
- Interest rate volatility assumptions are based on the Dickler, Jarrow and van Deventer papers on daily U.S. Treasury yields and forward rates from 1962 to 2011. In this chapter, we again retain the volatility assumptions used in Chapter 8 but expand the number of random risk factors driving interest rates to three factors
- The modeling period is 4 equal length periods of one year each.
- The HJM implementation is that of a "bushy tree" which we describe below

With two factors, the bushy tree has "up shifts," "mid shifts," and "down shifts" at each node in the tree. With three risk factors, there are 4 branches at each node of the tree. For simplicity and consistency with an n-factor HJM implementation, we abandon the terms up and down and instead label the shifts "shift 1," "shift 2," "shift 3" and "shift 4." The bushy tree starts with 1 branch at time zero and moves to 4 at time 1, 16 at time 2, and 64 at time 3. We split the tree into its upper and lower halves for better visibility:

Figure 2

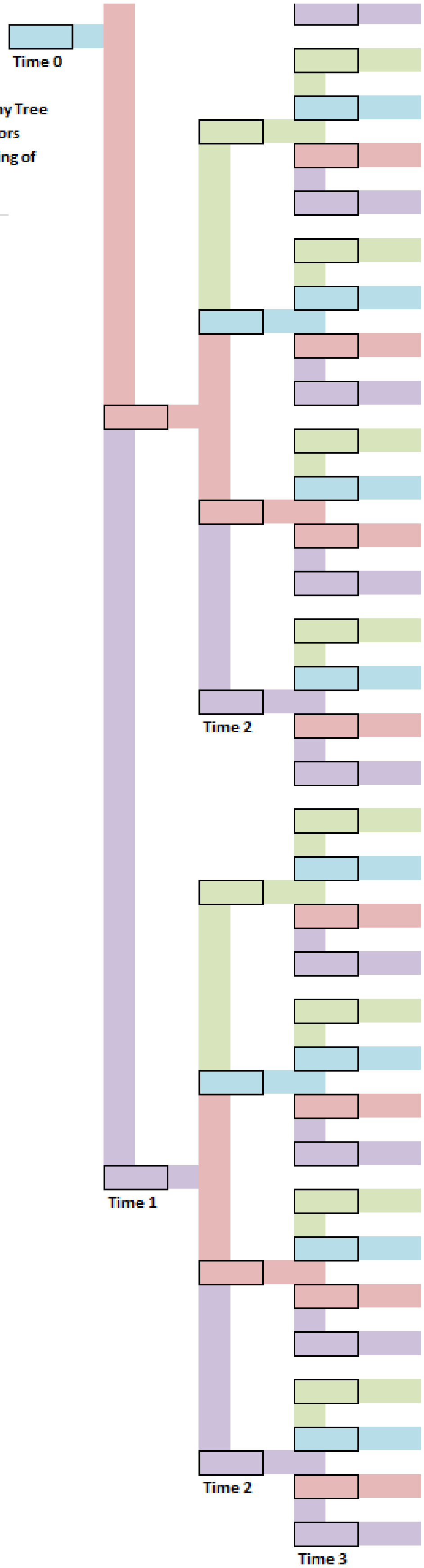


Example of Bushy Tree with 3 Risk Factors for HJM Modeling of No Arbitrage Forward Rate Movements

Scenario	Scenario Description		
1	Shift 1	Shift 1	Shift 1
2	Shift 1	Shift 1	Shift 2
3	Shift 1	Shift 1	Shift 3
4	Shift 1	Shift 1	Shift 4
5	Shift 1	Shift 2	Shift 1
6	Shift 1	Shift 2	Shift 2
7	Shift 1	Shift 2	Shift 3
8	Shift 1	Shift 2	Shift 4
9	Shift 1	Shift 3	Shift 1
10	Shift 1	Shift 3	Shift 2
11	Shift 1	Shift 3	Shift 3
12	Shift 1	Shift 3	Shift 4
13	Shift 1	Shift 4	Shift 1
14	Shift 1	Shift 4	Shift 2
15	Shift 1	Shift 4	Shift 3
16	Shift 1	Shift 4	Shift 4
17	Shift 2	Shift 1	Shift 1
18	Shift 2	Shift 1	Shift 2
19	Shift 2	Shift 1	Shift 3
20	Shift 2	Shift 1	Shift 4
21	Shift 2	Shift 2	Shift 1
22	Shift 2	Shift 2	Shift 2
23	Shift 2	Shift 2	Shift 3
24	Shift 2	Shift 2	Shift 4
25	Shift 2	Shift 3	Shift 1
26	Shift 2	Shift 3	Shift 2
27	Shift 2	Shift 3	Shift 3
28	Shift 2	Shift 3	Shift 4
29	Shift 2	Shift 4	Shift 1
30	Shift 2	Shift 4	Shift 2
31	Shift 2	Shift 4	Shift 3
32	Shift 2	Shift 4	Shift 4
33	Shift 3	Shift 1	Shift 1

Figure 3

Example of Bushy Tree
with 3 Risk Factors
for HJM Modeling of
No Arbitrage
Forward Rate
Movements



32	Shift 2	Shift 4	Shift 4
33	Shift 3	Shift 1	Shift 1
34	Shift 3	Shift 1	Shift 2
35	Shift 3	Shift 1	Shift 3
36	Shift 3	Shift 1	Shift 4
37	Shift 3	Shift 2	Shift 1
38	Shift 3	Shift 2	Shift 2
39	Shift 3	Shift 2	Shift 3
40	Shift 3	Shift 2	Shift 4
41	Shift 3	Shift 3	Shift 1
42	Shift 3	Shift 3	Shift 2
43	Shift 3	Shift 3	Shift 3
44	Shift 3	Shift 3	Shift 4
45	Shift 3	Shift 4	Shift 1
46	Shift 3	Shift 4	Shift 2
47	Shift 3	Shift 4	Shift 3
48	Shift 3	Shift 4	Shift 4
49	Shift 4	Shift 1	Shift 1
50	Shift 4	Shift 1	Shift 2
51	Shift 4	Shift 1	Shift 3
52	Shift 4	Shift 1	Shift 4
53	Shift 4	Shift 2	Shift 1
54	Shift 4	Shift 2	Shift 2
55	Shift 4	Shift 2	Shift 3
56	Shift 4	Shift 2	Shift 4
57	Shift 4	Shift 3	Shift 1
58	Shift 4	Shift 3	Shift 2
59	Shift 4	Shift 3	Shift 3
60	Shift 4	Shift 3	Shift 4
61	Shift 4	Shift 4	Shift 1
62	Shift 4	Shift 4	Shift 2
63	Shift 4	Shift 4	Shift 3
64	Shift 4	Shift 4	Shift 4

At each of the points in time on the lattice (time 0, 1, 2, 3 and 4) there are sets of zero coupon bond prices and forward rates. At time 0, there is one set of data. At time one, there are four sets of data, the "shift 1 set," the "shift 2 set, " the "shift 3 set" and the "shift 4 set." At time two, there are sixteen sets of data, and at time three there are $64=4^3$ sets of data.

As shown in Chapters 6 through 8, volatilities of the U.S. Treasury one year spot rate and 1 year forward rates with maturities in years 2, 3, …, 10 depend dramatically on the starting level of the one year U.S. Treasury spot rate. As before, we use the same volatility table as in Chapter 8 to measure total volatility, measured as the standard deviation of 1 year changes in continuously compounded spot and forward rates from 1963 to 2011. We use this table later to divide total volatility between three uncorrelated risk factors.

We will again use the zero coupon bond prices prevailing on March 31, 2011 as our other inputs, as we have done beginning in Chapter 6. We now partition total interest rate volatility among the three risk factors.

Risk Factor 1:
Annual Changes in the 1 Year U.S. Treasury Spot Rate

In the Chapters 6 and 7, the nature of the single factor shocking 1 year spot and forward rates was not specified. Chapter 8, we postulate that the first of the three factors driving changes in forward rates is the change in the 1 year spot rate of interest. For each of the 1 year U.S. Treasury forward rates $f_k(t)$, we run the regression

$$[Change\ in\ f_k(i)] = a + b[Change\ in\ 1\ year\ U.S.Treasury\ spot\ rate\ (i)] + e_k(i)$$

where the change in continuously compounded yields is measured over annual intervals from 1963 to 2011. We add an additional assumption that the second risk factor is the 1 year change in the 1 year forward rate maturing in year 10. To be consistent with the HJM assumptions that the individual risk factors are uncorrelated, we estimate the incremental explanatory power of this factor by adding the 1 year change in the 1 year forward rate maturing in year 10 as an explanatory variable in a regression where the dependent variable is the appropriate series of residuals from the first set of regressions, which we label the "Regression 1" set.

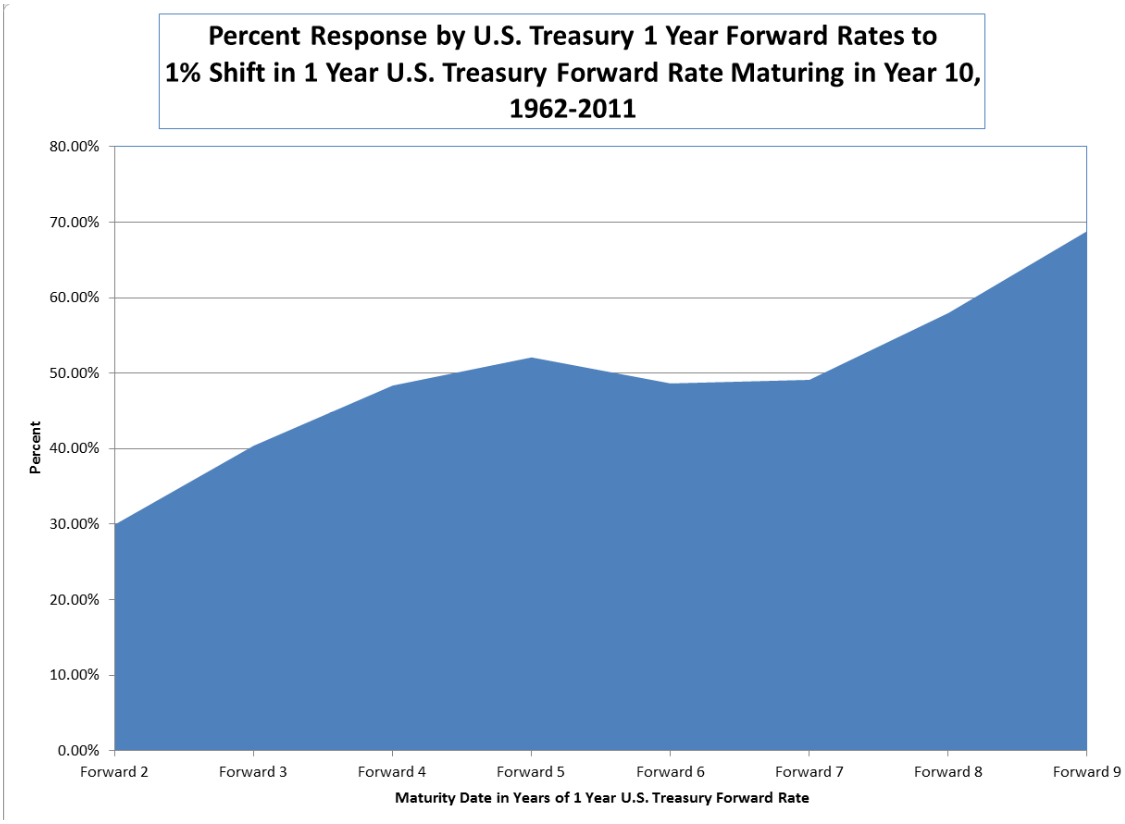
$$e_k(i) = a + b[change\ in\ 1\ year\ forward\ maturing\ in\ year\ 10\ (i)] + e_{2k}(i)$$

Because of the nature of the linear regression of changes in 1 year forward rates on the first risk factor (changes in the 1 year spot U.S. Treasury rate), we know that risk factor 1 and the residuals $e_k(i)$ are uncorrelated. The regression above will pick up the residual influence of the 1 year changes in the 1 year forward rate maturing in year 10.

