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Advanced Financial Risk Management:

Tools and Techniques for Integrated Credit Risk, Market Risk, Liquidity Risk and Interest Rate Risk Management

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Part 2: Risk Management Techniques for Interest Rate Analytics

Chapter 6:
Introduction to Heath, Jarrow and Morton interest rate modeling

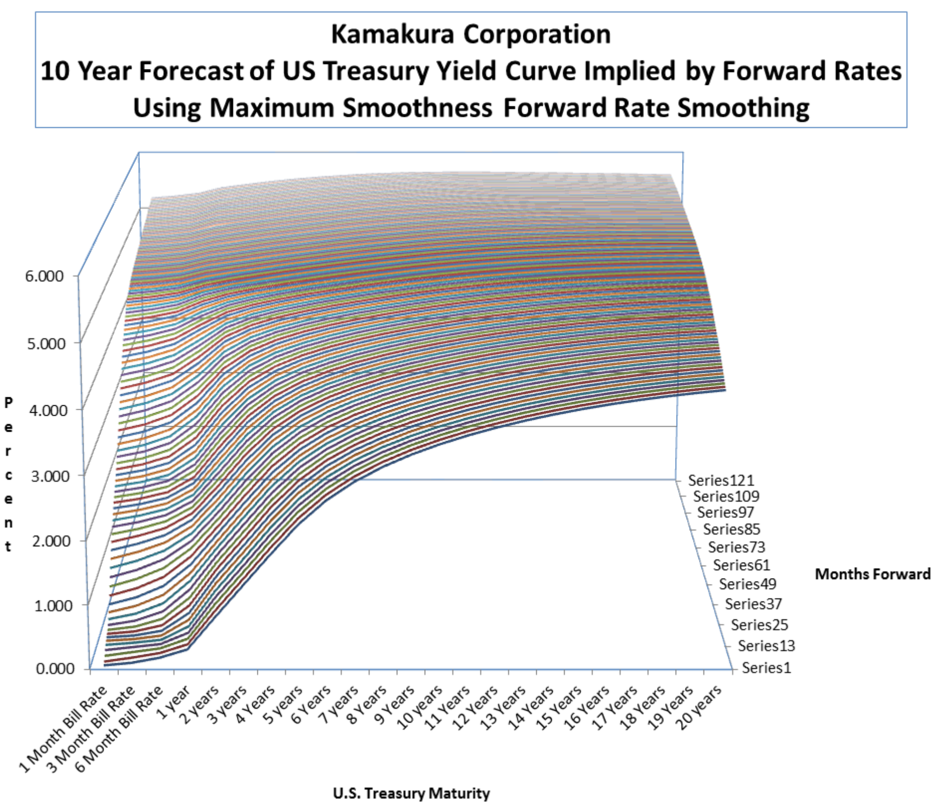
In Chapter 5 we mentioned the review by Dickler, Jarrow and van Deventer of 50 years of daily U.S. Treasury interest rate movements. They concluded that U.S. interest rates are driven by 5-10 random factors, a much larger number of factors than major financial institutions typically use for interest rate risk management. In a series of three papers, David Heath, Robert A. Jarrow and Andrew Morton (1990, 1990, 1992) introduced the "Heath Jarrow and Morton" ("HJM") framework for modeling interest rates driven by a large number of random factors while preserving the standard "no arbitrage" assumptions of modern finance. The HJM framework is a very powerful general solution for modeling interest rates driven by a large number of factors and for the valuation and simulation of cash flows on securities of all types. The authors believe that an understanding of the HJM approach is a fundamental requirement for a risk manager. In this chapter and the following three chapters, we lay out four worked examples of the use of the HJM approach with increasing realism. A full enterprise wide risk management infrastructure relies on a sophisticated implementation of the HJM framework with the 5-10 driving risk factors and a very large number of time steps. Such an implementation should make use of both the HJM implications for Monte Carlo simulation and the "bushy tree" approach for valuing securities like callable bonds and home mortgages. For expositional purposes, this and the following chapters focus on the bushy tree approach, but we explain why an HJM Monte Carlo simulation is essential for the simulation of cash flows, net income, and liquidity risk in Chapter 10.

The authors wish to thank Prof. Robert A. Jarrow for his encouragement and advice on this series of worked examples of the HJM approach. What follows is based heavily on Prof. Jarrow's classic book, *Modeling Fixed Income Securities and Interest Rate Options* (second edition, 2002), particularly chapters 4, 6, 8, 9 and 15. In our Chapter 9, we incorporate some modifications of Chapter 15 in Prof. Jarrow's book that would have been impossible without Prof. Jarrow's advice and support.

In Chapters 6 through 9, we use data from the Federal Reserve statistical release H15 published on April 1, 2011. U.S. Treasury yield curve data was smoothed using Kamakura Corporation's Kamakura Risk Manager version 7.3 to create zero coupon bonds via the maximum smoothness technique of Adams and van Deventer as described in Chapter 5. Most of the applications of the HJM approach by financial market participants have come in the form of the "Libor market model" and applications related to the interest rate swap market and deposits issued at the London interbank offered rate, or "Libor." In light of recent lawsuits alleging manipulation of Libor rates, this application of HJM is a complex special case that we will defer to Chapter 17. In this chapter, we assume away such complexities and assume away the potential default of the U.S. government. In this chapter, as in the original HJM publications, it is assumed that the underlying yield curve is "risk free" from a credit risk perspective.

