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Advanced Financial Risk Management:

Tools and Techniques for Integrated Credit Risk, Market Risk, Liquidity Risk and Interest Rate Risk Management

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Part 2: Risk Management Techniques for Interest Rate Analytics

Chapter 7: HJM interest rate modeling with rate and maturity-dependent volatility

In Chapter 6, we provided a worked example of interest rate modeling and valuation using the Heath Jarrow and Morton framework and the assumption that forward rate volatility was dependent on years to maturity using actual U.S. Treasury yields prevailing on March 31, 2011 and historical volatility from 1962-2011. In chapter, we increase the realism of the forward rate volatility assumption. Actual data makes it very clear that forward rate volatility depends not just on the time to maturity of the forward rate but also on the level of spot interest rates. This chapter shows how to incorporate that assumption into the analysis for enhanced accuracy and realism. This is a much more general assumption that that used by Ho and Lee [1986] (constant volatility) or by Vasicek [1977] and Hull and White [1990] (declining volatility).

We again use data from the Federal Reserve statistical release H15 published on April 1, 2011 for yields prevailing on March 31, 2011. U.S. Treasury yield curve data was smoothed using Kamakura Risk Manager version 7.3 to create zero coupon bonds via the maximum smoothness forward rate technique of Adams and van Deventer discussed in Chapter 5.

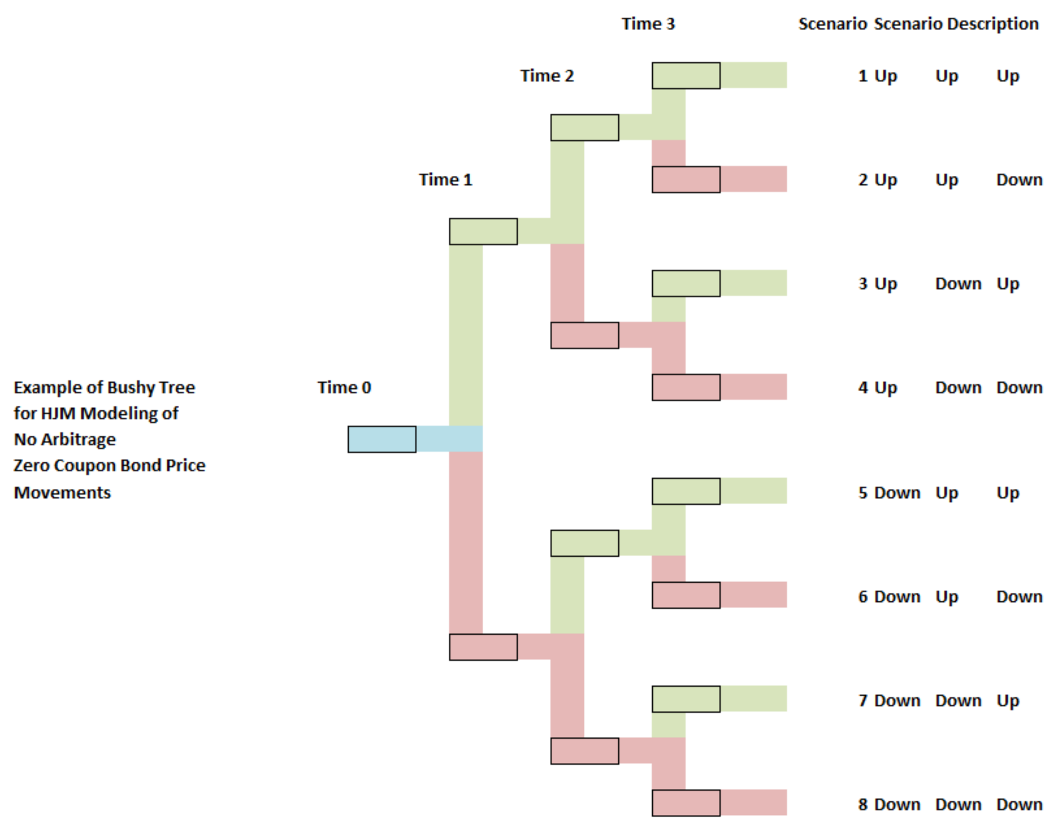
Objectives of the Example and Key Input Data

Following Jarrow (2002), we make the same modeling assumptions for our worked example here as in Chapter 6:

- Zero coupon bond prices for the U.S. Treasury curve on March 31, 2011 are the basic inputs.
- Interest rate volatility assumptions are based on the Dickler, Jarrow and van Deventer series on daily U.S. Treasury yields and forward rates from 1962 to 2011. In this chapter, we make the volatility assumptions more accurate than in Chapter 6.
- The modeling period is 4 equal length periods of one year each.
- The HJM implementation is that of a "bushy tree" which we describe below

The bushy tree looks the same as in Chapter 6, but the modification we make in volatility assumptions will change the "height" of the branches in the tree:

Figure 1

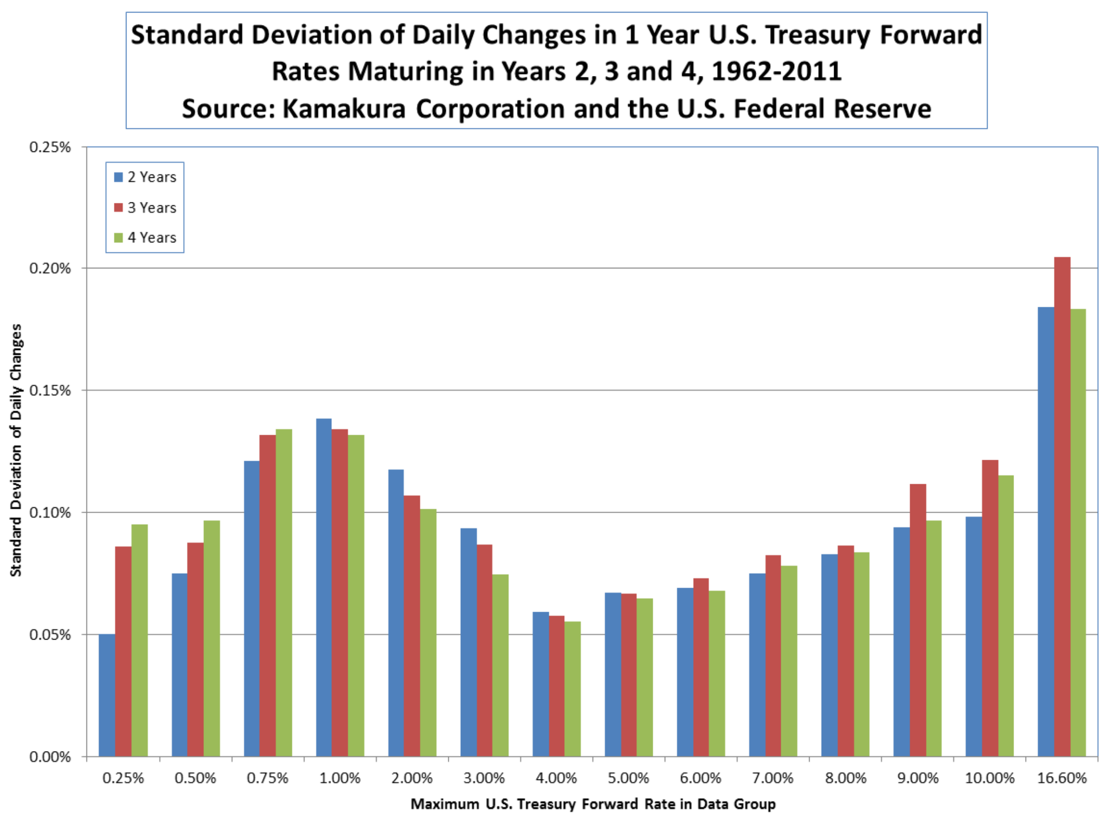


At each of the points in time on the lattice (time 0, 1, 2, 3 and 4) there are again sets of zero coupon bond prices and forward rates. At time 0, there is one set of data. At time one, there are two sets of data, the "up set" and the "down set." At time two, there are four sets of data (up up, up down, down up, and down down), and at time three there are $8=2^3$ sets of data.

In Chapter 6, we calculated the standard deviation of the daily changes in three 1 year forward rates (the 1 year U.S. Treasury rates maturing at time T=2, 3, and 4) over the period from 1962-2011. We then adjusted for the fact that we have measured volatility over a time period of 1 day (which we assume is 1/250th of a year based on 250 business days per year) and we need a volatility for which the length of the period Δ is 1, not 1/250. The resulting yield curve shifts implied by the Heath Jarrow and Martin approach resulted in negative interest rates in three of the scenarios. In this chapter, we re-examine our volatility assumptions to see whether volatility is dependent on the level of interest rates as well as on maturity. We hypothesize that interest rate volatility is dependent on the starting level of the spot 1 year U.S. Treasury yield (continuously compounded) and the maturity of the forward rate (the first 1 year forward rate matures at time t+2, the second 1 year forward rate matures at time t+3, and so on). We categorize the data by interest rate levels, grouping data for spot 1 year rates under 0.09%, between 0.09% and 0.25%, between 0.25% and 0.50%, between 0.50% and 0.75%, between 0.75% and 1.00%, and then in one percent intervals to 10%. The final group of data includes all of those observations in which the starting spot 1 year U.S. Treasury yield was over 10%.

The graph below shows the need for incorporating the level of interest rates in our selection of forward rate volatility. We plot the actual daily volatility of 1 year forward rates maturing in years 2, 3 and 4 by interest rate groups, which are determined by the highest level of starting 1 year U.S. Treasury yields in that group:

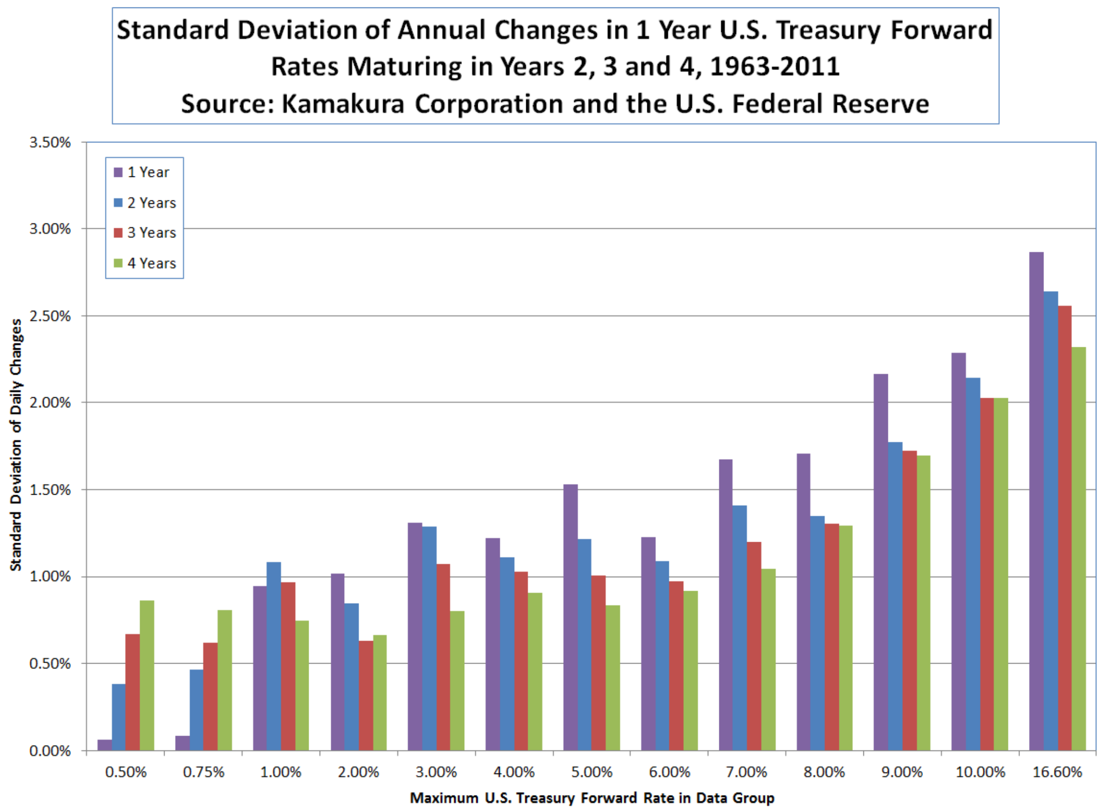
Figure 2



The graph makes it obvious that interest rate volatility depends in a complex but smooth way on both forward rate maturity and the starting level of the 1 year U.S. Treasury yield.

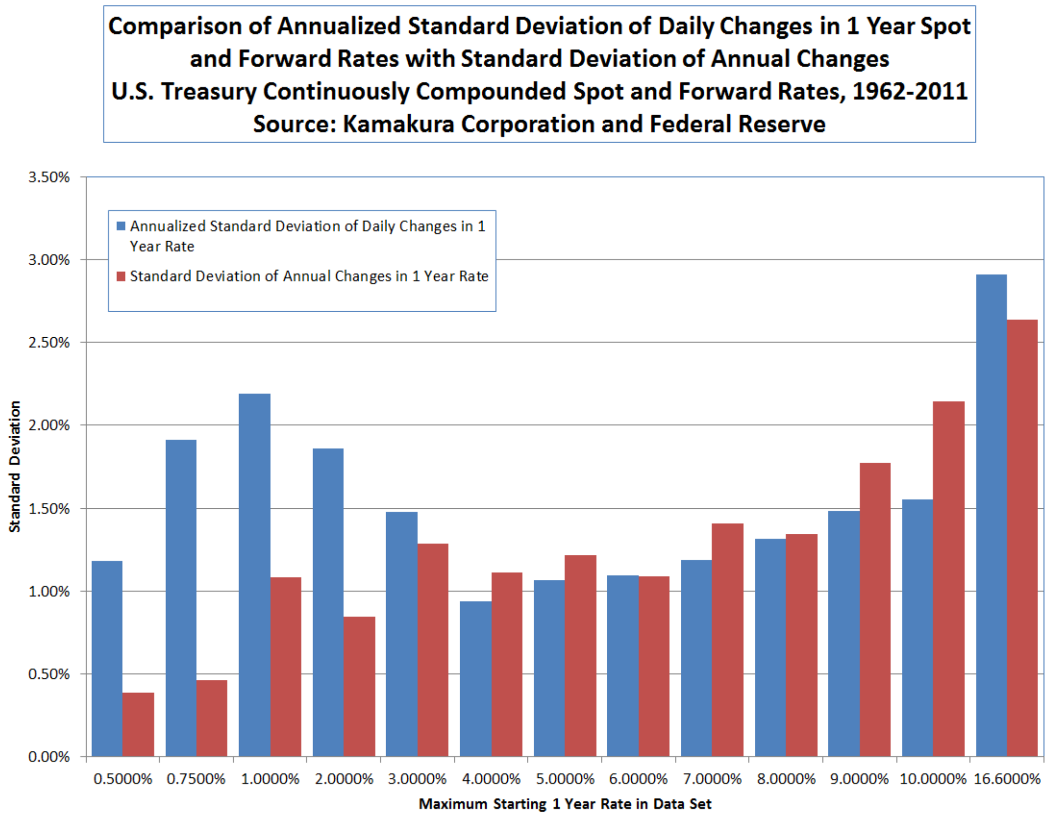
As suggested by Kamakura Senior Research Fellow Dr. Sean Patrick Klein, we then annualized these daily volatilities as we did in Chapter 6 and compared them to volatilities of the same forward rates calculated based on 1 year changes (not annualized daily changes) in the forward rate from 1963 to 2011. The results will surprise many. We start by plotting the forward rate volatilities using annual time intervals and find a striking difference in how volatilities change as rates rise:

Figure 3



Forward rate volatility rises in a smooth but complex way as the level of interest rates rises. When we plot the volatilities calculated based on annual time intervals against the annualized volatilities derived from daily time intervals, the results are very surprising:

Figure 4



We used a very standard financial methodology for annualizing the daily volatilities, but the result substantially overestimated the actual forward rate annual volatilities for forward rates when the 1 year U.S. Treasury was 3.00% or below. We conclude that this contributed strongly to the fact that our first worked example produced negative interest rates. The results also confirm that the level of interest rates does impact the level of forward rate volatility but in a more complex way than the lognormal interest rate movements assumed by Fischer Black with co-authors Derman and Toy (1990), and Karasinski (1991).

In this chapter, we use the exact volatilities for the 1 year changes in continuously compounded forward rates as shown in this table. There were no observations at the time this data was compiled for starting 1 year U.S. Treasury yields below 0.002499%, so we have set those volatilities by assumption. With the passage of time, these assumptions can be replaced with facts or with data from other low rate counties like Japan. The volatilities used were selected in look-up table fashion based on U.S. Treasury spot rate levels at that node on the bushy tree from this table:

Figure 5

Data Group	1 Year Rate in Data Group Minimum Spot Rate	Maximum Spot Rate	Number of Observations	Standard Deviation of Annual Changes in 1 Year Rate 1 Year	2 Years	3 Years	4 Years
1	0.00%	0.09%	Assumed	0.01000000%	0.02000000%	0.10000000%	0.20000000%
2	0.09%	0.25%	Assumed	0.04000000%	0.10000000%	0.20000000%	0.30000000%
3	0.25%	0.50%	346	0.06363856%	0.38419072%	0.67017474%	0.86385093%
4	0.50%	0.75%	86	0.08298899%	0.46375394%	0.61813347%	0.80963400%
5	0.75%	1.00%	20	0.94371413%	1.08296308%	0.96789760%	0.74672896%
6	1.00%	2.00%	580	1.01658509%	0.84549720%	0.63127387%	0.66276260%
7	2.00%	3.00%	614	1.30986113%	1.28856305%	1.07279248%	0.80122998%
8	3.00%	4.00%	1492	1.22296401%	1.11359303%	1.02810902%	0.90449832%
9	4.00%	5.00%	1672	1.53223304%	1.21409679%	1.00407389%	0.83492621%
10	5.00%	6.00%	2411	1.22653783%	1.09067681%	0.97034198%	0.91785966%
11	6.00%	7.00%	1482	1.67496041%	1.40755286%	1.20035444%	1.04528208%
12	7.00%	8.00%	1218	1.70521091%	1.34675823%	1.30359062%	1.29177866%
13	8.00%	9.00%	765	2.16438588%	1.77497758%	1.72454046%	1.69404480%
14	9.00%	10.00%	500	2.28886301%	2.14441843%	2.02862693%	2.02690844%
15	10.00%	16.60%	959	2.86758369%	2.64068702%	2.55664943%	2.32042833%
Grand Total			12145	1.73030695%	1.49119219%	1.37014684%	1.26364341%

We will continue to use the zero coupon bond prices prevailing on March 31, 2011 as our other inputs:

Number of Periods:	**4**
Length of Periods (Years)	**1**
Number of Risk Factors:	**1**
Volatility Term Structure:	**Empirical**

Parameter Inputs

Period Start	Period End	Zero Coupon Bond Prices
0	1	0.99700579
1	2	0.98411015
2	3	0.96189224
3	4	0.93085510

Our final assumption is that there is one random factor driving the forward rate curve. This implies (since all of our volatilities are positive) that all forward rates move up or down together. We relax this assumption when we move to 2 and 3 factor examples.

Key Implications and Notation of the HJM Approach

We now re-introduce the same notation used in Chapter 6:

Δ Length of time period, which is 1 in this example

$r(t,s_t)$ The simple 1 period risk free interest rate as of current time t in state s_t

$R(t,s_t)$ — $1+r(t,s_t)$, the total return, the value of $1 dollar invested at the risk free rate for 1 period

$\sigma(t,T,s_t)$ — Forward rate volatility at time t for the one period forward rate maturing at T in state s_t (a sequence of ups and downs). Because σ is now dependent on the spot U.S. Treasury rate, the state s_t is very important in determining its level.

$P(t,T,s_t)$ — Zero coupon bond price at time t maturing at time T in state s_t (i.e. up or down)

Index — = 1 if the state is "up" and = -1 if the state is "down"

$U(t,T,s_t)$ — = the total return that leads to bond price $P(t+\Delta,T,s_t=up)$ on a T maturity bond in an upshift state

$D(t,T,s_t)$ — = the total return that leads to bond price $P(t+\Delta, T, s_t=down)$ on a T maturity bond in a downshift state

$K(t,T,s_t)$ — is the sum of the forward volatilities as of time t for the 1 period forward rates from t+Δ to T-Δ, as shown here:

$$K(t,T,s_t) = \sum_{j=t+\Delta}^{T-\Delta} \sigma(t,j,s_t)\sqrt{\Delta}$$

We again use this trigonometric function, which is found in common spreadsheet software:

$$Cosh(x) = \frac{1}{2}[e^x + e^{-x}]$$

In the interests of saving space, we'll again arrange the bushy tree to look like a table by stretching it in the same way shown in Chapter 6.