There are many econometric approaches to this process that are more elegant, but we use this simple approach for clarity of exposition.

Our hypothesis that the 1 year change in the 1 year forward rate maturing in year 10 was important was right on the mark. The coefficients, shown below, are very statistically significant with t statistics ranging from 68 to 175. This graph shows that the impact of the change of the 1 year forward rate maturing in year 10 generally increases for rates maturing closest to year 10, as one would expect:

Figure 4



The third risk factor is "everything else" not explained by the changes in the 1 year spot rate or the changes in the 1 year forward rate maturing in year 10. Because of the sequential nature of the regression 1 set and the regression 2 set (which means correlation among the risk factors is zero), total volatility for the forward rate maturing in T-t=k years can be divided as follows between the three risk factors:

$$\sigma_{k,total}^2 = \sigma_{k,1}^2 + \sigma_{k,2}^2 + \sigma_{k,3}^2$$

We also know that the risk contribution of the first risk factor, the change in the spot 1 year U.S. treasury rate, is proportional to the regression coefficient α_k of changes in forward rate maturing in year k on changes in the 1 year spot rate. Because regression set 2 is done on the residuals of regression set 1, the risk contribution of risk factor 2 (the change in the 1 year forward rate maturing in year 10) is also proportional to the relevant regression coefficient. Note again that, by construction, risk factor 2 is not correlated with risk factor 1. We denote the total volatility of the 1 year U.S. Treasury spot rate by the subscript "1,total" and the total volatility of the 1 year forward rate maturing in year 10 by "10, total." Then the contribution of risk factors 1 and 2 to the volatility of the 1 year forward rate maturing in year k is

$$\sigma_{k,1}^2 = \alpha_{1k}^2 \sigma_{1,total}^2$$

$$\sigma_{k,2}^2 = \alpha_{10k}^2 \sigma_{10,total}^2$$

This allows us to solve for the volatility of risk factor 3 using this equation:

$$\sigma_{k,3} = \left[\sigma_{k,total}^2 - \alpha_{1k}^2 \sigma_{1,total}^2 - \alpha_{10k}^2 \sigma_{10,total}^2\right]^{1/2}$$

If ever the quantity in brackets is negative, $\sigma_{k,3}$ is set to zero and $\sigma_{k,2}$ is set to the value which makes the quantity in brackets exactly zero. Because the total volatility for each forward rate varies by the level of the 1 year spot U.S. Treasury spot rate, so will the values of $\sigma_{k,1}$, $\sigma_{k,2,}$ and $\sigma_{k,3}$. The regression coefficients used in this process are summarized in this table:

Figure 5

Dependent Variable: 1 Year Change in	Coefficient b_1 on 1 Year Change in 1 Year U.S. Treasury Rate	Standard Error of Regression 1	Dependent Variable: Regression 1 Residual on 1 Year Change in	Coefficient b_2 on 1 Year Change in 1 Year U.S. Treasury Forward Maturing in Year 10	Standard Error of Regression 2
Forward 2	0.774289	0.654783%	Forward 2	0.299553	0.556380%
Forward 3	0.636076	0.816101%	Forward 3	0.404013	0.670241%
Forward 4	0.539780	0.851190%	Forward 4	0.483604	0.643346%
Forward 5	0.439883	0.933073%	Forward 5	0.520913	0.714295%
Forward 6	0.353384	0.956797%	Forward 6	0.486439	0.775357%
Forward 7	0.309580	0.958593%	Forward 7	0.491257	0.773537%
Forward 8	0.309049	0.914843%	Forward 8	0.579650	0.625036%
Forward 9	0.322513	0.936120%	Forward 9	0.688282	0.497098%
Forward 10	0.327368	1.003651%			

Using the equations above, the look up table for risk factor 1 (changes in the 1 year spot rate) volatility is given here:

Figure 6

1 Year Rate in Data Group				Standard Deviation of Risk Factor 1, Sigma 1			
Data Group	Minimum Spot Rate	Maximum Spot Rate	Number of Observations	1 Year	2 Years	3 Years	4 Years
1	0.00%	0.09%	Assumed	0.01000000%	0.00774289%	0.00492507%	0.00265845%
2	0.09%	0.25%	Assumed	0.04000000%	0.03097155%	0.01970027%	0.01063382%
3	0.25%	0.50%	346	0.06363856%	0.04927462%	0.03134242%	0.01691802%
4	0.50%	0.75%	86	0.08298899%	0.06425743%	0.04087264%	0.02206225%
5	0.75%	1.00%	20	0.94371413%	0.73070713%	0.46478558%	0.25088212%
6	1.00%	2.00%	580	1.01658509%	0.78713029%	0.63127387%	0.34074924%
7	2.00%	3.00%	614	1.30986113%	1.01421060%	0.64511544%	0.34822064%
8	3.00%	4.00%	1492	1.22296401%	0.94692714%	0.60231802%	0.32511944%
9	4.00%	5.00%	1672	1.53223304%	1.18639064%	0.75463511%	0.40733721%
10	5.00%	6.00%	2411	1.22653783%	0.94969431%	0.60407816%	0.32606952%
11	6.00%	7.00%	1482	1.67496041%	1.29690282%	0.82492930%	0.44528063%
12	7.00%	8.00%	1218	1.70521091%	1.32032544%	0.83982788%	0.45332259%
13	8.00%	9.00%	765	2.16438588%	1.67585941%	1.06597465%	0.57539217%
14	9.00%	10.00%	500	2.28886301%	1.77224063%	1.12728047%	0.60848385%
15	10.00%	16.60%	959	2.86758369%	2.22033748%	1.41230431%	0.76233412%

The look up table for risk factor 2 (changes in the 1 year forward rate maturing in year 10) is given in this table:

Figure 7

	1 Year Rate in Data Group			Standard Deviation of Risk Factor 2, Sigma 2			
Data Group	Minimum Spot Rate	Maximum Spot Rate	Number of Observations	1 Year	2 Years	3 Years	4 Years
1	0.00%	0.09%	Assumed		0.01844038%	0.09987864%	0.14508126%
2	0.09%	0.25%	Assumed		0.09508293%	0.14140444%	0.16926147%
3	0.25%	0.50%	346		0.11460958%	0.15457608%	0.18502795%
4	0.50%	0.75%	86		0.21277322%	0.28697122%	0.34350527%
5	0.75%	1.00%	20		0.30003119%	0.40465767%	0.48437625%
6	1.00%	2.00%	580		0.28241222%	0.00000000%	0.45593183%
7	2.00%	3.00%	614		0.13927096%	0.18783734%	0.22484177%
8	3.00%	4.00%	1492		0.22509440%	0.30358902%	0.36339682%
9	4.00%	5.00%	1672		0.22361237%	0.30159018%	0.36100420%
10	5.00%	6.00%	2411		0.28294579%	0.38161427%	0.45679324%
11	6.00%	7.00%	1482		0.29059766%	0.39193450%	0.46914658%
12	7.00%	8.00%	1218		0.26551545%	0.47400436%	0.56738441%
13	8.00%	9.00%	765		0.46585233%	0.62830374%	0.75208116%
14	9.00%	10.00%	500		0.58305515%	0.78637737%	0.94129569%
15	10.00%	16.60%	959		0.58088433%	0.78344955%	0.93779108%

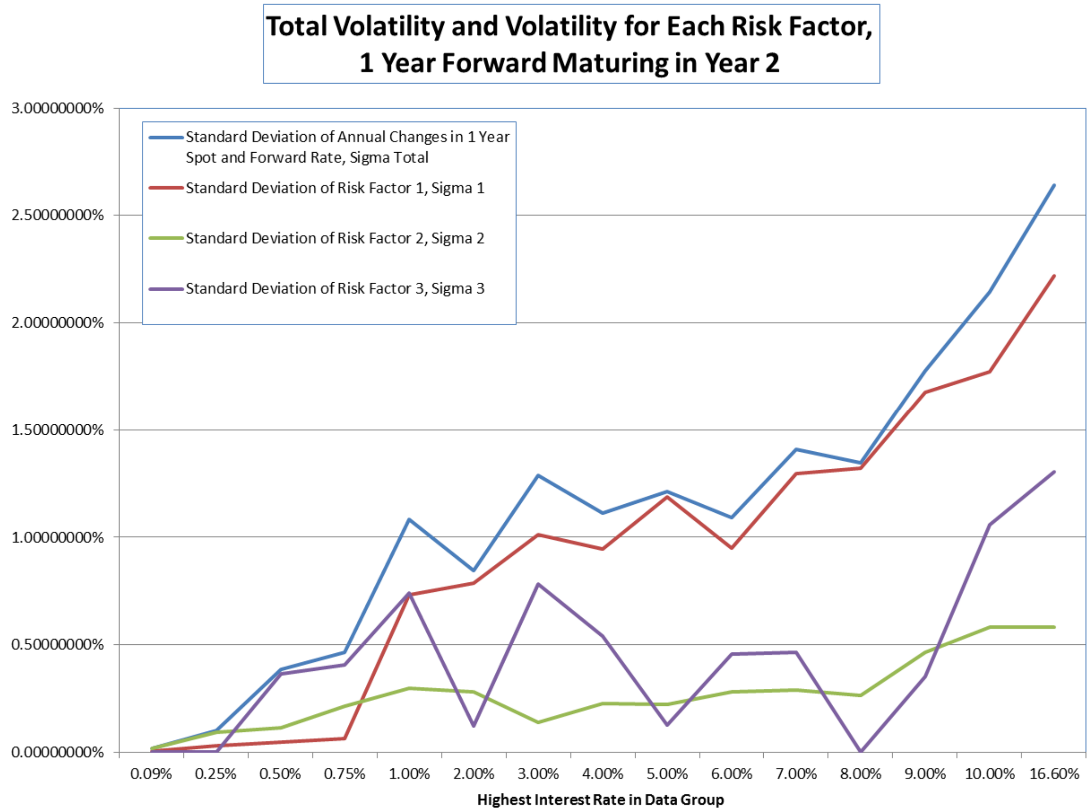
The look up table for the general "all other" risk factor 3 is as follows:

Figure 8

	1 Year Rate in Data Group			Standard Deviation of Risk Factor 3, Sigma 3			
Data Group	Minimum Spot Rate	Maximum Spot Rate	Number of Observations	1 Year	2 Years	3 Years	4 Years
1	0.00%	0.09%	Assumed	0.00000000%	0.00000000%	0.00000000%	0.13763852%
2	0.09%	0.25%	Assumed	0.00000000%	0.00000000%	0.14005958%	0.24746207%
3	0.25%	0.50%	346	0.00000000%	0.36337194%	0.65135096%	0.84363313%
4	0.50%	0.75%	86	0.00000000%	0.40702120%	0.54595415%	0.73281962%
5	0.75%	1.00%	20	0.00000000%	0.74084910%	0.74635937%	0.50994308%
6	1.00%	2.00%	580	0.00000000%	0.12463857%	0.00000000%	0.33951493%
7	2.00%	3.00%	614	0.00000000%	0.78254406%	0.83631746%	0.68568072%
8	3.00%	4.00%	1492	0.00000000%	0.54106464%	0.77592194%	0.76181187%
9	4.00%	5.00%	1672	0.00000000%	0.12847478%	0.58968940%	0.63313043%
10	5.00%	6.00%	2411	0.00000000%	0.45562957%	0.65652394%	0.72628160%
11	6.00%	7.00%	1482	0.00000000%	0.46346642%	0.77892862%	0.82112196%
12	7.00%	8.00%	1218	0.00000000%	0.00000000%	0.87713028%	1.06830036%
13	8.00%	9.00%	765	0.00000000%	0.35358489%	1.20123780%	1.40466564%
14	9.00%	10.00%	500	0.00000000%	1.05723235%	1.49203773%	1.68880657%
15	10.00%	16.60%	959	0.00000000%	1.30610213%	1.98193330%	1.98085391%

When we graph the relative contributions of each risk factor to the volatility of the forward rate maturing in year 2, we see an interesting pattern as the level of the 1 year spot Treasury rate rises:

Figure 9



The contribution of risk factor 1 rises steadily as the level of interest rates rises. The impact of risk factor 2 rises more gradually as the level interest rates rises. The "all other risk factor," risk factor 3, is volatile and spikes a number of times, due in part to the spike in total volatility when the 1 year spot U.S. Treasury rate is between 75 and 100 basis points. We will see below that this has an impact on our simulations and that this is a value for total volatility that one may want to override as a matter of professional judgment.

Alternative Specifications of the Interest Rate Volatility Surface

In Chapters 6 through 9, we have used a look-up table to specify the relevant levels of interest rate volatility. A common practice, sometimes used wisely and sometimes not, is to use the multi-variate equivalent of Chapter 5's yield curve smoothing techniques to create a smooth "interest rate volatility surface" for each of the three risk factors. Risk factor values would be chosen from this smoothed surface instead of the look-up table. This is a modest extension of the approach taken in these examples and we leave such an extension to the reader.

Key Implications and Notation of the HJM Approach

As in prior chapters, we confirm in this example that the zero coupon bond valuations are 100% consistent with the zero coupon bond price inputs, confirming that there are no arbitrage opportunities. We now introduce the same notation first used in Chapter 6 with slight modifications:

Δ	Length of time period, which is 1 in this example
$r(t,s_t)$	The simple 1 period risk free interest rate as of current time t in state s_t
$R(t,s_t)$	$1+r(t,s_t)$, the total return, the value of \$1 dollar invested at the risk free rate for 1 period
$f(t,T,s_t)$	The simple 1 period forward rate maturing at time T in state s_t as of time t. This rate is not adjusted for compounding or day count in any way.
$F(t,T,s_t)$	$1+f(t,T,s_t)$, the return on \$1 invested at the forward rate for one period.
$\sigma_1(t,T,s_t)$	Forward rate volatility due to risk factor 1, the change in the spot 1 year U.S. Treasury rate, at time t for the one period forward rate maturing at T in state s_t (a sequence of ups and downs). Because σ_1 is dependent on the spot 1 year U.S. Treasury rate, the state s_t is very important in determining its level.
$\sigma_2(t,T,s_t)$	Forward rate volatility due to risk factor 2, the change in the 1 year forward rate maturing in 10 years, at time t for the one period forward rate maturing at T in state s_t (a sequence of ups and downs). Because σ_2 is dependent on the spot 1 year U.S. Treasury rate and its volatility, the state s_t is very important in determining its level.
$\sigma_3(t,T,s_t)$	Forward rate volatility due to risk factor 3, all volatility remaining after the impacts of risk factors 1 and 2. Because σ_3 is dependent on the spot 1 year U.S. Treasury rate and its volatility, the state s_t is very important in determining its level.
$P(t,T,s_t)$	Zero coupon bond price at time t maturing at time T in state s_t (i.e. up or down)
Index(1)	An indicator of the state for risk factor 1, defined below.
Index(2)	An indicator of the state for risk factor 2, defined below.
Index(3)	An indicator of the state for risk factor 3, defined below.
$K(i,t,T,s_t)$	is the weighted sum of the forward volatilities for risk factor i (either 1, 2 or 3) as of time t for the 1 period forward rates from $t+\Delta$ to T, as shown here for i=1:

$$K(1,t,T,s_t) = \Delta \sum_{j=t+\Delta}^{T-t} \sigma_1(t,j,s_t)\sqrt{\Delta}$$

For i=2 or 3, this expression has a negative sign:

$$K(i,t,T,s_t) = -\Delta \sum_{j=t+\Delta}^{T-t} \sigma_i(t,j,s_t)\sqrt{\Delta}$$

Note that this is a slightly different definition of K than we used in Chapters 6 through 8. Note also that in Chapter 8, for i=2, we could have added a negative sign without changing the results of our analysis. For more than three factors, the positive sign on K for the first factor and negative signs for subsequent factors can be confirmed using the formulas that we discuss in Chapter 10, where we use the HJM framework for Monte Carlo simulation. As in Chapters 6 through 8, we again use the cosh(x) expression:

$$Cosh(x) = \frac{1}{2}[e^{x} + e^{-x}]$$

In the interests of saving space, we'll again arrange the bushy tree to look like a table by stretching the bushy tree as follows. Note that S-1 means "shift 1," S-2 means "shift 2," and so on. The "stretched" bushy tree is rearranged from the graph above as shown below:

Figure 10

	Map of Sequence of States			
	Current Time			
Row Number	0	1	2	3
1	Time 0	S -1	S-1, S-1	S-1, S-1, S-1
2		S-2	S-1, S-2	S-1, S-1, S-2
3		S-3	S-1, S-3	S-1, S-1, S-3
4		S-4	S-1, S-4	S-1, S-1, S-4
5			S-2, S-1	S-1, S-2, S-1
6			S-2, S-2	S-1, S-2, S-2
7			S-2, S-3	S-1, S-2, S-3
8			S-2, S-4	S-1, S-2, S-4
9			S-3, S-1	S-1, S-3, S-1
10			S-3, S-2	S-1, S-3, S-2
11			S-3, S-3	S-1, S-3, S-3
12			S-3, S-4	S-1, S-3, S-4
13			S-4, S-1	S-1, S-4, S-1
14			S-4, S-2	S-1, S-4, S-2
15			S-4, S-3	S-1, S-4, S-3
16			S-4, S-4	S-1, S-4, S-4
17				S-2, S-1, S-1
18				S-2, S-1, S-2
19				S-2, S-1, S-3
20				S-2, S-1, S-4
21				S-2, S-2, S-1
22				S-2, S-2, S-2
23				S-2, S-2, S-3
24				S-2, S-2, S-4
25				S-2, S-3, S-1
26				S-2, S-3, S-2
27				S-2, S-3, S-3
28				S-2, S-3, S-4
29				S-2, S-4, S-1
30				S-2, S-4, S-2
31				S-2, S-4, S-3
32				S-2, S-4, S-4
33				S-3, S-1, S-1
34				S-3, S-1, S-2
35				S-3, S-1, S-3
36				S-3, S-1, S-4
37				S-3, S-2, S-1
38				S-3, S-2, S-2
39				S-3, S-2, S-3
40				S-3, S-2, S-4
41				S-3, S-3, S-1
42				S-3, S-3, S-2
43				S-3, S-3, S-3
44				S-3, S-3, S-4
45				S-3, S-4, S-1
46				S-3, S-4, S-2
47				S-3, S-4, S-3
48				S-3, S-4, S-4
49				S-4, S-1, S-1
50				S-4, S-1, S-2
51				S-4, S-1, S-3
52				S-4, S-1, S-4
53				S-4, S-2, S-1
54				S-4, S-2, S-2
55				S-4, S-2, S-3
56				S-4, S-2, S-4
57				S-4, S-3, S-1
58				S-4, S-3, S-2
59				S-4, S-3, S-3
60				S-4, S-3, S-4
61				S-4, S-4, S-1
62				S-4, S-4, S-2
63				S-4, S-4, S-3
64				S-4, S-4, S-4

A completely populated zero coupon bond price tree would then be summarized like this; prices shown are for the zero coupon bond price maturing at time T=4 at times 0, 1, 2, and 3:

Figure 11

	Four Year Zero Coupon Bond Price			
	Current Time			
Row Number	0	1	2	3
1	0.93085510	0.89453539	0.89519397	0.94241274
2		0.96356381	0.95507979	0.94726825
3		0.94041258	0.93324679	0.95083214
4		0.93257021	0.89861721	0.92560481
5			0.94614101	0.96989377
6			0.98290319	0.97474129
7			0.97807468	0.98011229
8			0.96914668	0.96095793
9			0.95797643	0.95038062
10			0.96276439	0.98059970
11			0.96806939	0.96918220
12			0.93724224	0.94785242
13			0.95246679	0.94853984
14			0.95722721	0.95342692
15			0.96250170	0.95701398
16			0.93185185	0.93162263
17				0.96013753
18				0.99066684
19				0.97913212
20				0.95758337
21				0.97843924
22				0.99449944
23				0.99239106
24				0.98813866
25				0.97443935
26				1.00374797
27				0.99741350
28				0.97878128
29				0.98254587
30				0.98745663
31				0.99289769
32				0.97349346
33				0.97770406
34				0.98259062
35				0.98800487
36				0.96869626
37				0.97770406
38				0.98259062
39				0.98800487
40				0.96869626
41				0.96492120
42				0.99394354
43				0.98767094
44				0.969220711
45				0.956873373
46				0.987298898
47				0.975803393
48				0.954327901
49				0.966171633
50				0.996892813
51				0.985285603
52				0.963601426
53				0.974819161
54				0.979691298
55				0.985089571
56				0.965837939
57				0.974819161
58				0.979691298
59				0.985089571
60				0.965837939
61				0.956882114
62				0.977817245
63				0.973471088
64				0.952172548

In order to populate the trees with zero coupon bond prices and forward rates, we again need to select the pseudo probabilities.

Pseudo Probabilities

We set the pseudo probabilities in a manner consistent with Jarrow (2002), Chapter 7. Without loss of generality, we set the probability of shift 1 to 1/8, the probability of shift 2 to 1/8, the probability of shift 3 to ¼, and the probability of shift 4 to ½. The no arbitrage restrictions that stem from this set of pseudo probabilities are given below. We now demonstrate how to construct the bushy tree and use it for risk-neutral valuation.

