Advanced Financial Risk Management:

Tools and Techniques for Integrated Credit Risk, Market Risk, Liquidity Risk and Interest Rate Risk Management

April 2012

Donald R. van Deventer
Kenji Imai
Mark Mesler

Part 4: Risk Management Applications, Instrument by Instrument

Chapter 22

Forward and Futures Contracts

In this chapter, we continue the process of valuing the most common instruments on the balance sheet of major financial institutions around the world. Valuation enables us how to efficiently perform a multi-period simulation of cash flows and future values for these instruments, both of which are critical to the ultimate objective—measuring interest rate risk, market risk, liquidity risk and credit risk in a consistent framework, via the Jarrow-Merton put option introduced in Chapter 1.

Our focus in this chapter is on forward contracts and futures contracts for interest –rate-related instruments. Careful attention to this issue is important both from a correct hedging analysis and from an accounting perspective. This chapter lays the groundwork for the correlation requirements of those standards, both in terms of valuation and cash flow. In Chapter 20, we warned of the dangers of collusion and market manipulation in the market for credit default swaps. In Chapter 17, on credit spread modeling, we mentioned the legal actions alleging similar manipulation in the London Interbank Offered Market. On June 27, 2012, agencies of the U.S. government and the U.K.'s Financial Services Authority levied more than $450 million in penalties for Libor manipulation against Barclays PLC. We will assume perfectly competitive markets in the analysis below, but we warn readers, as we have throughout this volume, to make the appropriate adjustments given the nature of the pricing index and the nature of the counterparty. It is a mistake to assume (a) that everyone you trade with is Mother Theresa and that (b) you are the smartest one in the room. Lehman veteran Larry McDonald is often quoted as saying "If you are playing poker and you can't find the sucker at the table, it's probably you." Beware.

Financial forward and futures contracts are critical everyday tools of risk management and investment management. Their popularity is probably best summarized by the interest rate futures contracts traded on the Chicago Mercantile Exchange. Open interest in these contracts has long exceeded trillions of dollars, even for the most long-dated contracts. Eurodollar futures on June 28, 2012 traded at maturities as long as 9 years forward. Figure 1 from www.CMEGroup.com summarizes the open interest in interest rate futures as of May 2012:

Figure 1

CME Group Exchange Open Interest Report - Monthly

	Open Interest MAY 2012	Open Interest MAY 2011	% CHG	Open Interest APR 2012	% CHG
Interest Rate Futures					
30-YR BOND	741,599	740,953	0.1%	576,675	28.6%
10-YR NOTE	2,012,970	1,973,136	2.0%	1,847,462	9.0%
5-YR NOTE	1,300,391	1,555,969	-16.4%	1,332,831	-2.4%
2-YR NOTE	991,488	1,046,815	-5.3%	950,995	4.3%
FED FUND	553,694	751,787	-26.3%	532,994	3.9%
10-YR SWAP	8,633	13,823	-37.5%	7,973	8.3%
5-YR SWAP	10,210	28,377	-64.0%	8,880	15.0%
MINI EURODOLLAR	0	1	-100.0%	0	-
EUROYEN	263	825	-68.1%	516	-49.0%
EURODOLLARS	8,658,588	10,202,720	-15.1%	8,811,377	-1.7%
EMINI ED	7	11	-36.4%	7	0.0%
7-YR SWAP	36	631	-94.3%	39	-7.7%
3-MONTH OIS	0	0	-	0	-
3-YR NOTE	0	0	-	0	-
30-YR SWAP	122	640	-80.9%	140	-12.9%
1-MONTH EURODOLLAR	18,632	10,318	80.6%	18,050	3.2%
ULTRA T-BOND	386,551	378,832	2.0%	357,091	8.2%
OTR 10-YR NOTE	0	410	-100.0%	2	-100.0%
OTR 5-YR NOTE	0	47	-100.0%	0	-
OTR 2-YR NOTE	0	44	-100.0%	0	-
SOVEREIGN YIELD FUTURE	0	0	0.0%	0	-
EURO BONDS	294	0	-	532	-44.7%

This chapter illustrates the important distinctions between forward and futures contracts of different types, using the tools of Chapter 21 that we developed in the context of valuing options on zero-coupon and coupon-bearing bonds. Note that we will not delve into the detailed contract specifications of specific futures contracts because the information is overwhelmingly voluminous. Instead, we concentrate on analytical methods that can be applied to any forward or futures contract with appropriate adjustments for the details of the instrument. Daily cash flows and margin requirements are critical considerations with interest rate futures. We will use the 3 factor HJM bushy tree from Chapter 9 again in this chapter. In practice, we would use daily time steps and a Monte Carlo implementation of HJM. We use the bushy tree from Chapter 9 for ease of exposition. We start first with various forward contracts and then cover futures contracts.

Forward Contracts on Zero-Coupon Bonds

The most basic forward contract is a forward contract on the price of a zero-coupon bond. In this section and the rest of the chapter, we need to make an important distinction between the *value* of an existing contract on the books of an investment manager, insurance firm, or bank and the *price* of the forward contract quoted in the market. Before making this distinction clear, a real world example of a forward contract on zero-coupon bonds is useful. In this case, the best example is provided by the 90-day Bank Bill futures contract traded on the Sydney Futures Exchange. For purposes of this section, we will ignore daily mark-to-market requirements and analyze this contract as a forward, not a futures, contract. We will do the true futures analysis later in the chapter.