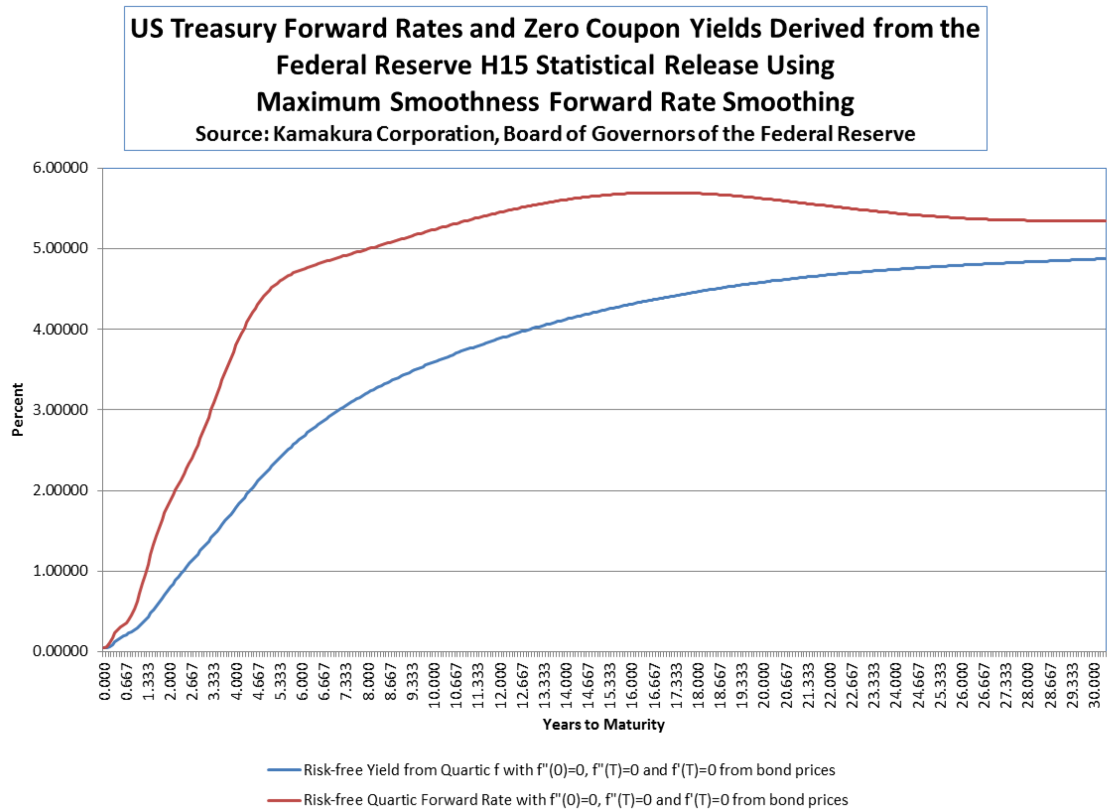
The smoothed U.S. Treasury yield curve and the implied forward yield curve monthly for ten years looked like this as of the March 31, 2011 data reported by the Federal Reserve on April 1, 2011:

Figure 1



The continuous forward rate curve and zero coupon bond yield curve that prevailed as of the close of business on March 31, 2011 were as follows:

Figure 2



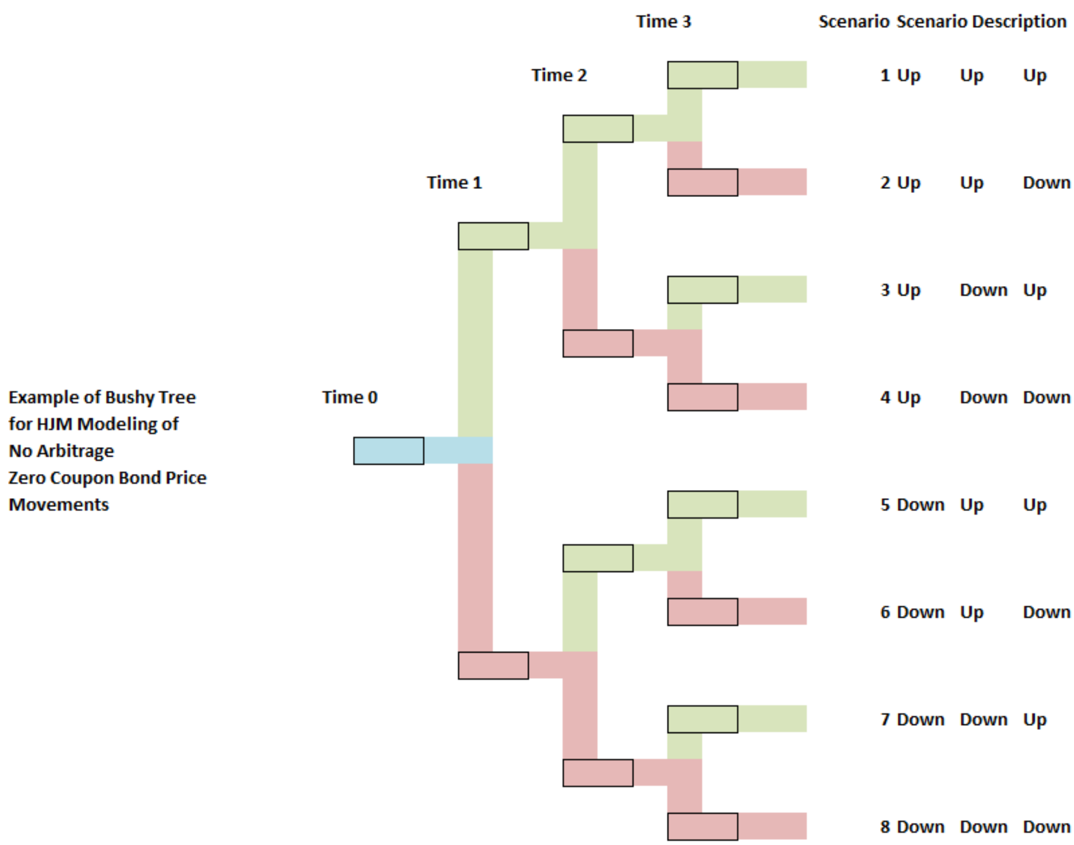
Objectives of the Example and Key Input Data