The Formula for Zero Coupon Bond Price Shifts with Three Risk Factors

Like the two risk factor case, with three risk factors it is convenient to calculate the forward rates first and then derive zero coupon bond prices from them. We do this using slightly modified versions (developed with Prof. Jarrow) of equations 15.40 and 15.42 in Jarrow (2002, page 297). We use this equation for the shift in forward rates:

(1)

$$F(t+\Delta, T, s_{t+\Delta})$$
$$= F(t,T,s_t)e^{[\mu(t,T,s_t)]\Delta^2}e^{\Delta[Index(1)\sigma_1(t,T,s_t)\sqrt{\Delta}+Index(2)\sqrt{2}\sigma_2(t,T,s_t)\sqrt{\Delta}+Index(3)2\sigma_3(t,T,s_t)\sqrt{\Delta}]}$$

where when T=t+Δ, the expression μ becomes

$$\mu(t,T) = \frac{1}{\Delta^2} log\left[\frac{1}{2}e^{[K(1,t,t+\Delta)]}\langle\frac{1}{2}e^{[K(2,t,t+\Delta)\sqrt{2}]}\cosh\big(2K(3,t,t+\Delta)\big)+\frac{1}{2}e^{-[K(2,t,t+\Delta)\sqrt{2}]}\rangle + \frac{1}{2}e^{-[K(1,t,t+\Delta)]}\right]$$

We use this expression to evaluate formula (1) when T>t+Δ:

$$\mu(t,T) = \frac{1}{\Delta^2} log\left[\frac{1}{2}e^{[K(1,t,t+\Delta)]}\langle\frac{1}{2}e^{[K(2,t,t+\Delta)\sqrt{2}]}\cosh\big(2K(3,t,t+\Delta)\big)+\frac{1}{2}e^{-[K(2,t,t+\Delta)\sqrt{2}]}\rangle + \frac{1}{2}e^{-[K(1,t,t+\Delta)]}\right] - \sum_{j=t+\Delta}^{T-\Delta}\mu(t,j)$$

The values for the pseudo probabilities plus Index (1), Index (2), and Index (3) are set as follows:

Type of Shift	Pseudo Probability	Index(1)	Index (2)	Index (3)
Shift 1	12.50%	-1	1	1
Shift 2	12.50%	-1	1	-1
Shift 3	25.00%	-1	-1	0
Shift 4	50.00%	1	0	0

Building the Bushy Tree for Zero Coupon Bonds Maturing at Time T=2

We now populate the bushy tree for the 2 year zero coupon bond. We calculate each element of equation (1). Note that, even as the bushy tree grows more complex, accuracy of valuation to 10 decimal places in common spreadsheet software is maintained. When t=0 and T=2, we know Δ=1 and

$P(0,2,s_t) = 0.9841101497$

The one period risk free rate is again

$R(0,s_t)=1/P(0,1,s_t)=1/0.997005786=1.003003206$

The 1 period spot rate for U.S. Treasuries is $r(0, s_t) = R(0,s_t)-1 = 0.3003206\%$. At this level of the spot rate for U.S. Treasuries, volatilities for risk factors 1, 2 and 3 are selected from data group 3 in the look up table above. The volatilities for risk factor 1 for the 1 year forward rates maturing in years 2, 3 and 4 are 0.000492746, 0.000313424 , and 0.00016918. The volatilities for risk factor 2 for the 1 year forward rates maturing in years 2, 3 and 4 are 0.001146096, 0.001545761, and 0.00185028. Finally, the volatilities for risk factor 3 are also taken from the appropriate risk factor table for data group 3: 0.003633719, 0.00651351, and 0.008436331.

The scaled sum of sigmas $K(1,t,T,s_t)$ for t=0 and T=2 becomes

$$K(1,t,T,s_t) = \Delta \sum_{j=t+\Delta}^{T-\Delta} \sigma_1(t,j,s_t)\sqrt{\Delta} = \Delta \sum_{j=1}^{1} \sigma_1(0,j,s_t)\sqrt{\Delta}$$

and therefore $K(1,0,T,s_t) = (1)(\sqrt{1})(0.000492746) = 0.000492746$. Similarly, remembering the minus sign, $K(2,0,T,s_t) = -0.001146096$ and $K(3,0,T, s_t) = -0.003633719$. We also can calculate that $\mu(t,t+\Delta) = 0.0000073730$.

Using formula 1 with these inputs and the fact that the variable Index(1)=-1, Index(2)=1, and Index(3)=1 for shift 1 gives us the forward returns for shift 1 of 1.021652722. For shifts 2, 3, and 4, the values of the three index variables change. Using these new index values, the forward returns in shifts 2, 3, and 4 are 1.006910523, 1.010972304, and 1.013610654. Note that the forward rates (as opposed to forward returns) are equal to these values minus 1. From the forward returns, we calculate the zero coupon bond prices:

$P(1,2,s_t = \text{shift } 1) = 0.9788061814 = 1/F(1,2,s_t = \text{shift } 1)$
$P(1,2,s_t = \text{shift } 2) = 0.9931369048 = 1/F(1,2,s_t = \text{shift } 2)$
$P(1,2,st = \text{shift } 3) = 0.9891467808 = 1/F(1,2,s_t = \text{shift } 3)$
$P(1,2,st = \text{shift } 4) = 0.9865721079 = 1/F(1,2,s_t = \text{shift } 4)$

We have fully populated the bushy tree for the zero coupon bond maturing at T=2), since all of the 4 shifts at time t=2 result in a riskless pay-off of the zero coupon bond at its face value of 1.

Two Year Zero Coupon Bond Price			
	Current Time		
Row Number	0	1	2
1	0.9841101497	0.9788061814	1
2		0.9931369048	1
3		0.9891467808	1
4		0.9865721079	1

Building the Bushy Tree for Zero Coupon Bonds Maturing at Time T=3

For the zero coupon bonds and 1 period forward returns (which equal 1 plus the forward rate) maturing at time T=3, we use the same volatilities listed above for risk factors 1, 2 and 3 to calculate

$K(1,0,3,s_t) = 0.000806170$

Remembering the minus sign for risk factors 2 and 3 gives us

$K(2,0,3,s_t) = -0.002691857$
$K(3,0,3,s_t) = -0.010147229$

$\mu(t,T) = 0.0000479070$

The resulting forward returns for shifts 1, 2, 3, and 4 are 1.038505795, 1.011797959, 1.020593023, and 1.023467874. Zero coupon bond prices are calculated from the 1 period forward returns, so

$P(1,3,s_t = \text{Shift 1 Shift 1}) = 1/[F(1,2,s_t = \text{Shift 1})F(1,3,s_t = \text{Shift 1})]$

The zero coupon bond prices for maturity in time T=3 for the four shifts are 0.9425139335, 0.9815565410, 0.9691882649, and 0.9639502449. We have now populated the second column of the zero coupon bond price table for the zero coupon bond maturing at T=3.

Figure 12

	Three Year Zero Coupon Bond Price			
	Current Time			
Row Number	0	1	2	3
1	0.9618922376	0.9425139335	0.9556681147	1
2		0.9815565410	0.9860553163	1
3		0.9691882649	0.9745742911	1
4		0.9639502449	0.9531258489	1
5			0.9779990715	1
6			0.9940520493	1
7			0.9919446211	1
8			0.9876941290	1
9			0.9811450109	1
10			0.9860487650	1
11			0.9914820691	1
12			0.9721055077	1
13			0.9783890415	1
14			0.9832790213	1
15			0.9886970636	1
16			0.9693749296	1

Building the Bushy Tree for Zero Coupon Bonds Maturing at Time T = 4

We now populate the bushy tree for the zero coupon bond maturing at T=4. Using the same volatilities as before, we find that
$K(1,0,4,s_t) = 0.000975351$
Again, remembering the minus sign, we calculate
$K(2,0,4,s_t) = -0.004542136$
$K(3,0,4,s_t) = -0.018583560$

$\mu(t,T) = 0.0001272605$

Using formula (1) with the correct values for Index (1), Index (2) and Index (3) leads to the following forward returns for Shifts 1, 2, 3, and 4: 1.0536351507, 1.0186731103, 1.0305990034, and 1.0336489801. The zero coupon bond price for Shift 1 is calculated as follows:

$P(1,4,s_t = \text{Shift } 1) = 1/[F(1,2,s_t = \text{Shift } 1)F(1,3,s_t = \text{Shift } 1)\ F(1,4,s_t = \text{Shift } 1)]$

A similar calculation gives us the zero coupon bond prices for all 4 shifts 1, 2, 3, and 4: 0.8945353929, 0.9635638078, 0.9404125772, and 0.9325702085.

We have now populated the column labeled 1 for T=1 for the zero coupon bond price maturing at time T=4:

Figure 13

	Four Year Zero Coupon Bond Price			
	Current Time			
Row Number	0	1	2	3
1	0.9308550992	0.8945353929	0.8951939726	0.9424127368
2		0.9635638078	0.9550797882	0.9472682532
3		0.9404125772	0.9332467863	0.9508321387
4		0.9325702085	0.8986172056	0.9256048053
5			0.9461410097	0.9698937713
6			0.9829031898	0.9747412918
7			0.9780746840	0.9801122898
8			0.9691466829	0.9609579282
9			0.9579764308	0.9503806226
10			0.9627643885	0.9805996987
11			0.9680693916	0.9691821954
12			0.9372422391	0.9478524225
13			0.9524667898	0.9485398381
14			0.9572272104	0.9534269227
15			0.9625017026	0.9570139788
16			0.9318518473	0.9316226298
17				0.9601375282
18				0.9906668429
19				0.9791321239
20				0.9575833729
21				0.9784392385
22				0.9944994413
23				0.9923910646
24				0.9881386595
25				0.9744393509
26				1.0037479721
27				0.9974134983
28				0.9787812762
29				0.9825458738
30				0.9874566295
31				0.9928976912
32				0.9734934642
33				0.9777040612
34				0.9825906175
35				0.9880048666
36				0.9686962602
37				0.9777040612
38				0.9825906175
39				0.9880048666
40				0.9686962602
41				0.9649211970
42				0.9939435367
43				0.9876709370
44				0.9692207112
45				0.9568733730
46				0.9872988979
47				0.9758033932
48				0.9543279009
49				0.9661716333
50				0.9968928133
51				0.9852856029
52				0.9636014261
53				0.9748191606
54				0.9796912981
55				0.9850895715
56				0.9658379388
57				0.9748191606
58				0.9796912981
59				0.9850895715
60				0.9658379388
61				0.9568821144
62				0.9778172454
63				0.9734710878
64				0.9521725485

Now we move to the third column, which displays the outcome of the T=4 zero coupon bond price after 16 scenarios: Shift 1 followed by Shifts 1, 2, 3, or 4; Shift 2 followed by Shifts 1, 2, 3 or 4 and so on. We calculate $P(2,4,s_t$ = shift 1), $P(2,4,s_t$=shift 2), $P(2,4,s_t$=shift 3) and $P(2,4,s_t$ = shift 4) after the initial state of "shift 2" as follows. After a shift = shift 2, the prevailing 1 period zero coupon bond price at time t=1 is 0.9931369048, which implies a 1 period (1 year) U.S. Treasury spot rate of 0.0069105228 (i.e. 69 basis points). This is data group 4. When t=1, T=4, and Δ=1 then the volatilities for the two remaining 1 period forward rates that are relevant are taken from the lookup table for data group 4 for risk factor 1: 0.0006425743, and 0.0004087264. For risk factor 2, the volatilities for the two remaining 1 period forward rates are also chosen from data group 4: 0.0021277322, and 0.0028697122. Finally, the volatilities for risk factor 3 are also taken from data group 4: 0.0040702120 and 0.0054595415.