Contract:	90-day Bank Bills, Sydney Futures Exchange
Quotation basis:	100 - yield in percent, quoted to two decimal places
Assumed maturity	
of underlying bill	90 days
Valuation formula:	

$$Bank\ Bill\ Contract\ Price = \frac{365(1000000)}{365 + \dfrac{90 * yield}{100}}$$

The bank bill contract assumes the underlying instrument pays simple interest on an actual/365 day basis and has a maturity of 90 days. Essentially, the price of the contract is the price of a 90-day zero-coupon bond with principal amount of A$ 1 million at maturity. The calculation is the same as those we used in Chapter 4. One of the key features of the Sydney Bank Bill contract, as opposed to the Eurodollar contracts we discuss below, is that the value of an 0.01% change in yield depends on the level of interest rates rather than being a constant dollar amount like the Eurodollar contract:

Yield (%)	**A$ Value of 0.01% Yield Increase**
9.00%	A$ 23.60

10.00%	A$ 23.49
11.00%	A$ 23.37
12.00%	A$ 23.26

Note that the quotation method (100 - yield) is irrelevant to valuation—only the underlying cash flow described by the valuation formula matters for valuation purposes.

For notational convenience, we do the valuation analysis assuming that the principal amount on the underlying instrument is A$1 instead of A$ 1 million. We know from Chapter 4 that the forward price of this bond in a no arbitrage world with perfect competition should be the ratio of zero-coupon bond prices:

$$Forward\ Bond\ Price(r,t,T_1,T_2) = \frac{P(r,t,T_2)}{P(r,t,T_1)}$$

where T_2 is the maturity date of the underlying instrument (in years), T_1 is the maturity date of the forward contract, and t is the current time. This result doesn't depend on any particular term structure model since it can be derived using arbitrage arguments alone. If financial institution has bought the forward contract at an exercise price of K, then at the maturity of the forward (time T= T_1), the gain will be $P(r,T_1,T_2)$ - K. Let’s assume that the forward contract we are analyzing is a 3 year forward contract on the one year zero coupon bond maturing at time T=4. We can use the bushy tree of Chapter 9 and the related 1 year zero coupon bond prices as of time T=3 to analyze cash flows to the purchaser of the forward contract for any given exercise price K. For an exercise price K=0.97, the cash flows at time T=3 on the 64 nodes are shown in Figure 2:

Figure 2

Gain or Loss on 3 Year Forward Contract on Zero Coupon Bond Maturing in Year 2				
State Name	**State Number**	**Zero Coupon**	**Forward Price**	**Forward Cash Flow**
Time 3 S-1, S-1, S-1	21	0.9424127	0.97	-0.0276
Time 3 S-1, S-1, S-2	22	0.9472683	0.97	-0.0227
Time 3 S-1, S-1, S-3	23	0.9508321	0.97	-0.0192
Time 3 S-1, S-1, S-4	24	0.9256048	0.97	-0.0444
Time 3 S-1, S-2, S-1	25	0.9698938	0.97	-0.0001
Time 3 S-1, S-2, S-2	26	0.9747413	0.97	0.0047
Time 3 S-1, S-2, S-3	27	0.9801123	0.97	0.0101
Time 3 S-1, S-2, S-4	28	0.9609579	0.97	-0.0090
Time 3 S-1, S-3, S-1	29	0.9503806	0.97	-0.0196
Time 3 S-1, S-3, S-2	30	0.9805997	0.97	0.0106
Time 3 S-1, S-3, S-3	31	0.9691822	0.97	-0.0008
Time 3 S-1, S-3, S-4	32	0.9478524	0.97	-0.0221
Time 3 S-1, S-4, S-1	33	0.9485398	0.97	-0.0215
Time 3 S-1, S-4, S-2	34	0.9534269	0.97	-0.0166
Time 3 S-1, S-4, S-3	35	0.957014	0.97	-0.0130
Time 3 S-1, S-4, S-4	36	0.9316226	0.97	-0.0384
Time 3 S-2, S-1, S-1	37	0.9601375	0.97	-0.0099
Time 3 S-2, S-1, S-2	38	0.9906668	0.97	0.0207
Time 3 S-2, S-1, S-3	39	0.9791321	0.97	0.0091
Time 3 S-2, S-1, S-4	40	0.9575834	0.97	-0.0124
Time 3 S-2, S-2, S-1	41	0.9784392	0.97	0.0084
Time 3 S-2, S-2, S-2	42	0.9944994	0.97	0.0245
Time 3 S-2, S-2, S-3	43	0.9923911	0.97	0.0224
Time 3 S-2, S-2, S-4	44	0.9881387	0.97	0.0181
Time 3 S-2, S-3, S-1	45	0.9744394	0.97	0.0044
Time 3 S-2, S-3, S-2	46	1.003748	0.97	0.0337
Time 3 S-2, S-3, S-3	47	0.9974135	0.97	0.0274
Time 3 S-2, S-3, S-4	48	0.9787813	0.97	0.0088
Time 3 S-2, S-4, S-1	49	0.9825459	0.97	0.0125
Time 3 S-2, S-4, S-2	50	0.9874566	0.97	0.0175
Time 3 S-2, S-4, S-3	51	0.9928977	0.97	0.0229
Time 3 S-2, S-4, S-4	52	0.9734935	0.97	0.0035
Time 3 S-3, S-1, S-1	53	0.9777041	0.97	0.0077
Time 3 S-3, S-1, S-2	54	0.9825906	0.97	0.0126
Time 3 S-3, S-1, S-3	55	0.9880049	0.97	0.0180
Time 3 S-3, S-1, S-4	56	0.9686963	0.97	-0.0013
Time 3 S-3, S-2, S-1	57	0.9777041	0.97	0.0077
Time 3 S-3, S-2, S-2	58	0.9825906	0.97	0.0126
Time 3 S-3, S-2, S-3	59	0.9880049	0.97	0.0180
Time 3 S-3, S-2, S-4	60	0.9686963	0.97	-0.0013
Time 3 S-3, S-3, S-1	61	0.9649212	0.97	-0.0051
Time 3 S-3, S-3, S-2	62	0.9939435	0.97	0.0239
Time 3 S-3, S-3, S-3	63	0.9876709	0.97	0.0177
Time 3 S-3, S-3, S-4	64	0.9692207	0.97	-0.0008
Time 3 S-3, S-4, S-1	65	0.9568734	0.97	-0.0131
Time 3 S-3, S-4, S-2	66	0.9872989	0.97	0.0173
Time 3 S-3, S-4, S-3	67	0.9758034	0.97	0.0058
Time 3 S-3, S-4, S-4	68	0.9543279	0.97	-0.0157
Time 3 S-4, S-1, S-1	69	0.9661716	0.97	-0.0038
Time 3 S-4, S-1, S-2	70	0.9968928	0.97	0.0269
Time 3 S-4, S-1, S-3	71	0.9852856	0.97	0.0153
Time 3 S-4, S-1, S-4	72	0.9636014	0.97	-0.0064
Time 3 S-4, S-2, S-1	73	0.9748192	0.97	0.0048
Time 3 S-4, S-2, S-2	74	0.9796913	0.97	0.0097
Time 3 S-4, S-2, S-3	75	0.9850896	0.97	0.0151
Time 3 S-4, S-2, S-4	76	0.9658379	0.97	-0.0042
Time 3 S-4, S-3, S-1	77	0.9748192	0.97	0.0048
Time 3 S-4, S-3, S-2	78	0.9796913	0.97	0.0097
Time 3 S-4, S-3, S-3	79	0.9850896	0.97	0.0151
Time 3 S-4, S-3, S-4	80	0.9658379	0.97	-0.0042
Time 3 S-4, S-4, S-1	81	0.9568821	0.97	-0.0131
Time 3 S-4, S-4, S-2	82	0.9778172	0.97	0.0078
Time 3 S-4, S-4, S-3	83	0.9734711	0.97	0.0035
Time 3 S-4, S-4, S-4	84	0.9521725	0.97	-0.0178

