

Linkages and Configuration Spaces

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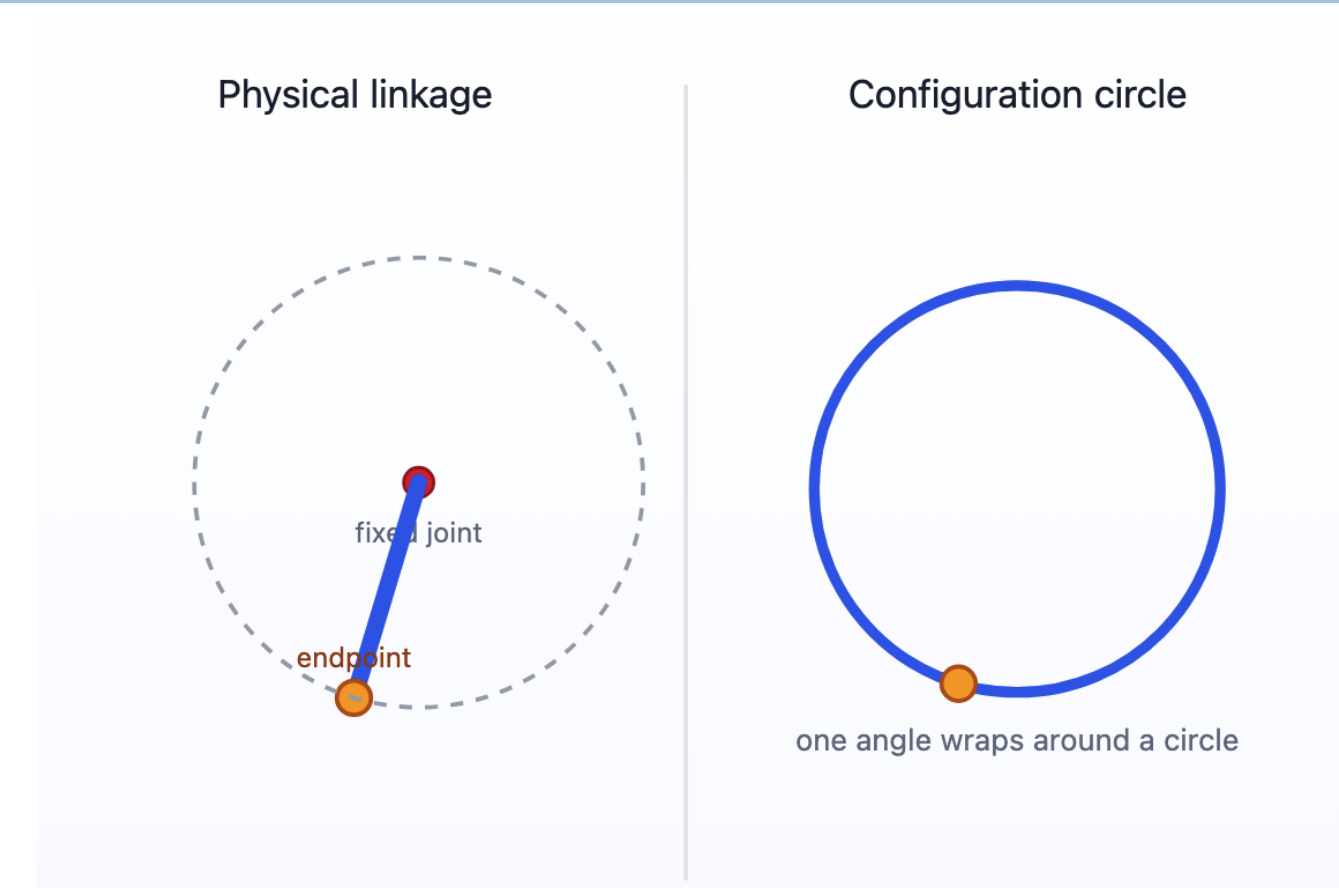
Introduction

A **linkage** is a set of stiff bars connected at joints, a bit like a robot arm or the folding lamp on your desk. Some points on a linkage are fixed: pinned in place, and others are free to move, as long as every bar keeps its exact length.

Configuration space: as the bars swing around, what is the full set of positions the moving point can reach? That set of possible positions is called the configuration space of the linkage.