At time 1 in the shift 2 state, the zero coupon bond prices for maturities at T=2, 3 and 4 are 0.9931369048, 0.9815565410, and 0.9635638078. We make the intermediate calculations as above for the zero coupon bond maturing at T=4:

$K(1,1,4,s_t)$ = 0.0010513007
As before, remembering the minus sign,
$K(2,1,4,s_t)$ = -0.0049974444
$K(3,1,4,s_t)$ = -0.0095297535
$\mu(t,T)$ = 0.0000474532

We can calculate the values of the 1 period forward return maturing at time T=4 in shifts 1, 2, 3, and 4 as follows: 1.0336715790, 1.0113427849, 1.0141808568, and 1.0191379141. Similarly, using the appropriate intermediate calculations, we can calculate the forward returns for maturity at T=3: 1.0224958583, 1.0059835405, 1.0081207950, and 1.0124591923. Since

$P(2,4,s_t$ = Shift 2 Shift 1) =1/[$F(1,3,s_t$ = Shift 2 Shift 1) $F(1,4,s_t$ = Shift 2 Shift 1)]

the zero coupon bond prices for maturity at T=4 after an initial shift 2 for subsequent shifts 1, 2, 3, and 4 are as follows:

0.9461410097
0.9829031898
0.9780746840
0.9691466829

We have correctly populated the fifth, sixth, seventh, and eighth rows of column 3 (t=2) of the bushy tree above for the zero coupon bond maturing at T=4 (note values have been rounded for display only). The remaining calculations are left to the reader.

If we combine all of these tables, we can create a consolidated table of the term structure of zero coupon bond prices in each scenario as in examples one, two and three.

Figure 14

State Name	State Number	Zero Coupon Bond Prices — Periods to Maturity 1	2	3	4
Time 0	**0**	0.9970057865	0.9841101497	0.9618922376	0.9308550992
Time 1 S-1	**1**	0.9788061814	0.9425139335	0.8945353929	
Time 1 S-2	**2**	0.9931369048	0.9815565410	0.9635638078	
Time 1 S-3	**3**	0.9891467808	0.9691882649	0.9404125772	
Time 1 S-4	**4**	0.9865721079	0.9639502449	0.9325702085	
Time 2 S-1, S-1	**5**	0.9556681147	0.8951939726		
Time 2 S-1, S-2	**6**	0.9860553163	0.9550797882		
Time 2 S-1, S-3	**7**	0.9745742911	0.9332467863		
Time 2 S-1, S-4	**8**	0.9531258489	0.8986172056		
Time 2 S-2, S-1	**9**	0.9779990715	0.9461410097		
Time 2 S-2, S-2	**10**	0.9940520493	0.9829031898		
Time 2 S-2, S-3	**11**	0.9919446211	0.9780746840		
Time 2 S-2, S-4	**12**	0.9876941290	0.9691466829		
Time 2 S-3, S-1	**13**	0.9811450109	0.9579764308		
Time 2 S-3, S-2	**14**	0.9860487650	0.9627643885		
Time 2 S-3, S-3	**15**	0.9914820691	0.9680693916		
Time 2 S-3, S-4	**16**	0.9721055077	0.9372422391		
Time 2 S-4, S-1	**17**	0.9783890415	0.9524667898		
Time 2 S-4, S-2	**18**	0.9832790213	0.9572272104		
Time 2 S-4, S-3	**19**	0.9886970636	0.9625017026		
Time 2 S-4, S-4	**20**	0.9693749296	0.9318518473		
Time 3 S-1, S-1, S-1	**21**	0.9424127368			
Time 3 S-1, S-1, S-2	**22**	0.9472682532			
Time 3 S-1, S-1, S-3	**23**	0.9508321387			
Time 3 S-1, S-1, S-4	**24**	0.9256048053			
Time 3 S-1, S-2, S-1	**25**	0.9698937713			
Time 3 S-1, S-2, S-2	**26**	0.9747412918			
Time 3 S-1, S-2, S-3	**27**	0.9801122898			
Time 3 S-1, S-2, S-4	**28**	0.9609579282			
Time 3 S-1, S-3, S-1	**29**	0.9503806226			
Time 3 S-1, S-3, S-2	**30**	0.9805996987			
Time 3 S-1, S-3, S-3	**31**	0.9691821954			
Time 3 S-1, S-3, S-4	**32**	0.9478524225			
Time 3 S-1, S-4, S-1	**33**	0.9485398381			
Time 3 S-1, S-4, S-2	**34**	0.9534269227			
Time 3 S-1, S-4, S-3	**35**	0.9570139788			
Time 3 S-1, S-4, S-4	**36**	0.9316226298			
Time 3 S-2, S-1, S-1	**37**	0.9601375282			
Time 3 S-2, S-1, S-2	**38**	0.9906668429			
Time 3 S-2, S-1, S-3	**39**	0.9791321239			
Time 3 S-2, S-1, S-4	**40**	0.9575833729			
Time 3 S-2, S-2, S-1	**41**	0.9784392385			
Time 3 S-2, S-2, S-2	**42**	0.9944994413			
Time 3 S-2, S-2, S-3	**43**	0.9923910646			
Time 3 S-2, S-2, S-4	**44**	0.9881386595			
Time 3 S-2, S-3, S-1	**45**	0.9744393509			
Time 3 S-2, S-3, S-2	**46**	1.0037479721			
Time 3 S-2, S-3, S-3	**47**	0.9974134983			
Time 3 S-2, S-3, S-4	**48**	0.9787812762			
Time 3 S-2, S-4, S-1	**49**	0.9825458738			
Time 3 S-2, S-4, S-2	**50**	0.9874566295			
Time 3 S-2, S-4, S-3	**51**	0.9928976912			
Time 3 S-2, S-4, S-4	**52**	0.9734934642			
Time 3 S-3, S-1, S-1	**53**	0.9777040612			
Time 3 S-3, S-1, S-2	**54**	0.9825906175			
Time 3 S-3, S-1, S-3	**55**	0.9880048666			
Time 3 S-3, S-1, S-4	**56**	0.9686962602			
Time 3 S-3, S-2, S-1	**57**	0.9777040612			
Time 3 S-3, S-2, S-2	**58**	0.9825906175			
Time 3 S-3, S-2, S-3	**59**	0.9880048666			
Time 3 S-3, S-2, S-4	**60**	0.9686962602			
Time 3 S-3, S-3, S-1	**61**	0.9649211970			
Time 3 S-3, S-3, S-2	**62**	0.9939435367			
Time 3 S-3, S-3, S-3	**63**	0.9876709370			
Time 3 S-3, S-3, S-4	**64**	0.9692207112			
Time 3 S-3, S-4, S-1	**65**	0.9568733730			
Time 3 S-3, S-4, S-2	**66**	0.9872988979			
Time 3 S-3, S-4, S-3	**67**	0.9758033932			
Time 3 S-3, S-4, S-4	**68**	0.9543279009			
Time 3 S-4, S-1, S-1	**69**	0.9661716333			
Time 3 S-4, S-1, S-2	**70**	0.9968928133			
Time 3 S-4, S-1, S-3	**71**	0.9852856029			
Time 3 S-4, S-1, S-4	**72**	0.9636014261			
Time 3 S-4, S-2, S-1	**73**	0.9748191606			
Time 3 S-4, S-2, S-2	**74**	0.9796912981			
Time 3 S-4, S-2, S-3	**75**	0.9850895715			
Time 3 S-4, S-2, S-4	**76**	0.9658379388			
Time 3 S-4, S-3, S-1	**77**	0.9748191606			
Time 3 S-4, S-3, S-2	**78**	0.9796912981			
Time 3 S-4, S-3, S-3	**79**	0.9850895715			
Time 3 S-4, S-3, S-4	**80**	0.9658379388			
Time 3 S-4, S-4, S-1	**81**	0.9568821144			
Time 3 S-4, S-4, S-2	**82**	0.9778172454			
Time 3 S-4, S-4, S-3	**83**	0.9734710878			
Time 3 S-4, S-4, S-4	**84**	0.9521725485			

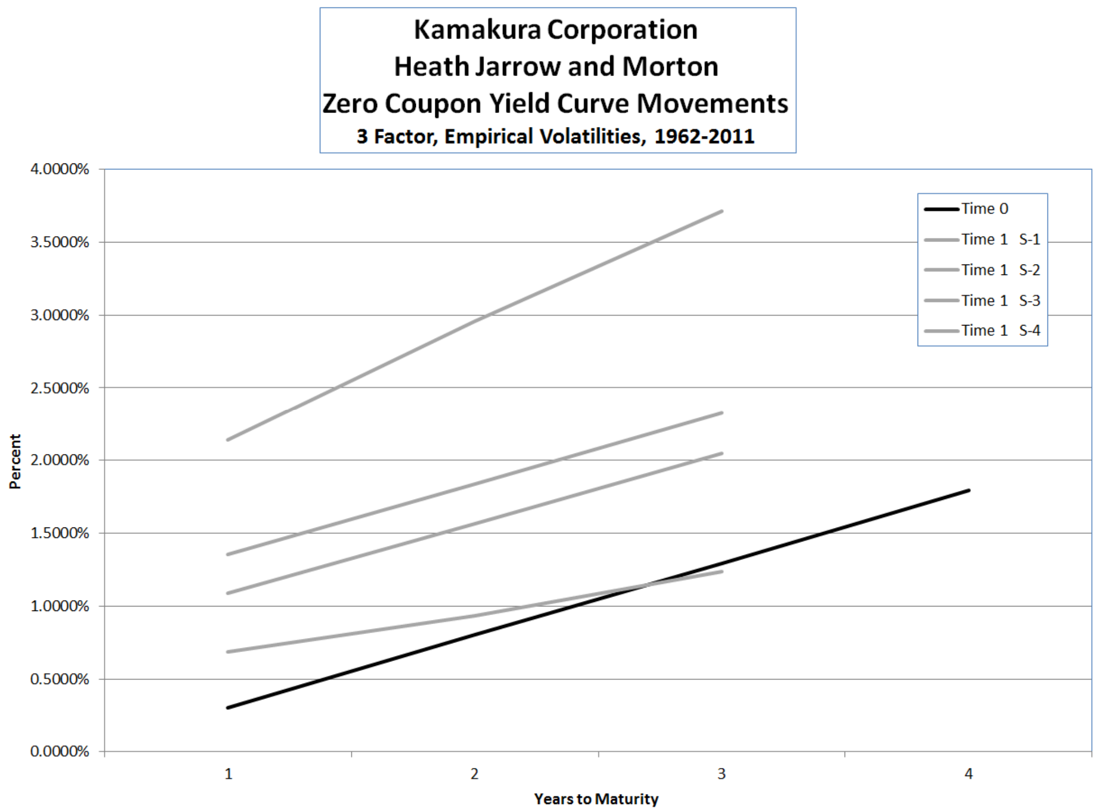
At any point in time t, the continuously compounded yield to maturity at time T can be calculated as y(T-t)=-ln[P(t,T)]/(T-t). Note that we have one incidence of negative rates (in state 46, shaded) on this bushy tree. This can be corrected by overriding the spike in volatility we discussed above in the data group for the 1 year spot U.S. Treasury rate levels between 0.75% and 1.00%. Note that yield curve curve shifts are much more complex than in the three prior examples using one and two risk factors:

Figure 15

State Name	State Number	Continuously Compounded Zero Yields Periods to Maturity 1	2	3	4
Time 0	0	0.2999%	0.8009%	1.2951%	1.7913%
Time 1 S-1	1	2.1422%	2.9602%	3.7150%	
Time 1 S-2	2	0.6887%	0.9308%	1.2372%	
Time 1 S-3	3	1.0913%	1.5648%	2.0479%	
Time 1 S-4	4	1.3519%	1.8358%	2.3270%	
Time 2 S-1, S-1	5	4.5345%	5.5357%		
Time 2 S-1, S-2	6	1.4043%	2.2980%		
Time 2 S-1, S-3	7	2.5755%	3.4543%		
Time 2 S-1, S-4	8	4.8008%	5.3449%		
Time 2 S-2, S-1	9	2.2247%	2.7682%		
Time 2 S-2, S-2	10	0.5966%	0.8622%		
Time 2 S-2, S-3	11	0.8088%	1.1085%		
Time 2 S-2, S-4	12	1.2382%	1.5670%		
Time 2 S-3, S-1	13	1.9035%	2.1466%		
Time 2 S-3, S-2	14	1.4049%	1.8973%		
Time 2 S-3, S-3	15	0.8554%	1.6226%		
Time 2 S-3, S-4	16	2.8291%	3.2407%		
Time 2 S-4, S-1	17	2.1848%	2.4350%		
Time 2 S-4, S-2	18	1.6862%	2.1857%		
Time 2 S-4, S-3	19	1.1367%	1.9110%		
Time 2 S-4, S-4	20	3.1104%	3.5291%		
Time 3 S-1, S-1, S-1	21	5.9312%			
Time 3 S-1, S-1, S-2	22	5.4173%			
Time 3 S-1, S-1, S-3	23	5.0418%			
Time 3 S-1, S-1, S-4	24	7.7308%			
Time 3 S-1, S-2, S-1	25	3.0569%			
Time 3 S-1, S-2, S-2	26	2.5583%			
Time 3 S-1, S-2, S-3	27	2.0088%			
Time 3 S-1, S-2, S-4	28	3.9825%			
Time 3 S-1, S-3, S-1	29	5.0893%			
Time 3 S-1, S-3, S-2	30	1.9591%			
Time 3 S-1, S-3, S-3	31	3.1303%			
Time 3 S-1, S-3, S-4	32	5.3556%			
Time 3 S-1, S-4, S-1	33	5.2831%			
Time 3 S-1, S-4, S-2	34	4.7692%			
Time 3 S-1, S-4, S-3	35	4.3937%			
Time 3 S-1, S-4, S-4	36	7.0827%			
Time 3 S-2, S-1, S-1	37	4.0679%			
Time 3 S-2, S-1, S-2	38	0.9377%			
Time 3 S-2, S-1, S-3	39	2.1089%			
Time 3 S-2, S-1, S-4	40	4.3342%			
Time 3 S-2, S-2, S-1	41	2.1797%			
Time 3 S-2, S-2, S-2	42	0.5516%			
Time 3 S-2, S-2, S-3	43	0.7638%			
Time 3 S-2, S-2, S-4	44	1.1932%			
Time 3 S-2, S-3, S-1	45	2.5893%			
Time 3 S-2, S-3, S-2	46	-0.3741%			
Time 3 S-2, S-3, S-3	47	0.2590%			
Time 3 S-2, S-3, S-4	48	2.1447%			
Time 3 S-2, S-4, S-1	49	1.7608%			
Time 3 S-2, S-4, S-2	50	1.2623%			
Time 3 S-2, S-4, S-3	51	0.7128%			
Time 3 S-2, S-4, S-4	52	2.6864%			
Time 3 S-3, S-1, S-1	53	2.2548%			
Time 3 S-3, S-1, S-2	54	1.7563%			
Time 3 S-3, S-1, S-3	55	1.2068%			
Time 3 S-3, S-1, S-4	56	3.1804%			
Time 3 S-3, S-2, S-1	57	2.2548%			
Time 3 S-3, S-2, S-2	58	1.7563%			
Time 3 S-3, S-2, S-3	59	1.2068%			
Time 3 S-3, S-2, S-4	60	3.1804%			
Time 3 S-3, S-3, S-1	61	3.5709%			
Time 3 S-3, S-3, S-2	62	0.6075%			
Time 3 S-3, S-3, S-3	63	1.2406%			
Time 3 S-3, S-3, S-4	64	3.1263%			
Time 3 S-3, S-4, S-1	65	4.4084%			
Time 3 S-3, S-4, S-2	66	1.2782%			
Time 3 S-3, S-4, S-3	67	2.4494%			
Time 3 S-3, S-4, S-4	68	4.6748%			
Time 3 S-4, S-1, S-1	69	3.4414%			
Time 3 S-4, S-1, S-2	70	0.3112%			
Time 3 S-4, S-1, S-3	71	1.4824%			
Time 3 S-4, S-1, S-4	72	3.7078%			
Time 3 S-4, S-2, S-1	73	2.5503%			
Time 3 S-4, S-2, S-2	74	2.0518%			
Time 3 S-4, S-2, S-3	75	1.5023%			
Time 3 S-4, S-2, S-4	76	3.4759%			
Time 3 S-4, S-3, S-1	77	2.5503%			
Time 3 S-4, S-3, S-2	78	2.0518%			
Time 3 S-4, S-3, S-3	79	1.5023%			
Time 3 S-4, S-3, S-4	80	3.4759%			
Time 3 S-4, S-4, S-1	81	4.4075%			
Time 3 S-4, S-4, S-2	82	2.2432%			
Time 3 S-4, S-4, S-3	83	2.6887%			
Time 3 S-4, S-4, S-4	84	4.9009%			

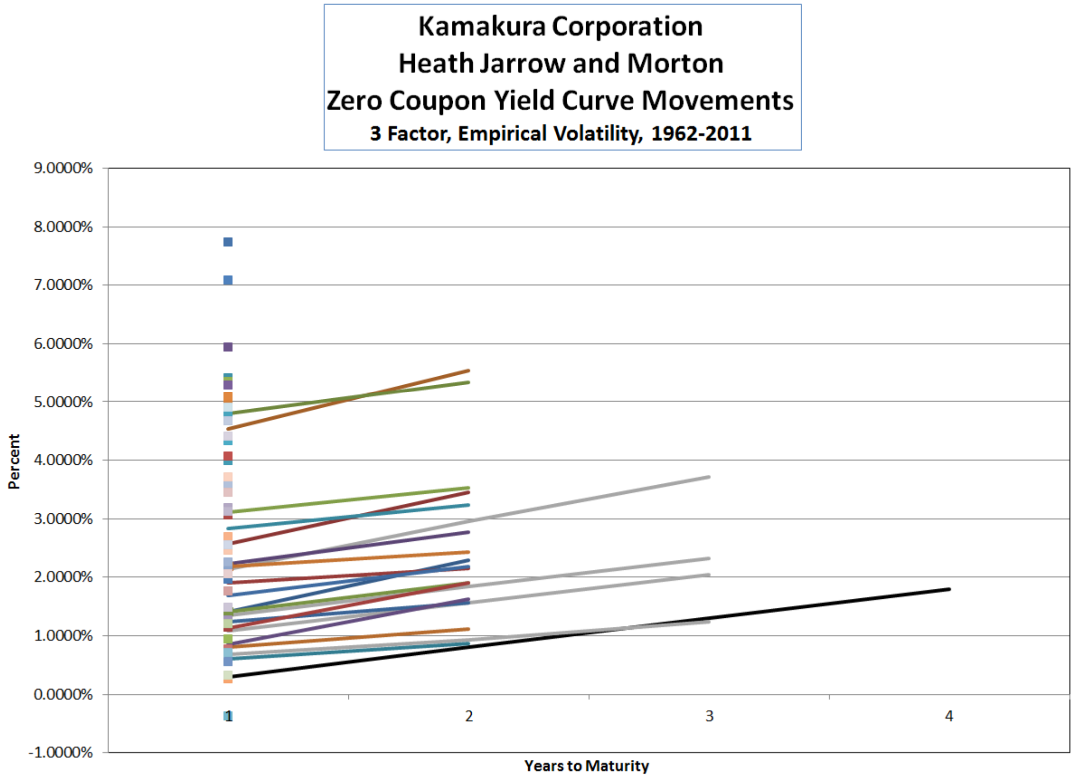
We can graph yield curve movements as shown below at time t=1. We plot the original spot Treasury yield curve and then shifted yield curves for shifts 1, 2, 3, and 4. These shifts are relative to the 1 period forward rates prevailing at time zero for maturity at times T=2, 3 and 4. Because these 1 period forward rates were much higher than yields as of time t=0, all four shifts produce yields higher than time zero yields with the exception of one yield for maturity at T=4.

Figure 16



When we add 16 yield curves prevailing at time t=3 and 64 “single point” yield curves prevailing at time t=4, two things are very clear. First, yield curve movements in a three factor model are much more complex and much more realistic than what we saw in the one-factor and two-factor examples. Second, in the low yield environment prevailing as of March 31, 2011, no arbitrage yield curve simulation shows “there is nowhere to go but up” from a yield curve perspective, with just a few exceptions.

Figure 17



Finally, we can display the 1 year U.S. Treasury spot rates and the associated term structure of 1 year forward rates in each scenario.

Figure 18 [Chapter09-Figure18.tif]