If we drop these cash flows into the cash flow table and multiply by the proper probability weighted discount factors, we find in Figure 3 that the value of this forward position is -0.0022:

Figure 3

Maturity of Cash Flow Received

	Current Time				
Row	0	1	2	3	4
1	0.0000	0.0000	0.0000	-0.0276	0.0000
2		0.0000	0.0000	-0.0227	0.0000
3		0.0000	0.0000	-0.0192	0.0000
4		0.0000	0.0000	-0.0444	0.0000
5			0.0000	-0.0001	0.0000
6			0.0000	0.0047	0.0000
7			0.0000	0.0101	0.0000
8			0.0000	-0.0090	0.0000
9			0.0000	-0.0196	0.0000
10			0.0000	0.0106	0.0000
11			0.0000	-0.0008	0.0000
12			0.0000	-0.0221	0.0000
13			0.0000	-0.0215	0.0000
14			0.0000	-0.0166	0.0000
15			0.0000	-0.0130	0.0000
16			0.0000	-0.0384	0.0000
17				-0.0099	0.0000
18				0.0207	0.0000
19				0.0091	0.0000
20				-0.0124	0.0000
21				0.0084	0.0000
22				0.0245	0.0000
23				0.0224	0.0000
24				0.0181	0.0000
25				0.0044	0.0000
26				0.0337	0.0000
27				0.0274	0.0000
28				0.0088	0.0000
29				0.0125	0.0000
30				0.0175	0.0000
31				0.0229	0.0000
32				0.0035	0.0000
33				0.0077	0.0000
34				0.0126	0.0000
35				0.0180	0.0000
36				-0.0013	0.0000
37				0.0077	0.0000
38				0.0126	0.0000
39				0.0180	0.0000
40				-0.0013	0.0000
41				-0.0051	0.0000
42				0.0239	0.0000
43				0.0177	0.0000
44				-0.0008	0.0000
45				-0.0131	0.0000
46				0.0173	0.0000
47				0.0058	0.0000
48				-0.0157	0.0000
49				-0.0038	0.0000
50				0.0269	0.0000
51				0.0153	0.0000
52				-0.0064	0.0000
53				0.0048	0.0000
54				0.0097	0.0000
55				0.0151	0.0000
56				-0.0042	0.0000
57				0.0048	0.0000
58				0.0097	0.0000
59				0.0151	0.0000
60				-0.0042	0.0000
61				-0.0131	0.0000
62				0.0078	0.0000
63				0.0035	0.0000
64				-0.0178	0.0000
Risk Neutral Value =					-0.0022

We know, however, that we can replicate the cash flows of this forward contract by making these transactions:

Step 1: Buy zero coupon bond maturing at time T=4

Step 2: Agree to issue a zero coupon bond with a face value of 0.97 maturing at time T=3. The proceeds of this loan for 0.97, at the current yield curve, are 0.9330.