Following Jarrow (2002), we want to show how to use the HJM framework to model random interest rates and value fixed income securities and other securities using the simulated random rates. For all of the examples in Chapters 6 through 9, we assume the following:

- Zero coupon bond prices for the U.S. Treasury curve on March 31, 2011 are the basic inputs.
- Interest rate volatility assumptions are based on the Dickler, Jarrow and van Deventer papers cited above on daily U.S. Treasury yields and forward rates from 1962 to 2011.
- The modeling period is 4 equal length periods of one year each.
- The HJM implementation is that of a "bushy tree" which we describe below

The HJM framework is usually implemented using Monte Carlo simulation or a "bushy tree" approach where a lattice of interest rates and forward rates is constructed. This lattice, in the general case, does not "recombine" like the popular "binomial" or "trinomial" trees used to replicate Black-Scholes options valuation or simple legacy 1 factor term structure models which we discuss in Chapter 13. In general, the bushy tree does not recombine because the interest rate volatility assumptions imply a path-dependent interest rate model, not one that is path independent like the simplest one factor term structure model implementations. Instead the bushy tree of "up shifts" and "down shifts" in interest rates looks like this:

Figure 3



At each of the points in time on the lattice (time 0, 1, 2, 3 and 4) there are sets of zero coupon bond prices and forward rates. At time 0, there is one set of data. At time one, there are two sets of data, the "up set" and the "down set." At time two, there are four sets of data (up up, up down, down up, and down down), and at time three there are $8=2^3$ sets of data.

In order to build a bushy tree like this, we need to specify how many factors we are modeling and what the interest volatility assumptions are. The number of ending nodes on a bushy tree for n periods is 2^n for a 1 factor model, 3^n for a 2 factor model, and 4^n for a 3 factor model. For this example, we will build a one factor bushy tree. "One factor" implies some important unrealistic restrictions on simulated yield curve movements, which we discuss below and in Chapters 7 through 9 and 13.

The simplest interest rate volatility function one can assume is that the one year spot U.S. Treasury rate and the three one year forward rates (the one year forwards maturing at times 2, 3 and 4) all have an identical interest rate volatility σ. This is the discrete time equivalent of the Ho and Lee model (1986). We start by examining the actual standard deviations of the daily changes from 1962 to 2011 for 10 interest rates: the continuously compounded 1 year zero coupon bond yield, the continuously compounded 1 year forward rate maturing in year 2, the continuously compounded 1 year forward rate maturity in year 3, and so on for each one year forward rate out to the one maturing in year 10. The daily volatilities are as follows:

	Analysis of Daily Changes in Spot and 1 Year Forward Rates (Continuously Compounded Basis)									
	Maturity in Years									
	1	2	3	4	5	6	7	8	9	10
Standard Deviation	0.0832%	0.0911%	0.0981%	0.0912%	0.1065%	0.0989%	0.1160%	0.0954%	0.1018%	0.1524%
Maximum Daily Change	1.0124%	1.0011%	1.0470%	0.7269%	0.8383%	1.0634%	1.6645%	1.0765%	1.0964%	1.7287%
Minimum Daily Change	-1.0012%	-0.9360%	-0.9856%	-0.9428%	-1.2709%	-1.3032%	-1.0771%	-0.8747%	-1.0972%	-2.3316%
Width of Range	2.0135%	1.9371%	2.0325%	1.6696%	2.1092%	2.3665%	2.7416%	1.9512%	2.1936%	4.0603%

The one year spot rate's volatility was 0.0832% over this period. The volatilities for the forward rates maturing in years 2, 3 and 4 were 0.0911%, 0.0981%, and 0.0912% respectively. We can reach two conclusions immediately:

- Interest rate volatility is not constant over the maturity horizon (contrary to the Ho and Lee [1986] assumptions)
- Interest rate volatility is not declining over the maturity horizon (contrary to the Vasicek [1977] assumptions).