A completely populated zero coupon bond price tree would then be summarized like this where the final shift in price is color coded for up or down shifts:

Four Year Zero Coupon Bond				
	Current Time			
Row Number	0	1	2	3
1	0.930855	0.951558	0.98003	0.989917
2		0.915743	0.940645	0.980778
3			0.945074	0.978909
4			0.917569	0.954003
5				0.977728
6				0.952853
7				0.963794
8				0.942566

The mapping of the sequence of up and down states is shown here, consistent with the stretched tree above and with Chapter 6:

Map of Sequence of States				
	Current Time			
Row Number	0	1	2	3
1	Time 0	up	up-up	up-up-up
2		dn	up-dn	up-up-dn
3			dn-up	up-dn-up
4			dn-dn	up-dn-dn
5				dn-up-up
6				dn-up-dn
7				dn-dn-up
8				dn-dn-dn

Pseudo Probabilities

We use the same pseudo probabilities, consistent with no arbitrage, as in Chapter 6. There is an upshift probability in each state of 1/2 and a downshift probability in each state of 1/2. We now demonstrate how to construct the bushy tree with our new volatility assumptions and how to use it for risk-neutral valuation.

The Formula for Zero Coupon Bond Price Shifts

The formula for the shift in the array of zero coupon bond prices is unchanged from Chapter 6:

(1)

$$P(t+\Delta, T, s_{t+\Delta}) = \frac{[P(t,T,s_t)R(t,s_t)e^{K(t,T,s_t)\Delta(Index)}]}{[\cosh(K(t,T,s_t)\Delta)]}$$

where for notational convenience $\cosh[K(t,T,s_t)\Delta]\equiv 1$ when $T-\Delta<t+\Delta$. This is equation 15.17 in Jarrow (2002, page 286). We now put this formula to use.

Building the Bushy Tree for Zero Coupon Bonds Maturing at Time T=2

We now populate the bushy tree for the 2 year zero coupon bond. We calculate each element of equation (1). When t=0 and T=2, we know Δ=1 and

$P(0,2,s_t) = 0.98411015$.

The one period risk free rate is again

$R(0,s_t)=1/P(0,1,s_t)=1/0.997005786=1.003003206$

The 1 period spot rate for U.S. Treasuries is r(0, st) =R(0,st)-1=0.3003206%. At this level of the spot rate for U.S. Treasuries, volatilities are selected from data group 3 in the look up table above. The volatilities for the 1 year forward rates maturing in years 2, 3 and 4 are 0.003841907, 0.006701747, and 0.008638509.

The scaled sum of sigmas $K(t,T,s_t)$ for t=0 and T=2 becomes

$$K(t,T,s_t) = \sum_{j=t+\Delta}^{T-\Delta} \sigma(t,j,s_t)\sqrt{\Delta} = \sum_{j=1}^{1} \sigma(0,j,s_t)\sqrt{\Delta}$$

and therefore $K(0,T,s_t) = (\sqrt{1})(0.003841907) = 0.003841907$.

Using formula 1 with these inputs and the fact that the variable Index=1 for an upshift gives

$P(1,2,s_t = \text{up}) = 0.990858$

For a downshift we set Index = -1 and recalculate formula 1 to get

P(1,2,st = down) = 0.983273

We have fully populated the bushy tree for the zero coupon bond maturing at T=2 (note values have been rounded to six decimal places for display only), since all of the up and down states at time t=2 result in a riskless pay-off of the zero coupon bond at its face value of 1. Note also that, as a result of our volatility assumptions, the price of the bond maturing at time T=2 after an upshift to time t=1 no longer results in a value higher than the par value of 1. The change in our volatility assumptions has eliminated the negative rates found in example one.

Two Year Zero Coupon Bond				
	Current Time			
Row Number	0	1	2	3
1	0.98411	0.990858	1	
2		0.983273	1	
3			1	
4			1	
5				
6				
7				
8				

Building the Bushy Tree for Zero Coupon Bonds Maturing at Time T = 4

We now populate the bushy tree for the 4 year zero coupon bond. We calculate each element of equation (1). When t=0 and T=4, we know Δ=1 and

$P(0,4,s_t)$ =0.930855099

The one period risk free rate is

$R(0,s_t)=1/P(0,1,s_t)=1/0.997005786=1.003003206$

The spot U.S. Treasury rate $r(0,s_t)$ is still 0.003003206 and the same volatilities from data group 3 are relevant. The scaled sum of sigmas $K(t,T,s_t)$ for t=0 and T=4 becomes

$$K(t,T,s_t) = \sum_{j=t+\Delta}^{T-\Delta} \sigma(t,j,s_t)\sqrt{\Delta} = \sum_{j=1}^{3} \sigma(0,j,s_t)\sqrt{\Delta}$$

and therefore $K(0,T,s_t) = (\sqrt{1})(0.003841907+0.006701747+0.008638509) = 0.019182164$.

Using formula 1 with these inputs and the fact that the variable Index=1 for an upshift gives

$P(1,4,s_t = up)$ =0.951558.

For a downshift we set Index = -1 and recalculate formula 1 to get

P(1,4,st = down) = 0.915743.