Step 3: At time T=3, sell zero coupon bond maturing at time T=4 and pay off the loan of 0.97 as it matures. There will be a gain or a loss from doing so that is identical to the pay off on the forward contract, so the net cash flow of this alternative strategy to the forward contract should also be -0.0022.

We can confirm that indeed

Forward(0,3,4,.97)=P(0,4)-P(0,3)K= 0.9308550992 - 0.9618922376(0.97) = -0.0022.

This result stems from no arbitrage arguments and holds for any term structure model. Leaving aside the existing forward contract purchased for K=0.97, what should the exercise price K of a new forward contract be today? Since no initial cash is necessary to buy a forward "at market," the value of the forward should be zero so that

0=P(0,4)-P(0,3) K or K = P(0,4)/P(0,3)

We can confirm this result by using the HJM cash flow table and solving for the level of K that makes the value of the existing futures contract zero.

Turning back to the Sydney "forward" contract, the observable market price (in zero-coupon bond terms) equals the forward bond price. In yield terms as quoted on the Sydney Futures Exchange, the quoted yield will be

$$Y* = \frac{36500}{90}\left[\frac{1000000}{Forward\ Zero\ Coupon\ Bond\ Price} - 1\right]$$

Later in this chapter, we deal with the implications of daily margin requirements and the possibility that there could be a default (a) either by the Exchange itself or (b) on the instrument underlying the forward contract in keeping with our interest in an integrated treatment of interest rate risk, market risk, liquidity risk and credit risk. The analysis in this section implicitly assumes that both the Exchange and the underlying instrument are default-free.

Forward Rate Agreements

The design of the Sydney Futures Exchange Bank Bill Contract is one of a relatively small number of contracts where the underlying instrument is a zero-coupon bond. It is most common for the underlying instrument to be expressed in term of an interest rate. Perhaps the most common forward contract with this structure is the forward rate agreement (FRA), where the counterparties agree on a strike rate K at maturity T_1 based on an underlying rate for an instrument maturing at T_2. The cash flow at time T_2 from the FRA is proportional to the FRA rate at maturity and the strike rate K:

$$Cash\ Flow = B\left(FRA\ Rate[T_1] - K\right)$$

where the coefficient B reflects the notional principal amount, the maturity of the underlying instrument, and the interest quotation method used for the FRA contract (i.e. actual/360 days, actual/365 days, etc.). In this section, we analyze the case where this interest differential is paid at the maturity of the underlying instrument, T_2. Conventional FRAs normally pay in an economically equivalent way, paying the present value of this interest rate differential at the maturity of the FRA contract, T_1.

We value the contract as if cash flowed at time T_1 by taking the simple present value of T_2 cash flow from the perspective of time T_1 (the standard FRA convention):

$$
\begin{aligned}
Present\ Value &= P(\tau = T_2 - T_1)B[FRA\ Rate - K] \\
&= P(\tau)B\left[\frac{1}{P(\tau)} - 1 - K\right] \\
&= B[1 - P(\tau)(1 + K)] \\
&= B - BP(\tau)(1 + K)
\end{aligned}
$$

Let's assume that B is 25 dollars per 1% rate change ($0.25 per basis point) and that the FRA exercise price is K=4%. You receive any excess of the actual spot rate over 4% and you pay any short fall of the actual spot rate under 4%. The cash flows that are paid at time T=4 on a one year underlying FRA are set out in Figure 4 for each of the relevant 64 nodes:

Figure 4

FRA at 4% on One Year Instrument				
State Name	State Number	Spot Rate at	Exercise Spot	Cash Flow Time T = 4
Time 3 S-1, S-1, S-1	21	6.11%	4.00%	52.77
Time 3 S-1, S-1, S-2	22	5.57%	4.00%	39.17
Time 3 S-1, S-1, S-3	23	5.17%	4.00%	29.28
Time 3 S-1, S-1, S-4	24	8.04%	4.00%	100.94
Time 3 S-1, S-2, S-1	25	3.10%	4.00%	-22.40
Time 3 S-1, S-2, S-2	26	2.59%	4.00%	-35.22
Time 3 S-1, S-2, S-3	27	2.03%	4.00%	-49.27
Time 3 S-1, S-2, S-4	28	4.06%	4.00%	1.57
Time 3 S-1, S-3, S-1	29	5.22%	4.00%	30.53
Time 3 S-1, S-3, S-2	30	1.98%	4.00%	-50.54
Time 3 S-1, S-3, S-3	31	3.18%	4.00%	-20.51
Time 3 S-1, S-3, S-4	32	5.50%	4.00%	37.54
Time 3 S-1, S-4, S-1	33	5.43%	4.00%	35.63
Time 3 S-1, S-4, S-2	34	4.88%	4.00%	22.12
Time 3 S-1, S-4, S-3	35	4.49%	4.00%	12.29
Time 3 S-1, S-4, S-4	36	7.34%	4.00%	83.49
Time 3 S-2, S-1, S-1	37	4.15%	4.00%	3.79
Time 3 S-2, S-1, S-2	38	0.94%	4.00%	-76.45
Time 3 S-2, S-1, S-3	39	2.13%	4.00%	-46.72
Time 3 S-2, S-1, S-4	40	4.43%	4.00%	10.74
Time 3 S-2, S-2, S-1	41	2.20%	4.00%	-44.91
Time 3 S-2, S-2, S-2	42	0.55%	4.00%	-86.17
Time 3 S-2, S-2, S-3	43	0.77%	4.00%	-80.83
Time 3 S-2, S-2, S-4	44	1.20%	4.00%	-69.99
Time 3 S-2, S-3, S-1	45	2.62%	4.00%	-34.42
Time 3 S-2, S-3, S-2	46	-0.37%	4.00%	-109.33
Time 3 S-2, S-3, S-3	47	0.26%	4.00%	-93.52
Time 3 S-2, S-3, S-4	48	2.17%	4.00%	-45.80
Time 3 S-2, S-4, S-1	49	1.78%	4.00%	-55.59
Time 3 S-2, S-4, S-2	50	1.27%	4.00%	-68.24
Time 3 S-2, S-4, S-3	51	0.72%	4.00%	-82.12
Time 3 S-2, S-4, S-4	52	2.72%	4.00%	-31.93
Time 3 S-3, S-1, S-1	53	2.28%	4.00%	-42.99
Time 3 S-3, S-1, S-2	54	1.77%	4.00%	-55.71
Time 3 S-3, S-1, S-3	55	1.21%	4.00%	-69.65
Time 3 S-3, S-1, S-4	56	3.23%	4.00%	-19.21
Time 3 S-3, S-2, S-1	57	2.28%	4.00%	-42.99
Time 3 S-3, S-2, S-2	58	1.77%	4.00%	-55.71
Time 3 S-3, S-2, S-3	59	1.21%	4.00%	-69.65
Time 3 S-3, S-2, S-4	60	3.23%	4.00%	-19.21
Time 3 S-3, S-3, S-1	61	3.64%	4.00%	-9.11
Time 3 S-3, S-3, S-2	62	0.61%	4.00%	-84.77
Time 3 S-3, S-3, S-3	63	1.25%	4.00%	-68.79
Time 3 S-3, S-3, S-4	64	3.18%	4.00%	-20.61
Time 3 S-3, S-4, S-1	65	4.51%	4.00%	12.68
Time 3 S-3, S-4, S-2	66	1.29%	4.00%	-67.84
Time 3 S-3, S-4, S-3	67	2.48%	4.00%	-38.01
Time 3 S-3, S-4, S-4	68	4.79%	4.00%	19.64
Time 3 S-4, S-1, S-1	69	3.50%	4.00%	-12.47
Time 3 S-4, S-1, S-2	70	0.31%	4.00%	-92.21
Time 3 S-4, S-1, S-3	71	1.49%	4.00%	-62.66
Time 3 S-4, S-1, S-4	72	3.78%	4.00%	-5.57
Time 3 S-4, S-2, S-1	73	2.58%	4.00%	-35.42
Time 3 S-4, S-2, S-2	74	2.07%	4.00%	-48.18
Time 3 S-4, S-2, S-3	75	1.51%	4.00%	-62.16
Time 3 S-4, S-2, S-4	76	3.54%	4.00%	-11.57
Time 3 S-4, S-3, S-1	77	2.58%	4.00%	-35.42
Time 3 S-4, S-3, S-2	78	2.07%	4.00%	-48.18
Time 3 S-4, S-3, S-3	79	1.51%	4.00%	-62.16
Time 3 S-4, S-3, S-4	80	3.54%	4.00%	-11.57
Time 3 S-4, S-4, S-1	81	4.51%	4.00%	12.65
Time 3 S-4, S-4, S-2	82	2.27%	4.00%	-43.29
Time 3 S-4, S-4, S-3	83	2.73%	4.00%	-31.87
Time 3 S-4, S-4, S-4	84	5.02%	4.00%	25.57

If we drop these into the cash flow table, we get the value -15.4927 in Figure 5:

Figure 5

	Maturity of Cash Flow Received				
	Current Time				
Row	0	1	2	3	4
1	0.0000	0.0000	0.0000	0.0000	52.7655
2		0.0000	0.0000	0.0000	39.1679
3		0.0000	0.0000	0.0000	29.2759
4		0.0000	0.0000	0.0000	100.9367
5			0.0000	0.0000	-22.3981
6			0.0000	0.0000	-35.2169
7			0.0000	0.0000	-49.2719
8			0.0000	0.0000	1.5707
9			0.0000	0.0000	30.5250
10			0.0000	0.0000	-50.5397
11			0.0000	0.0000	-20.5056
12			0.0000	0.0000	37.5414
13			0.0000	0.0000	35.6299
14			0.0000	0.0000	22.1202
15			0.0000	0.0000	12.2920
16			0.0000	0.0000	83.4900
17				0.0000	3.7937
18				0.0000	-76.4473
19				0.0000	-46.7184
20				0.0000	10.7387
21				0.0000	-44.9103
22				0.0000	-86.1725
23				0.0000	-80.8318
24				0.0000	-69.9907
25				0.0000	-34.4222
26				0.0000	-109.3349
27				0.0000	-93.5170
28				0.0000	-45.8032
29				0.0000	-55.5895
30				0.0000	-68.2432
31				0.0000	-82.1172
32				0.0000	-31.9293
33				0.0000	-42.9890
34				0.0000	-55.7054
35				0.0000	-69.6481
36				0.0000	-19.2117
37				0.0000	-42.9890
38				0.0000	-55.7054
39				0.0000	-69.6481
40				0.0000	-19.2117
41				0.0000	-9.1149
42				0.0000	-84.7666
43				0.0000	-68.7926
44				0.0000	-20.6082
45				0.0000	12.6759
46				0.0000	-67.8388
47				0.0000	-38.0085
48				0.0000	19.6447
49				0.0000	-12.4680
50				0.0000	-92.2078
51				0.0000	-62.6646
52				0.0000	-5.5663
53				0.0000	-35.4218
54				0.0000	-48.1758
55				0.0000	-62.1597
56				0.0000	-11.5740
57				0.0000	-35.4218
58				0.0000	-48.1758
59				0.0000	-62.1597
60				0.0000	-11.5740
61				0.0000	12.6520
62				0.0000	-43.2850
63				0.0000	-31.8703
64				0.0000	25.5745
Risk Neutral Value =					-15.4927