The HJM approach is so general that the Ho and Lee and Vasicek models can be replicated as special cases. We discuss that replication in Chapter 13. There is no need to limit ourselves to such unrealistic volatility assumptions, however, because of the power of the HJM approach.

Instead, we assume that forward rates have their actual volatilities over the period from 1962-2011, but we need to adjust for the fact that we have measured volatility over a time period of 1 day (which we assume is $1/250^{th}$ of a year based on 250 business days per year) and we need a period for which the length of the period Δ is 1, not 1/250. We know that the variance changes as the length of the time period:

$$Measured\ Variance = \Delta\sigma^2$$

and therefore

$$[Measured\ Volatility]\frac{1}{\sqrt{\Delta}}=\sigma$$

Therefore the 1 year forward rate, which has a volatility of 0.00091149 over a one day period has an estimated annual volatility of $0.00091149/[(1/250)^{1/2}]=0.01441187$. We revisit this method of adjustment for changes in the time period repeatedly in this book. It's an adjustment that should be avoided whenever possible.

For convenience, we use the annual volatilities from this chart in our HJM implementation:

Interest Rate Volatility Assumptions for an Annual Period Model		
	Daily	Annual
1 Year Forward Rate Maturing in Year 2	0.00091149	0.01441187
1 Year Forward Rate Maturing in Year 3	0.00098070	0.01550626
1 Year Forward Rate Maturing in Year 4	0.00091243	0.01442681

We will use the zero coupon bond prices prevailing on March 31, 2011 as our other inputs:

Number of Periods:	4
Length of Periods (Years)	1
Number of Risk Factors:	1
Volatility Term Structure:	Empirical

Parameter Inputs

Period Start	Period End	Zero Coupon Bond Prices	Forward Rate Volatility
0	1	0.99700579	
1	2	0.98411015	0.01441187
2	3	0.96189224	0.01550626
3	4	0.93085510	0.01442681

Our final assumption is that there is one random factor driving the forward rate curve. This implies (since all of our volatilities are positive) that all forward rates move up or down together. This implication of the model, as we will see in Chapters 7 through 9, is grossly unrealistic. We relax this assumption when we move to 2 and 3 factor examples.

Key Implications and Notation of the HJM Approach

The Heath Jarrow and Morton conclusions are very complex to derive but their implications are very straightforward. Once the zero coupon bond prices and volatility assumptions are made, the mean of the distribution of forward rates (in a Monte Carlo simulation) and the structure of a bushy tree are completely determined by the constraints that there be no arbitrage in the economy. Modelers who are unaware of this insight would choose means of the distributions for forward rates such that Monte Carlo or bushy tree valuation would provide different prices for the zero coupon bonds on March 31, 2011 than those used as input. This would create the appearance of an arbitrage opportunity, but it is in fact a big error that calls into question the validity of the calculation, as it should. Even more than two decades after the publication of the HJM conclusions, financial market participants perform risk analysis that contains such errors.

We show in this example that the zero coupon bond valuations using a bushy tree are 100% consistent with the inputs. We now introduce our notation:

Δ	Length of time period, which is 1 in this example
$r(t,s_t)$	The simple 1 period uncompounded risk free interest rate as of current time t in state s_t
$R(t,s_t)$	$1+r(t,s_t)$, the total return, the value of $1 dollar invested at the risk free rate for 1 period with no compounding
$\sigma\ (t,T,s_t)$	Forward rate volatility at time t for forward maturing at T in state s (a sequence of ups and downs)
$P(t,T,s_t)$	Zero coupon bond price at time t maturing at time T in state s_t (i.e. up or down)
Index	= 1 if the state is "up" and = -1 if the state is "down"
$U(t,T,s_t)$	= the total return that leads to bond price $P(t+\Delta,T,s_t=up)$ on a T maturity bond in an upshift state one period from current time t.
$D(t,T,s_t)$	= the total return that leads to bond price $P(t+\Delta, T, s_t=down)$ on a T maturity bond in a downshift state one period from current time t.
$K(t,T,s_t)$	is the sum of the forward volatilities for the 1 period forward rates from $t+\Delta$ to $T-\Delta$, as shown here:

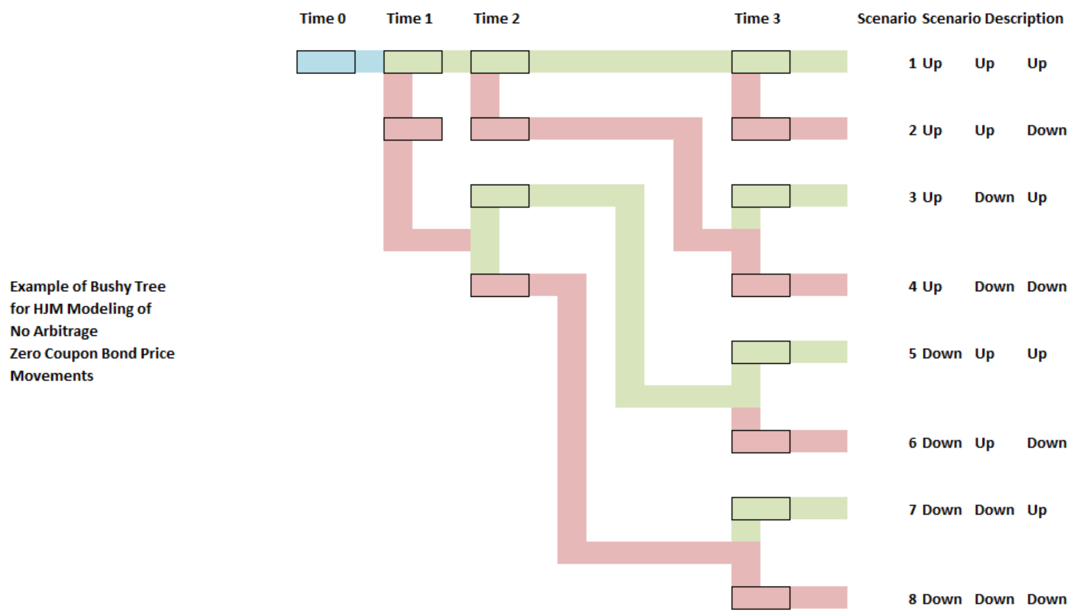
$$K(t,T,s_t) = \sum_{j=t+\Delta}^{T-\Delta} \sigma(t,j,s_t)\sqrt{\Delta}$$