Spot Rate Process

State	Row	0	1	2	3
S-1, S-1, S-1	1	0.3003%	2.1653%	4.6388%	6.1106%
S-1, S-1, S-2	2		0.6911%	1.4142%	5.5667%
S-1, S-1, S-3	3		1.0972%	2.6089%	5.1710%
S-1, S-1, S-4	4		1.3611%	4.9179%	8.0375%
S-1, S-2, S-1	5			2.2496%	3.1041%
S-1, S-2, S-2	6			0.5984%	2.5913%
S-1, S-2, S-3	7			0.8121%	2.0291%
S-1, S-2, S-4	8			1.2459%	4.0628%
S-1, S-3, S-1	9			1.9217%	5.2210%
S-1, S-3, S-2	10			1.4149%	1.9784%
S-1, S-3, S-3	11			0.8591%	3.1798%
S-1, S-3, S-4	12			2.8695%	5.5017%
S-1, S-4, S-1	13			2.2088%	5.4252%
S-1, S-4, S-2	14			1.7005%	4.8848%
S-1, S-4, S-3	15			1.1432%	4.4917%
S-1, S-4, S-4	16			3.1593%	7.3396%
S-2, S-1, S-1	17				4.1517%
S-2, S-1, S-2	18				0.9421%
S-2, S-1, S-3	19				2.1313%
S-2, S-1, S-4	20				4.4295%
S-2, S-2, S-1	21				2.2036%
S-2, S-2, S-2	22				0.5531%
S-2, S-2, S-3	23				0.7667%
S-2, S-2, S-4	24				1.2004%
S-2, S-3, S-1	25				2.6231%
S-2, S-3, S-2	26				-0.3734%
S-2, S-3, S-3	27				0.2593%
S-2, S-3, S-4	28				2.1679%
S-2, S-4, S-1	29				1.7764%
S-2, S-4, S-2	30				1.2703%
S-2, S-4, S-3	31				0.7153%
S-2, S-4, S-4	32				2.7228%
S-3, S-1, S-1	33				2.2804%
S-3, S-1, S-2	34				1.7718%
S-3, S-1, S-3	35				1.2141%
S-3, S-1, S-4	36				3.2315%
S-3, S-2, S-1	37				2.2804%
S-3, S-2, S-2	38				1.7718%
S-3, S-2, S-3	39				1.2141%
S-3, S-2, S-4	40				3.2315%
S-3, S-3, S-1	41				3.6354%
S-3, S-3, S-2	42				0.6093%
S-3, S-3, S-3	43				1.2483%
S-3, S-3, S-4	44				3.1757%
S-3, S-4, S-1	45				4.5070%
S-3, S-4, S-2	46				1.2864%
S-3, S-4, S-3	47				2.4797%
S-3, S-4, S-4	48				4.7858%
S-4, S-1, S-1	49				3.5013%
S-4, S-1, S-2	50				0.3117%
S-4, S-1, S-3	51				1.4934%
S-4, S-1, S-4	52				3.7773%
S-4, S-2, S-1	53				2.5831%
S-4, S-2, S-2	54				2.0730%
S-4, S-2, S-3	55				1.5136%
S-4, S-2, S-4	56				3.5370%
S-4, S-3, S-1	57				2.5831%
S-4, S-3, S-2	58				2.0730%
S-4, S-3, S-3	59				1.5136%
S-4, S-3, S-4	60				3.5370%
S-4, S-4, S-1	61				4.5061%
S-4, S-4, S-2	62				2.2686%
S-4, S-4, S-3	63				2.7252%
S-4, S-4, S-4	64				5.0230%

Valuation in the Heath Jarrow and Morton Framework

Prof. Jarrow, quoted in Chapter 6, described valuation as the expected value of cash flows using the risk neutral probabilities. Note that column 1 denotes the riskless 1 period interest rate in each scenario. For the state number 84 (three consecutive shift 4's), cash flows at time T=4 would be discounted by the one year spot rates at time t=0, by the one year spot rate at time t=1 in state 4 ("shift 4"), by the one year spot rate in state 20 ("shift 4 shift 4") at time t=2, and by the one year spot rate at time t=3 in state 84 ("shift 4 shift 4 shift 4"). The discount factor is

Discount Factor (0,4, Shift 4-Shift 4-Shift 4)
=1/(1.003003)(1.013519)(1.031104)(1.049009)

These discount factors are displayed here for each potential cash flow date:

Figure 19

	Discount Factors				
	Current Time				
Row Number	0	1	2	3	4
1	1	0.9970057865	0.9758754267	0.9326130292	0.8789063973
2		0.9970057865	0.9758754267	0.9326130292	0.8834347151
3		0.9970057865	0.9758754267	0.9326130292	0.8867584412
4		0.9970057865	0.9758754267	0.9326130292	0.8632311013
5			0.9901632408	0.9622671525	0.9332969176
6			0.9901632408	0.9622671525	0.9379615274
7			0.9901632408	0.9622671525	0.9431298622
8			0.9901632408	0.9622671525	0.9246982493
9			0.9861850641	0.9510631021	0.9038719431
10			0.9861850641	0.9510631021	0.9326121914
11			0.9861850641	0.9510631021	0.9217534253
12			0.9861850641	0.9510631021	0.9014674653
13			0.9836181004	0.9301320945	0.8822673463
14			0.9836181004	0.9301320945	0.8868129805
15			0.9836181004	0.9301320945	0.8901494165
16			0.9836181004	0.9301320945	0.8665321079
17				0.9683787302	0.9297767604
18				0.9683787302	0.9593406994
19				0.9683787302	0.9481707228
20				0.9683787302	0.9273033706
21				0.9842737987	0.9630521061
22				0.9842737987	0.9788597428
23				0.9842737987	0.9767845229
24				0.9842737987	0.9725989919
25				0.9821871007	0.9570817609
26				0.9821871007	0.9858683106
27				0.9821871007	0.9796466721
28				0.9821871007	0.9613463439
29				0.9779784197	0.9609086610
30				0.9779784197	0.9657112740
31				0.9779784197	0.9710325149
32				0.9779784197	0.9520555997
33				0.9675905555	0.9460172157
34				0.9675905555	0.9507454014
35				0.9675905555	0.9559841777
36				0.9675905555	0.9373013525
37				0.9724265646	0.9507454014
38				0.9724265646	0.9554972185
39				0.9724265646	0.9607621782
40				0.9724265646	0.9419859764
41				0.9777848079	0.9434852873
42				0.9777848079	0.9718628902
43				0.9777848079	0.9657296374
44				0.9777848079	0.9476892869
45				0.9586759325	0.9173314731
46				0.9586759325	0.9464996916
47				0.9586759325	0.9354792279
48				0.9586759325	0.9148911903
49				0.9623611704	0.9298060639
50				0.9623611704	0.9593709346
51				0.9623611704	0.9482006060
52				0.9623611704	0.9273325962
53				0.9671710430	0.9428168643
54				0.9671710430	0.9475290547
55				0.9671710430	0.9527501083
56				0.9671710430	0.9341304867
57				0.9725003275	0.9480119529
58				0.9725003275	0.9527501083
59				0.9725003275	0.9579999309
60				0.9725003275	0.9392777118
61				0.9534947268	0.9123820502
62				0.9534947268	0.9323435872
63				0.9534947268	0.9281995489
64				0.9534947268	0.9078915040

When taking expected values, we can calculate the probability of each scenario coming about since the probabilities of shifts 1, 2, 3, and 4 are 1/8, 1/8, ¼ and 1/2:

Figure 20

	Probability of Each State				
	Current Time				
Row Number	0	1	2	3	4
1	100.0000%	12.5000%	1.562500%	0.1953%	0.1953%
2		12.5000%	1.562500%	0.1953%	0.1953%
3		25.0000%	3.125000%	0.3906%	0.3906%
4		50.0000%	6.250000%	0.7813%	0.7813%
5			1.562500%	0.1953%	0.1953%
6			1.562500%	0.1953%	0.1953%
7			3.125000%	0.3906%	0.3906%
8			6.250000%	0.7813%	0.7813%
9			3.125000%	0.3906%	0.3906%
10			3.125000%	0.3906%	0.3906%
11			6.250000%	0.7813%	0.7813%
12			12.500000%	1.5625%	1.5625%
13			6.250000%	0.7813%	0.7813%
14			6.250000%	0.7813%	0.7813%
15			12.500000%	1.5625%	1.5625%
16			25.000000%	3.1250%	3.1250%
17				0.1953%	0.1953%
18				0.1953%	0.1953%
19				0.3906%	0.3906%
20				0.7813%	0.7813%
21				0.1953%	0.1953%
22				0.1953%	0.1953%
23				0.3906%	0.3906%
24				0.7813%	0.7813%
25				0.3906%	0.3906%
26				0.3906%	0.3906%
27				0.7813%	0.7813%
28				1.5625%	1.5625%
29				0.7813%	0.7813%
30				0.7813%	0.7813%
31				1.5625%	1.5625%
32				3.1250%	3.1250%
33				0.3906%	0.3906%
34				0.3906%	0.3906%
35				0.7813%	0.7813%
36				1.5625%	1.5625%
37				0.3906%	0.3906%
38				0.3906%	0.3906%
39				0.7813%	0.7813%
40				1.5625%	1.5625%
41				0.7813%	0.7813%
42				0.7813%	0.7813%
43				1.5625%	1.5625%
44				3.1250%	3.1250%
45				1.5625%	1.5625%
46				1.5625%	1.5625%
47				3.1250%	3.1250%
48				6.2500%	6.2500%
49				0.7813%	0.7813%
50				0.7813%	0.7813%
51				1.5625%	1.5625%
52				3.1250%	3.1250%
53				0.7813%	0.7813%
54				0.7813%	0.7813%
55				1.5625%	1.5625%
56				3.1250%	3.1250%
57				1.5625%	1.5625%
58				1.5625%	1.5625%
59				3.1250%	3.1250%
60				6.2500%	6.2500%
61				3.1250%	3.1250%
62				3.1250%	3.1250%
63				6.2500%	6.2500%
64				12.5000%	12.5000%
Total	100.0000%	100.0000%	100.0000%	100.0000%	100.0000%

It is again convenient to calculate the "probability weighted discount factors" for use in calculating the expected present value of cash flows. The values shown are simply the probability of occurrence of that scenario times the corresponding discount factor:

Figure 21

	Probability Weighted Discount Factors				
	Current Time				
Row Number	0	1	2	3	4
1	1	0.1246257233	0.0152480535	0.0018215098	0.0017166141
2		0.1246257233	0.0152480535	0.0018215098	0.0017254584
3		0.2492514466	0.0304961071	0.0036430196	0.0034639002
4		0.4985028932	0.0609922142	0.0072860393	0.0067439930
5			0.0154713006	0.0018794280	0.0018228455
6			0.0154713006	0.0018794280	0.0018319561
7			0.0309426013	0.0037588561	0.0036841010
8			0.0618852026	0.0075177121	0.0072242051
9			0.0308182833	0.0037150902	0.0035307498
10			0.0308182833	0.0037150902	0.0036430164
11			0.0616365665	0.0074301805	0.0072011986
12			0.1232731330	0.0148603610	0.0140854291
13			0.0614761313	0.0072666570	0.0068927136
14			0.0614761313	0.0072666570	0.0069282264
15			0.1229522625	0.0145333140	0.0139085846
16			0.2459045251	0.0290666280	0.0270791284
17				0.0018913647	0.0018159702
18				0.0018913647	0.0018737123
19				0.0037827294	0.0037037919
20				0.0075654588	0.0072445576
21				0.0019224098	0.0018809611
22				0.0019224098	0.0019118354
23				0.0038448195	0.0038155645
24				0.0076896391	0.0075984296
25				0.0038366684	0.0037386006
26				0.0038366684	0.0038510481
27				0.0076733367	0.0076534896
28				0.0153466734	0.0150210366
29				0.0076404564	0.0075070989
30				0.0076404564	0.0075446193
31				0.0152809128	0.0151723830
32				0.0305618256	0.0297517375
33				0.0037796506	0.0036953797
34				0.0037796506	0.0037138492
35				0.0075593012	0.0074686264
36				0.0151186024	0.0146453336
37				0.0037985413	0.0037138492
38				0.0037985413	0.0037324110
39				0.0075970825	0.0075059545
40				0.0151941651	0.0147185309
41				0.0076389438	0.0073709788
42				0.0076389438	0.0075926788
43				0.0152778876	0.0150895256
44				0.0305557752	0.0296152902
45				0.0149793114	0.0143333043
46				0.0149793114	0.0147890577
47				0.0299586229	0.0292337259
48				0.0599172458	0.0571806994
49				0.0075184466	0.0072641099
50				0.0075184466	0.0074950854
51				0.0150368933	0.0148156345
52				0.0300737866	0.0289791436
53				0.0075560238	0.0073657568
54				0.0075560238	0.0074025707
55				0.0151120475	0.0148867204
56				0.0302240951	0.0291915777
57				0.0151953176	0.0148126868
58				0.0151953176	0.0148867204
59				0.0303906352	0.0299374978
60				0.0607812705	0.0587048570
61				0.0297967102	0.0285119391
62				0.0297967102	0.0291357371
63				0.0595934204	0.0580124718
64				0.1191868409	0.1134864380

We now use the HJM bushy trees we have generated to value representative securities.