Note that we can replicate the net present value that will be received under this transaction at time T=3. We know B = $2500 (since rates are expressed as a decimal) and the FRA exercise price is 0.04. We replicate this time T=3 net present value pay-off with the following steps:

Step 1: Buy $2500 in principal amount of the zero coupon bond maturing at time T=3. The cost of this investment is P(0,3)2500.

Step 2: Borrow in the form of a zero coupon bond with principal $2500 (1 + 0.04) to mature at time T=4. The proceeds of this borrowing are P(0,4)[2500](1.04).

Step 3: At time T=3 either liquidate the position or invest the proceeds of the bond purchased in step 1 ($2500) in the one year spot rate maturing at time T=4.

No arbitrage requires that the net cost of this strategy be the same as -15. 4927 and we can confirm that is the case:

P(0,3)2500-P(0,4)[2500](1.04) = -15.4927.

We can also solve for the implied "price" of new FRA transactions K, recognizing that an FRA transaction at time zero is costless and therefore should have a value of zero to avoid arbitrage possibilities:

K =P(0,3)/P(0,4)-1

Again, this formula holds for any term structure model because the replication of the FRA is independent of the term structure model selected. What if the rate differential between the FRA rate and *K* is paid on an undiscounted basis at time T_1? We answer that question in the next section.

Eurodollar Futures-type Forward Contracts

What if the market convention regarding FRA payment was that the forward rate agreement pays its cash flow:

$$Cash\ Flow = B\left(FRA\ Rate[\,T_1\,] - K\right)$$

at the T_1 maturity of the FRA contract instead its present value at time T_1 or payment in cash at the T_2 maturity of the underlying instrument? An FRA that pays in this manner is essentially equivalent to a Eurodollar Futures contract with no mark-to-market margin (i.e. a Eurodollar futures-type forward). Such a Eurodollar futures-type forward, if modeled after the Chicago Mercantile Exchange Eurodollar futures contract, would pay $25 for every one basis point differential between the settlement yield on the futures-type forward and the strike price. Unlike the Sydney Futures Exchange Bank Bill future discussed earlier in this chapter, the dollar value of a basis point change in quoted yield is the same for all levels of interest rates under this type of contract. The cash flow at time T_1 under this kind of contract is:

$$Cash\ \ Flow\ = B\left(\frac{1}{P(\ \tau\ )} - 1 - K\right) = \frac{B}{P(\tau)} - (1 + K)B$$

The cash flows are the same as before but they arrive at time T=3 instead of time T=4. We can calculate the value of this kind of structure by dropping the cash flows into a different column, the column for time T=3, in the cash flow table. The value is -15.4146, as shown in Figure 6:

Figure 6

	Maturity of Cash Flow Received				
	Current Time				
Row	0	1	2	3	4
1	0.0000	0.0000	0.0000	52.7655	0.0000
2		0.0000	0.0000	39.1679	0.0000
3		0.0000	0.0000	29.2759	0.0000
4		0.0000	0.0000	100.9367	0.0000
5			0.0000	-22.3981	0.0000
6			0.0000	-35.2169	0.0000
7			0.0000	-49.2719	0.0000
8			0.0000	1.5707	0.0000
9			0.0000	30.5250	0.0000
10			0.0000	-50.5397	0.0000
11			0.0000	-20.5056	0.0000
12			0.0000	37.5414	0.0000
13			0.0000	35.6299	0.0000
14			0.0000	22.1202	0.0000
15			0.0000	12.2920	0.0000
16			0.0000	83.4900	0.0000
17				3.7937	0.0000
18				-76.4473	0.0000
19				-46.7184	0.0000
20				10.7387	0.0000
21				-44.9103	0.0000
22				-86.1725	0.0000
23				-80.8318	0.0000
24				-69.9907	0.0000
25				-34.4222	0.0000
26				-109.3349	0.0000
27				-93.5170	0.0000
28				-45.8032	0.0000
29				-55.5895	0.0000
30				-68.2432	0.0000
31				-82.1172	0.0000
32				-31.9293	0.0000
33				-42.9890	0.0000
34				-55.7054	0.0000
35				-69.6481	0.0000
36				-19.2117	0.0000
37				-42.9890	0.0000
38				-55.7054	0.0000
39				-69.6481	0.0000
40				-19.2117	0.0000
41				-9.1149	0.0000
42				-84.7666	0.0000
43				-68.7926	0.0000
44				-20.6082	0.0000
45				12.6759	0.0000
46				-67.8388	0.0000
47				-38.0085	0.0000
48				19.6447	0.0000
49				-12.4680	0.0000
50				-92.2078	0.0000
51				-62.6646	0.0000
52				-5.5663	0.0000
53				-35.4218	0.0000
54				-48.1758	0.0000
55				-62.1597	0.0000
56				-11.5740	0.0000
57				-35.4218	0.0000
58				-48.1758	0.0000
59				-62.1597	0.0000
60				-11.5740	0.0000
61				12.6520	0.0000
62				-43.2850	0.0000
63				-31.8703	0.0000
64				25.5745	0.0000
Risk Neutral Value =					-15.4146