We will also see the rare appearance of a trigonometric function in finance, one found in common spreadsheet software:

$$Cosh(x) = \frac{1}{2}[e^{x} + e^{-x}]$$

Note that the current times that will be relevant in building a bushy tree of zero coupon bond prices are current times t=0, 1, 2, and 3. We'll be interested in maturity dates T=2, 3, and 4. We know that at time zero, there are 4 zero coupon bonds outstanding. At time 1, only the bonds maturity at T = 2, 3, and 4 will remain outstanding. At time 2, only the bonds maturing at times T = 3 and 4 will remain, and at time 3 only the bond maturing at time 4 will remain. For each of the boxes below, we need to fill in the relevant bushy tree (one for each of the four zero coupon bonds) with each up shift and down shift of the zero coupon bond price as we step forward one more period (by $\Delta = 1$) on the tree. In the interests of saving space, we'll arrange the tree to look like a table by stretching the bushy tree as follows:

Figure 4



A completely populated zero coupon bond price tree would then be summarized like this where the final shift in price is color coded for up or down shifts:

	Four Year Zero Coupon Bond			
	Current Time			
Row Number	0	1	2	3
1	0.930855	0.975026	1.002895	1.010619
2		0.892275	0.944645	0.981905
3			0.944617	0.979758
4			0.889752	0.951921
5				0.981876
6				0.953979
7				0.951893
8				0.924847

The mapping of the sequence of up and down states is shown here, consistent with the stretched tree above:

Map of Sequence of States				
	Current Time			
Row Number	0	1	2	3
1	Time 0	up	up-up	up-up-up
2		dn	up-dn	up-up-dn
3			dn-up	up-dn-up
4			dn-dn	up-dn-dn
5				dn-up-up
6				dn-up-dn
7				dn-dn-up
8				dn-dn-dn

In order to populate the trees with zero coupon bond prices and forward rates, there is one more piece of information which we need to supply.

Pseudo Probabilities

In Chapter 7 of Jarrow (2002), Prof. Jarrow shows that a necessary and sufficient condition for no arbitrage is that, at every node in the tree, the one period return on a zero coupon bond neither dominates nor is dominated by a one period investment in the risk free rate. Written as the total return on a $1 investment, an up shift U must be greater than the risk free total return and the down shift D must be less than the risk free total return R:

$U(t,T,s_t) > R(t,s_t) > D(t,T,s_t)$ for all points in time t and states s_t.

Prof. Jarrow adds "This condition is equivalent to the following statement: there exists a unique number $\pi(t,s_t)$ strictly between 0 and 1 such that

$R(t,s_t)= \pi(t,s_t)U(t,T,s_t) + (1- \pi(t,s_t))D(t,T,s_t)$"

Solving for $\pi(t,s_t)$ gives us the relationship

$$\pi(t, s_t) = \frac{[R(t, s_t) - D(t, T, s_t)]}{[U(t, T, s_t) - D(t, T, s_t)]}$$

If the computed $\pi(t,s_t)$ values are between 0 and 1 everywhere on the bushy tree, then the tree is arbitrage free.

Professor Jarrow continues as follows:

"Note that each $\pi(t,s_t)$ can be interpreted as a pseudo probability of the up state occurring over the time interval (t, t+Δ)…We call these $\pi(t,s_t)$s pseudo probabilities because they are 'false' probabilities. They are not the actual probabilities generating the evolution of the money market account and the [T] period zero coupon bond prices. Nonetheless, these pseudo probabilities have an important use…" in valuation.

Prof. Jarrow goes on to explain on page 129 that "risk neutral valuation" is computed by "taking the expected cash flow, using the pseudo probabilities, and discounting at the spot rate of interest." He adds "this is called risk neutral valuation because it is the value that the random cash flow 'x' would have in an economy populated by risk-neutral investors, having the pseudo probabilities as their beliefs."

We now demonstrate how to construct the bushy tree and use it for risk-neutral valuation.