Valuation of a Zero Coupon Bond Maturing at Time T=4

A riskless zero coupon bond pays $1 in each of the 64 nodes of the bushy tree that prevail at time T=4. Its present value is simply the sum of the probability weighted discount factors in the last column on the right hand side. Their sum is 0.9308550992, which is the value we should get in a no-arbitrage economy, the value observable in the market and used as an input to create the tree.

Valuation of a Coupon-Bearing Bond Paying Annual Interest

Next we value a bond with no credit risk that pays $3 in interest at every scenario at times T=1, 2, 3, and 4 plus principal of 100 at time T=4. The valuation is calculated by multiplying each cash flow by the matching probability weighted discount factor, to get a value of 104.7070997370 . It will surprise a few (primarily those readers who skipped Chapters 6 through 8) that this is the same value that we arrived at in Chapters 6 through 8, even though the volatilities used and number of risk factors used are different. The values are the same because, by construction, our valuations for the zero coupon bond prices at time zero for maturities at T = 1, 2, 3, and 4 continue to match the inputs. Multiplying these zero coupon bond prices times 3, 3, 3, and 103 also leads to a value of 104.7070997370 as it should.

Figure 22

	Maturity of Cash Flow Received				
	Current Time				
Row Number	0	1	2	3	4
1	0.00	3.00	3.00	3.00	103.00
2		3.00	3.00	3.00	103.00
3		3.00	3.00	3.00	103.00
4		3.00	3.00	3.00	103.00
5			3.00	3.00	103.00
6			3.00	3.00	103.00
7			3.00	3.00	103.00
8			3.00	3.00	103.00
9			3.00	3.00	103.00
10			3.00	3.00	103.00
11			3.00	3.00	103.00
12			3.00	3.00	103.00
13			3.00	3.00	103.00
14			3.00	3.00	103.00
15			3.00	3.00	103.00
16			3.00	3.00	103.00
17				3.00	103.00
18				3.00	103.00
19				3.00	103.00
20				3.00	103.00
21				3.00	103.00
22				3.00	103.00
23				3.00	103.00
24				3.00	103.00
25				3.00	103.00
26				3.00	103.00
27				3.00	103.00
28				3.00	103.00
29				3.00	103.00
30				3.00	103.00
31				3.00	103.00
32				3.00	103.00
33				3.00	103.00
34				3.00	103.00
35				3.00	103.00
36				3.00	103.00
37				3.00	103.00
38				3.00	103.00
39				3.00	103.00
40				3.00	103.00
41				3.00	103.00
42				3.00	103.00
43				3.00	103.00
44				3.00	103.00
45				3.00	103.00
46				3.00	103.00
47				3.00	103.00
48				3.00	103.00
49				3.00	103.00
50				3.00	103.00
51				3.00	103.00
52				3.00	103.00
53				3.00	103.00
54				3.00	103.00
55				3.00	103.00
56				3.00	103.00
57				3.00	103.00
58				3.00	103.00
59				3.00	103.00
60				3.00	103.00
61				3.00	103.00
62				3.00	103.00
63				3.00	103.00
64				3.00	103.00
Risk Neutral Value =					**104.7070997370**

Valuation of a Digital Option on the 1 Year U.S. Treasury Rate

Now we value a digital option that pays \$1 at time T=3 if (at that time) the one year U.S. Treasury rate (for maturity at T=4) is over 8%. If we look at the table of the term structure of one year spot rates over time, this happens in only one scenario, in row 4. Note also that we have a slightly negative spot rate in row 26. This can be easily over-ridden by changing the relevant interest rate volatility assumptions:

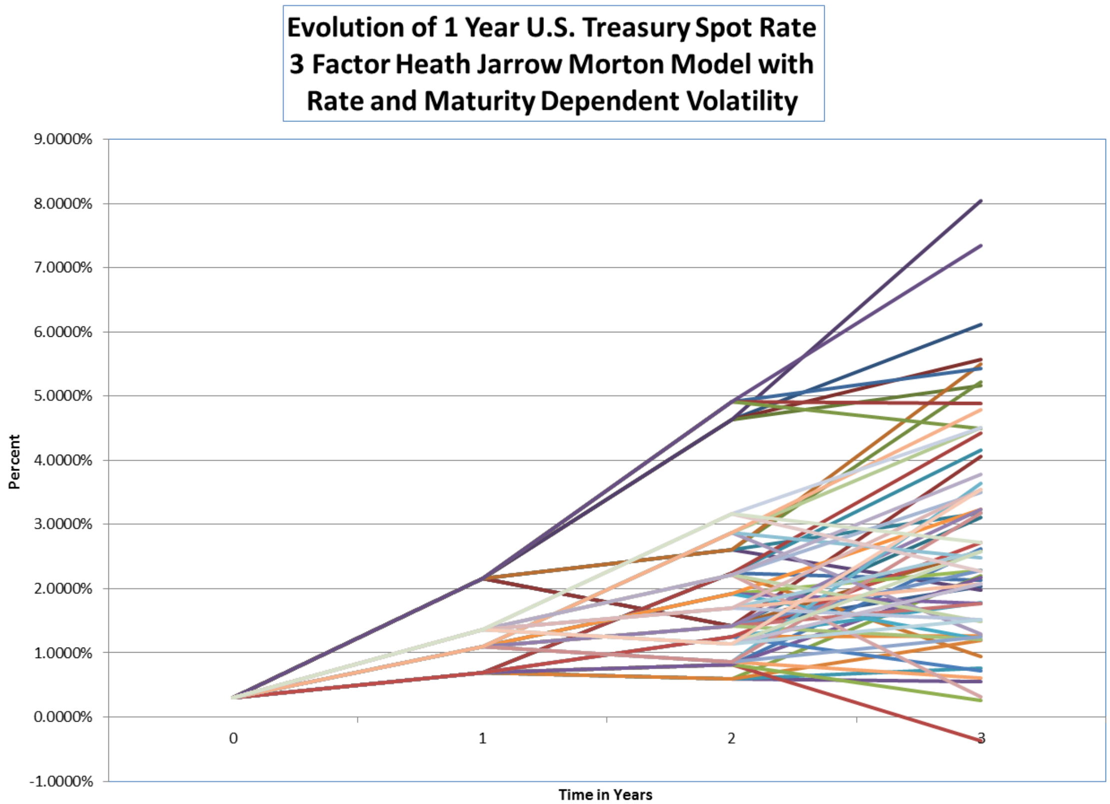
Figure 23

Spot Rate Process

State	Row	0	1	2	3
S-1, S-1, S-1	1	0.3003%	2.1653%	4.6388%	6.1106%
S-1, S-1, S-2	2		0.6911%	1.4142%	5.5667%
S-1, S-1, S-3	3		1.0972%	2.6089%	5.1710%
S-1, S-1, S-4	4		1.3611%	4.9179%	8.0375%
S-1, S-2, S-1	5			2.2496%	3.1041%
S-1, S-2, S-2	6			0.5984%	2.5913%
S-1, S-2, S-3	7			0.8121%	2.0291%
S-1, S-2, S-4	8			1.2459%	4.0628%
S-1, S-3, S-1	9			1.9217%	5.2210%
S-1, S-3, S-2	10			1.4149%	1.9784%
S-1, S-3, S-3	11			0.8591%	3.1798%
S-1, S-3, S-4	12			2.8695%	5.5017%
S-1, S-4, S-1	13			2.2088%	5.4252%
S-1, S-4, S-2	14			1.7005%	4.8848%
S-1, S-4, S-3	15			1.1432%	4.4917%
S-1, S-4, S-4	16			3.1593%	7.3396%
S-2, S-1, S-1	17				4.1517%
S-2, S-1, S-2	18				0.9421%
S-2, S-1, S-3	19				2.1313%
S-2, S-1, S-4	20				4.4295%
S-2, S-2, S-1	21				2.2036%
S-2, S-2, S-2	22				0.5531%
S-2, S-2, S-3	23				0.7667%
S-2, S-2, S-4	24				1.2004%
S-2, S-3, S-1	25				2.6231%
S-2, S-3, S-2	26				-0.3734%
S-2, S-3, S-3	27				0.2593%
S-2, S-3, S-4	28				2.1679%
S-2, S-4, S-1	29				1.7764%
S-2, S-4, S-2	30				1.2703%
S-2, S-4, S-3	31				0.7153%
S-2, S-4, S-4	32				2.7228%
S-3, S-1, S-1	33				2.2804%
S-3, S-1, S-2	34				1.7718%
S-3, S-1, S-3	35				1.2141%
S-3, S-1, S-4	36				3.2315%
S-3, S-2, S-1	37				2.2804%
S-3, S-2, S-2	38				1.7718%
S-3, S-2, S-3	39				1.2141%
S-3, S-2, S-4	40				3.2315%
S-3, S-3, S-1	41				3.6354%
S-3, S-3, S-2	42				0.6093%
S-3, S-3, S-3	43				1.2483%
S-3, S-3, S-4	44				3.1757%
S-3, S-4, S-1	45				4.5070%
S-3, S-4, S-2	46				1.2864%
S-3, S-4, S-3	47				2.4797%
S-3, S-4, S-4	48				4.7858%
S-4, S-1, S-1	49				3.5013%
S-4, S-1, S-2	50				0.3117%
S-4, S-1, S-3	51				1.4934%
S-4, S-1, S-4	52				3.7773%
S-4, S-2, S-1	53				2.5831%
S-4, S-2, S-2	54				2.0730%
S-4, S-2, S-3	55				1.5136%
S-4, S-2, S-4	56				3.5370%
S-4, S-3, S-1	57				2.5831%
S-4, S-3, S-2	58				2.0730%
S-4, S-3, S-3	59				1.5136%
S-4, S-3, S-4	60				3.5370%
S-4, S-4, S-1	61				4.5061%
S-4, S-4, S-2	62				2.2686%
S-4, S-4, S-3	63				2.7252%
S-4, S-4, S-4	64				5.0230%

The evolution of the spot rate can be displayed graphically as follows:

Figure 24



The cash flow payoff in the one relevant scenario can be input in the table above and multiplied by the appropriate probability weighted discount factor to find that this option has a value of only $ 0.0072860393. That is because the probability of row 4 occurring is only 0.7813% and the payoff is discounted by the simulated spot rates prior to time T=3.

Conclusion

This chapter, the fourth example of HJM interest rate simulation, shows how to simulate zero coupon bond prices, forward rates and zero coupon bond yields in an HJM framework with three risk factors and rate-dependent and maturity-dependent interest rate volatility. The results show a rich twist in simulated yield curves and a pull of rates upward from a very low rate environment. Monte Carlo simulation, an alternative to the bushy tree framework, can be done in a fully consistent way. The technique can be generalized to any user-specified number of risk factors, a long-standing feature of the Kamakura Risk Manager enterprise risk management system. We explain how to do this in Chapter 10. Also in Chapter 10, we discuss the implications of this type of analysis for the Basel-specified Internal Capital Adequacy Assessment Process ("ICAAP") for interest rate risk.