The valuation formulas for this instrument are more complex than the simple no arbitrage results given above.

Futures on Zero-Coupon Bonds:

The Sydney Futures Exchange Bank Bill Contract

From this point on in the chapter, we deal with true futures contracts. A 'true' futures contract has daily mark-to-market requirements, with cash flow on an existing contract occurring continuously through its life. Ending cash flow is essentially zero, since the full amount of the change in observable futures prices over the life of the contract will have already been paid in cash through the continuous mark-to-market requirements. We will ignore initial margin payments and the return of these initial payments since they represent a simple zero-coupon bond type adjustment to the formulas which follow.

Like the previous three sections, there is an important distinction between the market value of an existing futures position *V* and the observable market price of a futures contract *Q*. The first three sections showed the distinct differences between *V* and *Q* for three different types of forward contracts. In this section and the remainder of the chapter, we note the following:

> *The market value V of an existing futures position will always be zero since (a) the market price of a new futures contract must be zero to avoid arbitrage and (b) a previously taken futures position is continuously made equivalent to a new position by continuous margin payments. The observable market price Q of the future contract and the initial strike rate K on the futures transaction, put in place at time* t_0*, are related by the fact that cash payments of at least M=K-Q, the cumulative net loss (gain) on the futures contract, must have been made to (withdrawn from) the margin account since initiation of the transaction at time* t_0*. The balance of the margin account is unknown since parties with a positive margin balance may use the cash proceeds for any purpose.*

> *This important comment notwithstanding, a trader is always thinking about their gains or losses in terms of the original rate at which the futures contract was purchased K and the current market price Q. It is critical to note that this differential will have already impacted the financial institution's balance sheet (i.e. the cash account) and the trader's focus on the differential K-Q reflects only a partial view of the impact of the futures contract on the*

institution. Most sophisticated institutions are taking both views of the impact of a futures contract, the trader's view and the true daily cash flow pattern via the margin accounts.

These comments require a careful vocabulary to bridge the gap between the jargon of market participants and financial reality. To a market participant, the 'value of a futures contract' is *Q-K*, even though the margin account balance would be *Max(0,K-Q)* rather than always being *Q-K*. These same market participants are rarely aware of the margin account balance, which is normally managed by an operational group. To a financial economist, the value is zero. Fortunately, the interest rate risk of a futures position to both groups is the change in *Q* that results from a change in interest rates, as captured by movements in the single stochastic factor *r* (if the Vasicek term structure model applies) or by movements in the 3 factors driving our HJM model from Chapter 9. Taking care to remember this distinction between viewpoints, we ignore the market value of an existing futures position *V* and concentrate on determining the observable market price *Q*. For an excellent introduction to this topic see Chen [1992] and Jarrow and Turnbull [1996, especially chapters 6 and 13].

In modeling interest rate futures contracts, the most important difference versus the forward rate contracts discussed earlier in the chapter is easy to summarize. Let's assume at time T=0 one takes a position in a futures contract maturing at time T=3 on the zero coupon bond maturing at time T=4. Obviously, at time T=3, the price of the maturing futures contract will be forced by arbitrage to exactly equal the time T=3 price of the zero coupon bond maturing at time T=4. This "boundary condition" is no different than it is for forward contracts. The difference comes in what happens at times T=1 and T=2. In the case of a forward contract maturing at time T=3, nothing happens at times 1 and 2, which is why the cash flows above were zero for those time periods. We write the price of a futures contract as of time t with maturity at T=3 on the zero coupon bond maturing at T=4 as F(t,3,4). The time zero price of the futures contract is F(0,3,4). At time 1, there are four nodes on the bushy tree and the futures price will be take on one of 4 values, $F(1,3,4,s_i)$ for i=1, 2, 3, and 4. At each of these four nodes, there will be cash flow equal to $F(1,3,4,s_i)-F(0,3,4)$. This cash flow either represents the withdrawal of a gain from a margin account or the contribution of a loss back into the margin account. After this cash flow has occurred at time T=1, of course, an existing futures position and a new futures position are identical. Both have zero dollars in the margin account.

At time T=2, there are 16 nodes and there will be 16 gains or losses on the futures position relative to the 4 futures values on the nodes at T=1.