The Formula for Zero Coupon Bond Price Shifts

Prof. Jarrow demonstrates that the values for the risk neutral probability of an upshift $\pi(t,s_t)$ are determined by a set of mathematical limit conditions for an assumed evolution of interest rates. These limit conditions allow multiple solutions. Without loss of generality, one can always construct a tree whose limit is the assumed interest rate evolution with $\pi(t,s_t)$ =1/2 for all points in current time t and all up and down states s_t. Then, using our notation above and remembering that the variable "Index" is 1 in an upstate and -1 in a downstate, the shifts in zero coupon bond prices can be written as follows:

(1)

$$P(t + \Delta, T, s_{t+\Delta}) = \frac{[P(t,T,s_t)R(t,s_t)e^{K(t,T,s_t)\Delta(Index)}]}{[\cosh(K(t,T,s_t)\Delta)]}$$

where for notational convenience $\cosh[K(t,T,s_t)\Delta] \equiv 1$ when T-Δ<t+Δ. This is equation 15.17 in Jarrow (2002, page 286). We now put this formula to use.

Building the Bushy Tree for Zero Coupon Bonds Maturing at Time T=2

We now populate the bushy tree for the 2 year zero coupon bond. We calculate each element of equation (1). When t=0 and T=2, we know Δ=1 and

$P(0,2,s_t) = 0.98411015.$

The one period risk free return (remember, this is one plus the risk free rate) is

$R(0,s_t)=1/P(0,1,s_t)=1/0.997005786=1.003003206$

The scaled sum of sigmas $K(t,T,s_t)$ for t=0 and T=2 becomes

$$K(t,T,s_t) = \sum_{j=t+\Delta}^{T-\Delta} \sigma(t,j,s_t)\sqrt{\Delta} = \sum_{j=1}^{1} \sigma(0,j,s_t)\sqrt{\Delta}$$

and therefore $K(0,T,s_t) = (\sqrt{1})(0.01441187) = 0.01441187$

Using formula 1 with these inputs and the fact that the variable Index=1 for an upshift gives

$P(1,2,s_t = \text{up}) = 1.001290$

For a downshift we set Index = -1 and recalculate formula 1 to get

P(1,2,st = down) = 0.972841

We have fully populated the bushy tree for the zero coupon bond maturing at T=2 (note values have been rounded to six decimal places for display only), since all of the up and down states at time t=2 result in a riskless pay-off of the zero coupon bond at its face value of 1. Note also that, as a result of our volatility assumptions, the price of the bond maturing at time T=2 after an upshift to time t=1 has resulted in a value higher than the par value of 1. This implies negative interest rates. Negative nominal interest rates have appeared in Switzerland, Japan, Hong Kong and (selectively) in the United States. The academic argument is that this cannot occur, because one could “costlessly” store cash in order to avoid a negative interest rate. The assumption that cash can be stored costlessly is an assumption that is not true. In fact, on August 5, 2011, Bank of New York Mellon announced that it would change a fee (i.e. creating a negative return) on demand deposit balances in excess of $50 million), according to the *Wall Street Journal*. We will return to this issue in later chapters.

	Two Year Zero Coupon Bond			
	Current Time			
Row Number	0	1	2	3
1	0.98411	1.00129	1	
2		0.972841	1	
3			1	
4			1	
5				
6				
7				
8				

Building the Bushy Tree for Zero Coupon Bonds Maturing at Time T = 4

We now populate the bushy tree for the 4 year zero coupon bond. We calculate each element of equation (1). When t=0 and T=4, we know Δ=1 and

$P(0,4,s_t)$ =0.930855099

The one period risk free return (one plus the risk free rate) is

$R(0,s_t)=1/P(0,1,s_t)=1/0.997005786=1.003003206$

The scaled sum of sigmas $K(t,T,s_t)$ for t=0 and T=4 becomes

$$K(t,T,s_t) = \sum_{j=t+\Delta}^{T-\Delta} \sigma(t,j,s_t)\sqrt{\Delta} = \sum_{j=1}^{3} \sigma(0,j,s_t)\sqrt{\Delta}$$

and therefore $K(0,T,s_t)$ =(√1)(0.01441187+0.015506259+0.01442681)= 0.044344939.

Using formula 1 with these inputs and the fact that the variable Index=1 for an upshift gives

$P(1,4,s_t = up)$ =0.975026

For a downshift we set Index = -1 and recalculate formula 1 to get

P(1,4,st = down) = 0.892275

We have correctly populated the first two columns of the bushy tree for the zero coupon bond maturing at T=4 (note values have been rounded to six decimal places for display only).

Four Year Zero Coupon Bond				
	Current Time			
Row Number	0	1	2	3
1	0.930855	0.975026	1.002895	1.010619
2		0.892275	0.944645	0.981905
3			0.944617	0.979758
4			0.889752	0.951921
5				0.981876
6				0.953979
7				0.951893
8				0.924847

Now we move to the third column, which displays the outcome of the T=4 zero coupon bond price after 4 scenarios: up-up, up-down, down-up, and down-down. We calculate $P(2,4,s_t = up)$ and $P(2,4,s_t = down)$ after the initial "down" state as follows. When t=1, T=4, and Δ=1 then

$P(1,4,s_t = down)$ = 0.892275, as shown in the second row of the second column.