We have correctly populated the first two columns of the bushy tree for the zero coupon bond maturing at T=4 (note values have been rounded to six decimal places for display only).

	Four Year Zero Coupon Bond			
	Current Time			
Row Number	0	1	2	3
1	0.930855	0.951558	0.98003	0.989917
2		0.915743	0.940645	0.980778
3			0.945074	0.978909
4			0.917569	0.954003
5				0.977728
6				0.952853
7				0.963794
8				0.942566

Now we move to the third column, which displays the outcome of the T=4 zero coupon bond price after 4 scenarios: up-up, up-down, down-up, and down-down. We calculate $P(2,4,s_t = \text{up})$ and $P(2,4,s_t = \text{down})$ after the initial "down" state as follows. When t=1, T=4, and Δ=1 then the one period risk free rate is

$R(1,s_t = \text{down})=1/P(1,2,s_t = \text{down})=1/\ 0.983273 =1.017011098$.

The U.S. Treasury spot rate is $r(1,s_t = \text{down})=R(1,s_t = \text{down}) -1 = 1.7011098\%$, which is in data group 6. The volatilities for the two remaining 1 period forward rates that are relevant are taken from the lookup table for data group 6: 0.008454972 and 0.006312739.

The scaled sum of sigmas $K(t,T,s_t)$ for t=1 and T=4 becomes

$$K(t,T,s_t) = \sum_{j=t+\Delta}^{T-\Delta} \sigma(t,j,s_t)\sqrt{\Delta} = \sum_{j=2}^{3} \sigma(0,j,s_t)\sqrt{\Delta}$$

and therefore $K(1,4,s_t) =(\sqrt{1})(\ 0.008454972+0.006312739)=\ 0.014767711$.

Using formula 1 with these inputs and the fact that the variable Index=1 for an upshift gives

$P(2,4,s_t = \text{up}) = 0.945074$.

For a downshift we set Index = -1 and recalculate formula 1 to get

$P(2,4,s_t = \text{down}) = 0.917569$.

We have correctly populated the third and fourth rows of column 3 (t=2) of the bushy tree for the zero coupon bond maturing at T=4 (note values have been rounded to six decimal places for display only). The remaining calculations are left to the reader.

	Four Year Zero Coupon Bond			
	Current Time			
Row Number	0	1	2	3
1	0.930855	0.951558	0.98003	0.989917
2		0.915743	0.940645	0.980778
3			0.945074	0.978909
4			0.917569	0.954003
5				0.977728
6				0.952853
7				0.963794
8				0.942566

In a similar way, the bushy tree for the price of the zero coupon bond maturing at time T=3 can be calculated as follows:

	Three Year Zero Coupon Bond			
	Current Time			
Row Number	0	1	2	3
1	0.961892	0.974953	0.994604	1
2		0.954609	0.973293	1
3			0.979056	1
4			0.96264	1
5				1
6				1
7				1
8				1

If we combine all of these tables, we can create a table of the term structure of zero coupon bond prices in each scenario as in example one:

Figure 6

State Name	State Number	Zero Coupon Bond Prices Periods to Maturity 1	2	3	4
Time 0	**0**	0.99701	0.98411	0.96189	0.93086
Time 1 up	1	0.99086	0.97495	0.95156	
Time 1 down	2	0.98327	0.95461	0.91574	
Time 2 up-up	3	0.9946	0.98003		
Time 2 up-down	4	0.97329	0.94065		
Time 2 down-up	5	0.97906	0.94507		
Time 2 down-down	6	0.96264	0.91757		
Time 3 up-up-up	7	0.98992			
Time 3 up-up-down	8	0.98078			
Time 3 up-down-up	9	0.97891			
Time 3 up-down-down	10	0.954			
Time 3 down-up-up	11	0.97773			
Time 3 down-up-down	12	0.95285			
Time 3 down-down-up	13	0.96379			
Time 3 down-down-down	14	0.94257			