The analytical derivation of the futures price evolution is nicely done by Chen [1992] in the Vasicek framework. As the number of risk factors driving interest rates in an HJM framework gets larger and larger for greater realism, futures prices and their evolution in general can be most usefully derived by numerical methods. See Jarrow and Turnbull [1996] for an excellent summary of the general principles that govern the evolution of futures prices. The need for numerical methods is particularly true in light of the "cheapest to deliver option" embedded in many bond futures contracts. We turn to that issue now.

Futures on Coupon-Bearing Bonds:

Dealing with the Cheapest to Deliver Option

In this section, we consider the contract specifications of the Singapore Exchange's Japanese government bond (JGB) contract. Let's assume the coupon rate on the futures contract is 2%:

Notional Principal	¥50,000,000
Maturity	10 years
Coupon Rate	2%

In actual operation, there will be a number of JGB issues that are 'deliverable' under the Singapore contract. The seller of the futures contract has a 'delivery option' which allows the seller of the futures contract to deliver the 'cheapest to deliver' bond if the seller of the futures contract chooses to hold the contract to maturity. If there are N deliverable bonds, the exchange will calculate a delivery factor F_i which specifies the ratio of the principal amount on the ith deliverable bond to the notional principal of ¥50,000,000 that is necessary to satisfy delivery requirements.

One of the bonds at any given time is the 'cheapest to deliver'. After multiplying the yen coupon amounts and yen principal amount on this bond times the delivery factor, we will have the j maturities and cash flow amounts underlying the bond future. The market price of the bond future (as long as only one bond is deliverable) is the sum of zero-coupon bond futures prices (assuming one yen of principal on each zero-coupon bond future) times these cash flow amounts C_i:

$$Q_{Bond}(t,T_0) = \sum_{i=1}^{j} Q(r,t,T_i)C_i$$

T_0 is the maturity date of the bond futures contract. The sensitivity analysis of bond futures, assuming only one bond is deliverable, is the sum of the sensitivities for the zero-coupon bond futures portfolio which replicates the cash flows on the bond.

What about the value of the delivery option? Let us assume that there are two deliverable bonds and the one factor Vasicek model is a realistic description of rate movements [MODEL RISK ALERT: It is not realistic]. Using the insights of Jamshidian's [1989] approach to coupon-bearing bond options valuation, we know that there will be a level of the short rate *r*, which we label *s**, such that the two bonds are equally attractive to deliver as of time T_0. Above *s**, bond one is cheaper to deliver. Below *s**, bond two is cheaper to deliver. We can solve the partial differential equation for futures valuation subject to the boundary conditions that the futures price converges to bond one's deliverable value above *s** and to bond two's deliverable value below *s**. We also impose the condition that the first and second derivatives of the futures price are smooth at a short rate level of *s**. The result is a special weighted average of the futures prices that would have prevailed if each bond was the only deliverable bond under the futures contract. There are other options embedded in common U.S. and Japanese bond tracks that can be analyzed in a related way. The process is much more complex in a realistic HJM framework with the 5-10 risk factors that we showed were necessary for realism in Chapter 3.

Eurodollar and Euroyen Futures Contracts

Money market futures have evolved to a fairly standard structure around the world. The Singapore Exchange futures contracts in U.S. dollars, yen, and Singapore Dollars are all on three month instruments with a constant currency amount paid at maturity per basis point change in the price of the contract. The "per basis point" value of rate changes under each contract is as follows:

U.S. Dollars	US$ 25.00
Yen (Tokyo and London)	2500 yen
Singapore Dollars	SG$ 25.00

In each case, the futures price must be equal to the underlying three-month interest rates, multiplied by the notional principal amount and the appropriate day count factor to convert the rate to an annual basis. At maturity, the futures price (actually, the futures rate times notional principal) *Q* must meet this boundary condition:

$$Q(r,T_1,T_1,T_2) = B\left[\frac{1}{P(r,T_1,T_2)} - 1\right]$$

B represents the notional principal and annualization factor, T_1 is the maturity of the futures contract in years, and T_2 is the maturity of the underlying three-month instrument so T_2 is roughly 0.25 greater than T_1.

The market price of the futures contract in the Vasicek model context (ignoring the annualization factor and notional principal embedded in *B*) is one over the forward bond price, multiplied by an adjustment factor, minus one. This formula reduces to the forward rate if interest rate volatility σ is zero. In general, the formulas for these money market futures contracts (when they can be obtained) are highly dependent on the term structure model selected.

Defaultable Forward and Futures Contracts

Jarrow and Turnbull [1995] show that, in special circumstances, the adjustment made in the price of a defaultable instrument is a simple ratio times the value of the same instrument if it were not defaultable. Many market participants simulate future values and cash flows of financial forwards and futures on the implicit assumption that default will not occur. The realization that default of the futures exchange itself is a possibility, however, should not come as a surprise to anyone who is familiar with the high correlation in the default probabilities between major financial institutions that we discussed in the Introduction in the 2007-2010 credit crisis. How should this risk be analyzed? We need to use the simulation technique outlined in Chapters 19 and 20 to handle this accurately. Accuracy requires us to use simulation methods outlined in those chapters, although explicit formulaic solutions can be obtained for highly simplified assumptions.

A careful Monte Carlo simulation like that specified in Chapters 19 and 20 is essential to recognizing the slim, but real possibility, of the default of the futures exchanges in times of crisis, like the collapse of futures specialist MF Global on October 30, 2011. We return to this theme in Chapters 36 through 41 in detail.