The one period risk free return (one plus the risk free rate) is

$R(1,s_t = down)=1/P(1,2,s_t = down)$=1/ 0.972841 = 1.027917

The scaled sum of sigmas $K(t,T,s_t)$ for t=1 and T=4 becomes

$$K(t,T,s_t) = \sum_{j=t+\Delta}^{T-\Delta} \sigma(t,j,s_t)\sqrt{\Delta} = \sum_{j=2}^{3} \sigma(0,j,s_t)\sqrt{\Delta}$$

and therefore $K(1,4,s_t)$ =(√1)(0.01441187+0.015506259)= 0.029918

Using formula 1 with these inputs and the fact that the variable Index=1 for an upshift gives

$P(2,4,s_t = \text{up}) = 0.944617$

For a downshift we set Index = -1 and recalculate formula 1 to get

$P(2,4,s_t = \text{down}) = 0.889752$

We have correctly populated the third and fourth rows of column 3 (t=2) of the bushy tree for the zero coupon bond maturing at T=4 (note values have been rounded to six decimal places for display only). The remaining calculations are left to the reader.

	Four Year Zero Coupon Bond			
	Current Time			
Row Number	0	1	2	3
1	0.930855	0.975026	1.002895	1.010619
2		0.892275	0.944645	0.981905
3			0.944617	0.979758
4			0.889752	0.951921
5				0.981876
6				0.953979
7				0.951893
8				0.924847

In a similar way, the bushy tree for the zero coupon bond price maturing at time T=3 can be calculated as follows:

	Three Year Zero Coupon Bond			
	Current Time			
Row Number	0	1	2	3
1	0.961892	0.993637	1.006657	1
2		0.935925	0.978056	1
3			0.975917	1
4			0.948189	1
5				1
6				1
7				1
8				1

If we combine all of these tables, we can create a table of the term structure of zero coupon bond prices in each scenario:

		Zero Coupon Bond Prices			
		Periods to Maturity			
State Name	State Number	1	2	3	4
Time 0	0	0.997006	0.98411	0.961892	0.930855
Time 1 up	1	1.00129	0.993637	0.975026	
Time 1 down	2	0.972841	0.935925	0.892275	
Time 2 up-up	3	1.006657	1.002895		
Time 2 up-down	4	0.978056	0.944645		
Time 2 down-up	5	0.975917	0.944617		
Time 2 down-down	6	0.948189	0.889752		
Time 3 up-up-up	7	1.010619			
Time 3 up-up-down	8	0.981905			
Time 3 up-down-up	9	0.979758			
Time 3 up-down-down	10	0.951921			
Time 3 down-up-up	11	0.981876			
Time 3 down-up-down	12	0.953979			
Time 3 down-down-up	13	0.951893			
Time 3 down-down-down	14	0.924847			

At any point in time t, the continuously compounded yield to maturity at time T can be calculated as y(T-t)=-ln[P(t,T)]/(T-t). When we display the term structure of zero coupon yields in each scenario, we can see that our volatility assumptions have indeed produced some implied negative yields in this particular simulation. We discuss how to remedy that problem in the next chapter.

		Continuously Compounded Zero Yields			
		Periods to Maturity			
State Name	**State Number**	1	2	3	4
Time 0	0	0.2999%	0.8009%	1.2951%	1.7913%
Time 1 up	1	-0.1289%	0.3192%	0.8430%	
Time 1 down	2	2.7534%	3.3110%	3.7994%	
Time 2 up-up	3	-0.6635%	-0.1445%		
Time 2 up-down	4	2.2188%	2.8473%		
Time 2 down-up	5	2.4377%	2.8488%		
Time 2 down-down	6	5.3201%	5.8406%		
Time 3 up-up-up	7	-1.0563%			
Time 3 up-up-down	8	1.8261%			
Time 3 up-down-up	9	2.0449%			
Time 3 up-down-down	10	4.9273%			
Time 3 down-up-up	11	1.8290%			
Time 3 down-up-down	12	4.7114%			
Time 3 down-down-up	13	4.9303%			
Time 3 down-down-down	14	7.8127%			

Finally, we can display the 1 year U.S. Treasury spot rates and the associated term structure of 1 year forward rates in each scenario.

State Name	State Number	1 Year Forward Rates, Simple Interest — Periods to Maturity 1	2	3	4
Time 0	0	0.3003%	1.3104%	2.3098%	3.3343%
Time 1 up	1	-0.1288%	0.7702%	1.9087%	
Time 1 down	2	2.7917%	3.9443%	4.8920%	
Time 2 up-up	3	-0.6613%	0.3752%		
Time 2 up-down	4	2.2436%	3.5368%		
Time 2 down-up	5	2.4677%	3.3136%		
Time 2 down-down	6	5.4642%	6.5678%		
Time 3 up-up-up	7	-1.0508%			
Time 3 up-up-down	8	1.8428%			
Time 3 up-down-up	9	2.0660%			
Time 3 up-down-down	10	5.0507%			
Time 3 down-up-up	11	1.8459%			
Time 3 down-up-down	12	4.8242%			
Time 3 down-down-up	13	5.0539%			
Time 3 down-down-down	14	8.1260%			