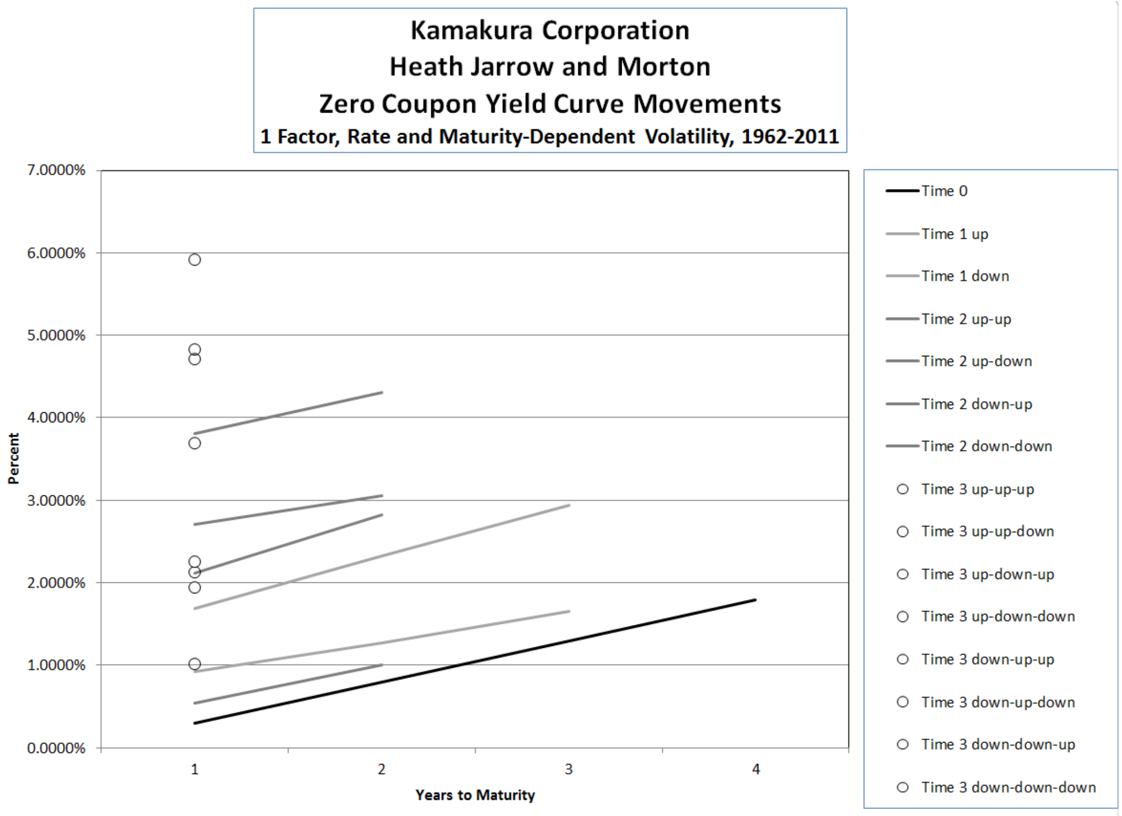
At any point in time t, the continuously compounded yield to maturity at time T can be calculated as $y(T-t)=-\ln[P(t,T)]/(T-t)$. Unlike example one, no negative rates appear in the bushy tree for example two because of the revised approach to interest rate volatility.

Figure 7

State Name	State Number	Continuously Compounded Zero Yields Periods to Maturity 1	2	3	4
Time 0	**0**	0.2999%	0.8009%	1.2951%	1.7913%
Time 1 up	1	0.9184%	1.2683%	1.6552%	
Time 1 down	2	1.6868%	2.3227%	2.9340%	
Time 2 up-up	3	0.5411%	1.0086%		
Time 2 up-down	4	2.7070%	3.0595%		
Time 2 down-up	5	2.1166%	2.8246%		
Time 2 down-down	6	3.8076%	4.3014%		
Time 3 up-up-up	7	1.0135%			
Time 3 up-up-down	8	1.9410%			
Time 3 up-down-up	9	2.1317%			
Time 3 up-down-down	10	4.7088%			
Time 3 down-up-up	11	2.2524%			
Time 3 down-up-down	12	4.8295%			
Time 3 down-down-up	13	3.6878%			
Time 3 down-down-down	14	5.9150%			

We can graph yield curve movements as shown below. Note that at time T=3, only the bond maturing in time T=4 remains so there is only one data point observable on the yield curve.

Figure 8



Finally, we can display the 1 year U.S. Treasury spot rates and the associated term structure of 1 year forward rates in each scenario.

Figure 9

State Name	State Number	1 Year Forward Rates, Simple Interest Periods to Maturity 1	2	3	4
Time 0	0	0.3003%	1.3104%	2.3098%	3.3343%
Time 1 up	1	0.9227%	1.6313%	2.4586%	
Time 1 down	2	1.7011%	3.0027%	4.2442%	
Time 2 up-up	3	0.5426%	1.4871%		
Time 2 up-down	4	2.7440%	3.4708%		
Time 2 down-up	5	2.1392%	3.5958%		
Time 2 down-down	6	3.8810%	4.9120%		
Time 3 up-up-up	7	1.0186%			
Time 3 up-up-down	8	1.9599%			
Time 3 up-down-up	9	2.1546%			
Time 3 up-down-down	10	4.8214%			
Time 3 down-up-up	11	2.2779%			
Time 3 down-up-down	12	4.9480%			
Time 3 down-down-up	13	3.7566%			
Time 3 down-down-down	14	6.0934%			

Valuation in the Heath Jarrow and Morton Framework

Valuation using our revised volatility assumptions is identical in methodology to Chapter 6. Note that column 1 denotes the riskless 1 period interest rate in each scenario. For the scenario number 14 (three consecutive downshifts in zero coupon bond prices), cash flows at time T=4 would be discounted by the one year spot rates at time t=0, by the one year spot rate at time t=1 in scenario 2 ("down"), by the one year spot rate in scenario 6 ("down down") at time t=2, and by the one year spot rate at time t=3 in scenario 14 ("down down down"). The discount factor is

Discount Factor (0,4, down down down) =1/(1.003003)(1.017011)(1.038810)(1.060934)

These discount factors are displayed here for each potential cash flow date:

	Discount Factors				
	Current Time				
Row Number	0	1	2	3	4
1	1	0.997006	0.987891	0.98256	0.972653
2		0.997006	0.987891	0.98256	0.963673
3			0.980329	0.961507	0.941228
4			0.980329	0.961507	0.917281
5				0.959798	0.938421
6				0.959798	0.914546
7				0.943704	0.909536
8				0.943704	0.889503

When taking expected values, we can calculate the probability of each scenario coming about since the probability of an upshift is ½:

	Probability of Each State				
	Current Time				
Row Number	0	1	2	3	4
1	100.00%	50.00%	25.00%	12.50%	12.50%
2		50.00%	25.00%	12.50%	12.50%
3			25.00%	12.50%	12.50%
4			25.00%	12.50%	12.50%
5				12.50%	12.50%
6				12.50%	12.50%
7				12.50%	12.50%
8				12.50%	12.50%

As in Chapter 6, it is convenient to calculate the “probability weighted discount factors” for use in calculating the expected present value of cash flows:

	Probability Weighted Discount Factors				
	Current Time				
Row Number	0	1	2	3	4
1	1	0.498503	0.246973	0.12282	0.121582
2		0.498503	0.246973	0.12282	0.120459
3			0.245082	0.120188	0.117654
4			0.245082	0.120188	0.11466
5				0.119975	0.117303
6				0.119975	0.114318
7				0.117963	0.113692
8				0.117963	0.111188

We now use the HJM bushy trees we have generated to value representative types of securities .

Valuation of a Zero Coupon Bond Maturing at Time T=4

A riskless zero coupon bond pays $1 in each of the 8 nodes of the bushy tree that prevail at time T=4:

	Maturity of Cash Flow Received				
	Current Time				
Row Number	0	1	2	3	4
1	0	0	0	0	1
2		0	0	0	1
3			0	0	1
4			0	0	1
5				0	1
6				0	1
7				0	1
8				0	1
Risk Neutral Value =					0.93085510

When we multiply this vector of 1s times the probability weighted discount factors in the time T=4 column in the previous table and add them, we get a zero coupon bond price of 0.93085510, which is the value we should get in a no-arbitrage economy, the value observable in the market and used as an input to create the tree.

Valuation of a Coupon-Bearing Bond Paying Annual Interest

Next we value a bond with no credit risk that pays $3 in interest at every scenario at times T=1, 2, 3, and 4 plus principal of 100 at time T=4. The valuation is calculated by multiplying each cash flow by the matching probability weighted discount factor, to get a value of 104.70709974. It will surprise many that this is the same value that we arrived at in Chapter 6, even though the volatilities used are different. The values are the same because, by construction, our valuations for the zero coupon bond prices at time zero for maturities at T = 1, 2, 3, and 4 continue to match the inputs. Multiplying these zero coupon bond prices times 3, 3, 3, and 103 also leads to a value of 104.70709974 as it should.

	Maturity of Cash Flow Received				
	Current Time				
Row Number	0	1	2	3	4
1	0	3	3	3	103
2		3	3	3	103
3			3	3	103
4			3	3	103
5				3	103
6				3	103
7				3	103
8				3	103
Risk Neutral Value =					104.70709974

Valuation of a Digital Option on the 1 Year U.S. Treasury Rate

Now we value a digital option that pays $1 at time T=3 if (at that time) the one year U.S. Treasury rate (for maturity at T=4) is over 4%. If we look at the table of the term structure of one year spot and forward rates above, this happens in three scenarios, scenarios 10 (up down down), 12 (down up down), and 14 (down down down). The cash flow can be input in the table below and multiplied by the probability weighted discount factors to find that this option has a value of 0.35812611:

Maturity of Cash Flow Received					
	Current Time				
Row Number	0	1	2	3	4
1	0	0	0	0	0
2		0	0	0	0
3			0	0	0
4			0	1	0
5				0	0
6				1	0
7				0	0
8				1	0
Risk Neutral Value =					0.35812611

Conclusion

This chapter, an extension of Chapter 6, shows how to simulate zero coupon bond prices, forward rates and zero coupon bond yields in an HJM framework with rate-dependent and maturity-dependent interest rate volatility. Monte Carlo simulation, an alternative to the bushy tree framework, can be done in a fully consistent way. We discuss HJM Monte Carlo simulation in Chapter 10.

In Chapter 8, we enrich the realism of the simulation by recognizing one very important flaw in the current example. In the current example, because all of the sigmas are positive, when the single risk factor shocks forward rates, they will all move up together, move down together, or remain unchanged together. This means that rates are 100% correlated with each other. If one examines the actual correlation of one year changes in forward rates, however, correlations fall well below 100%:

Correlation of 1 Year U.S. Treasury Spot and 1 Year U.S. Treasury Forward Rates, 1963-2012
Source: Kamakura Corporation and Federal Reserve

	U.S. Treasury 1 Year Spot or 1 Year Forward Rate Maturing in Year									
Maturity	1	2	3	4	5	6	7	8	9	10
1	1									
2	0.898447	1								
3	0.803277	0.962341	1							
4	0.739121	0.912764	0.969297	1						
5	0.632113	0.8427	0.888022	0.959259	1					
6	0.538515	0.776475	0.838138	0.895129	0.960935	1				
7	0.487825	0.726944	0.809716	0.842848	0.889794	0.973991	1			
8	0.504655	0.733078	0.820333	0.85747	0.880224	0.934078	0.970495	1		
9	0.512063	0.71558	0.790776	0.846346	0.854053	0.845016	0.862327	0.954766	1	
10	0.491524	0.67311	0.734645	0.804338	0.809255	0.758401	0.755353	0.878461	0.979529	1

In the next two chapters, we introduce a second and third risk factor in order to match both observed volatilities and correlations.