Valuation in the Heath Jarrow and Morton Framework

Prof. Jarrow in a quote above described valuation as the expected value of cash flows using the risk neutral probabilities. Note that column 1 denotes the riskless 1 period interest rate in each scenario. For the scenario number 14 (three consecutive downshifts in zero coupon bond prices), cash flows at time T=4 would be discounted by the one year spot rates at time t=0, by the one year spot rate at time t=1 in scenario 2 (“down”), by the one year spot rate in scenario 6 (“down down”) at time t=2, and by the one year spot rate at time t=3 in scenario 14 (“down down down”). The discount factor is

Discount Factor (0,4, down down down) =1/(1.003003)(1.027917)(1.054642)(1.081260)

These discount factors are displayed here for each potential cash flow date:

	Discount Factors				
	Current Time				
Row Number	0	1	2	3	4
1	1	0.997006	0.998292	1.004938	1.01561
2		0.997006	0.998292	1.004938	0.986754
3			0.969928	0.976385	0.956622
4			0.969928	0.976385	0.929442
5				0.94657	0.929414
6				0.94657	0.903007
7				0.919676	0.875433
8				0.919676	0.85056

When taking expected values, we can calculate the probability of each scenario coming about since the probability of an upshift is ½:

	Probability of Each State				
	Current Time				
Row Number	0	1	2	3	4
1	100.00%	50.00%	25.00%	12.50%	12.50%
2		50.00%	25.00%	12.50%	12.50%
3			25.00%	12.50%	12.50%
4			25.00%	12.50%	12.50%
5				12.50%	12.50%
6				12.50%	12.50%
7				12.50%	12.50%
8				12.50%	12.50%

It is convenient to calculate the “probability weighted discount factors” for use in calculating the expected present value of cash flows:

Probability Weighted Discount Factors					
	Current Time				
Row Number	0	1	2	3	4
1	1	0.498503	0.249573	0.125617	0.126951
2		0.498503	0.249573	0.125617	0.123344
3			0.242482	0.122048	0.119578
4			0.242482	0.122048	0.11618
5				0.118321	0.116177
6				0.118321	0.112876
7				0.114959	0.109429
8				0.114959	0.10632

We now use the HJM bushy trees we have generated to value representative securities.

Valuation of a Zero Coupon Bond Maturing at Time T=4

A riskless zero coupon bond pays $1 in each of the 8 nodes of the bushy tree that prevail at time T=4:

Maturity of Cash Flow Received					
	Current Time				
Row Number	0	1	2	3	4
1	0	0	0	0	1
2		0	0	0	1
3			0	0	1
4			0	0	1
5				0	1
6				0	1
7				0	1
8				0	1
Risk Neutral Value =				0.93085510	

When we multiply this vector of 1s times the probability weighted discount factors in the time T=4 column in the previous table and add them, we get a zero coupon bond price of 0.93085510, which is the value we should get in a no-arbitrage economy, the value observable in the market and used as an input to create the tree. While this is a very important confirmation of the accuracy of our interest rate simulation, very few interest rate analysts take this important step in the “model audit” process. The valuation of security whose net present value (price plus accrued interest) is used as an input should replicate the input value. If this is not the case, the calculation is wrong.

Valuation of a Coupon-Bearing Bond Paying Annual Interest

Next we value a bond with no credit risk that pays $3 in interest at every scenario at times T=1, 2, 3, and 4 plus principal of 100 at time T=4. The valuation is calculated by multiplying each cash flow by the matching probability weighted discount factor, to get a net present value (which is price plus accrued interest) of 104.70709974:

	Maturity of Cash Flow Received				
	Current Time				
Row Number	0	1	2	3	4
1	0	3	3	3	103
2		3	3	3	103
3			3	3	103
4			3	3	103
5				3	103
6				3	103
7				3	103
8				3	103
Risk Neutral Value =					104.70709974

Valuation of a Digital Option on the 1 Year U.S. Treasury Rate

Now we value a digital option that pays $1 at time T=3 if (at that time) the one year U.S. Treasury rate (for maturity at T=4) is over 8%. If we look at the table of the term structure of one year spot and forward rates above, this happens in only one scenario, scenario 14 (down down down). The cash flow can be input in the table below and multiplied by the probability weighted discount factors to find that this option has a value of 0.11495946:

Maturity of Cash Flow Received					
	Current Time				
Row Number	0	1	2	3	4
1	0	0	0	0	0
2		0	0	0	0
3			0	0	0
4			0	0	0
5				0	0
6				0	0
7				0	0
8				1	0
Risk Neutral Value =					0.11495946

Conclusion

The Dickler, Jarrow and van Deventer studies of movements in U.S. Treasury yields and forward rates from 1962 to 2011 confirm that 5-10 factors are needed to accurately model interest rate movements. Popular one factor models (Ho and Lee, Vasicek, Hull and White, Black Derman and Toy) cannot replicate the actual movements in yields that have occurred. The interest rate volatility assumptions in these models (constant, constant proportion, declining, etc.) are also inconsistent with observed volatility.

In order to handle a large number of driving factors and complex interest rate volatility structures, the Heath Jarrow and Morton framework is necessary. This chapter shows how to simulate zero coupon bond prices, forward rates and zero coupon bond yields in an HJM framework with maturity-dependent interest rate volatility. Monte Carlo simulation, an alternative to the bushy tree framework, can be done in a fully consistent way.

In the next chapter, we enrich the realism of the simulation by allowing for interest rate volatility that is also a function of the level of interest rates.