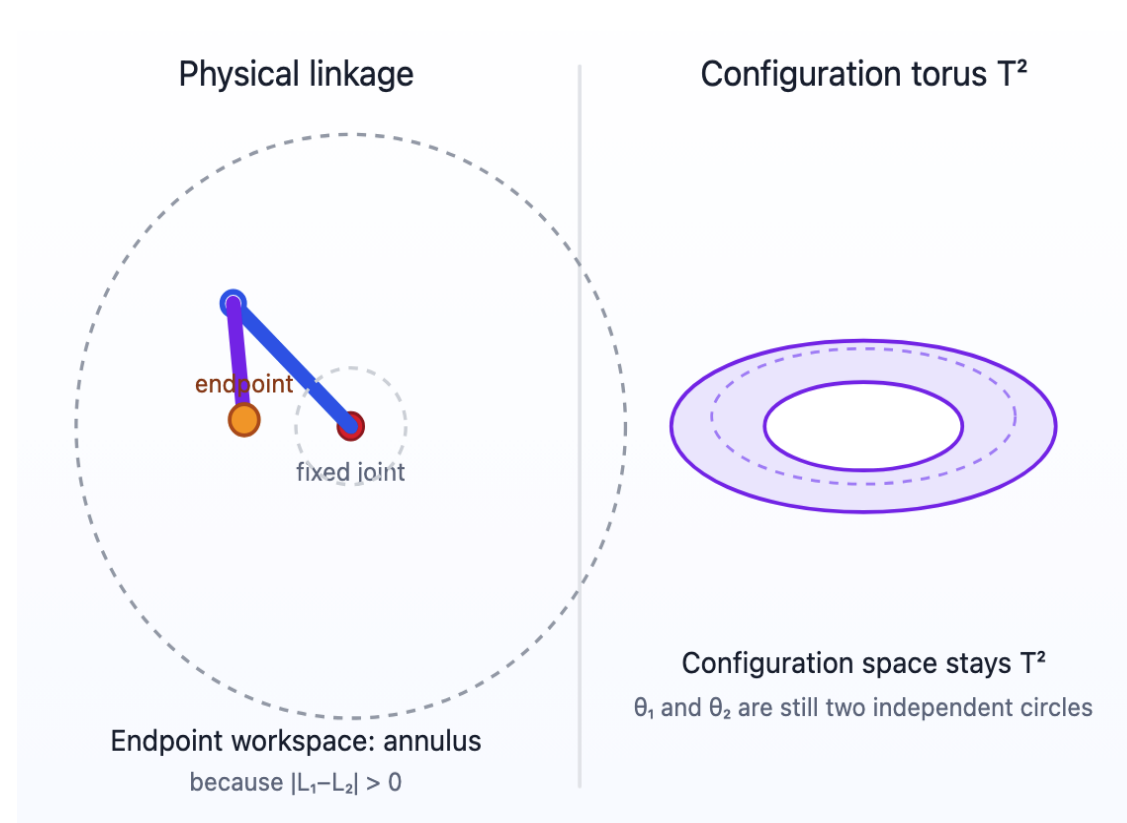
Why mathematicians care: robot arms, drafting tools, human joints, and even protein molecules are all linkages. Knowing the configuration space tells engineers every position a mechanism can reach and every position it can't before they ever build it.

One Rotating Bar -> Circle



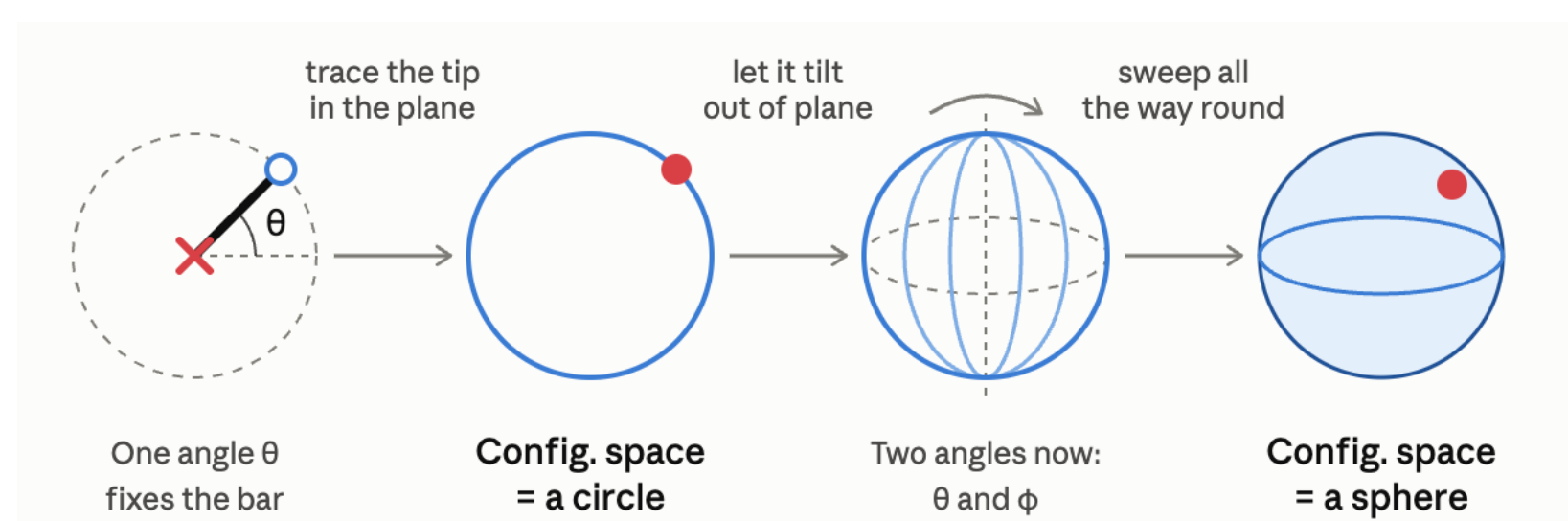
- **One rotating bar:** A single rigid bar rotates around one fixed joint. The endpoint traces a circle. One angle θ completely describes the configuration.
- **Configuration space:** S^1

Two Rotating Bars -> Torus/Sphere



- **Two independent joint angles**, θ_1 and θ_2 , describe the arm. The configuration space is always $S^1 \times S^1 = T^2$ for unrestricted revolute joints. The endpoint workspace changes with bar lengths. Assume outside joint is shorter than inside joint.

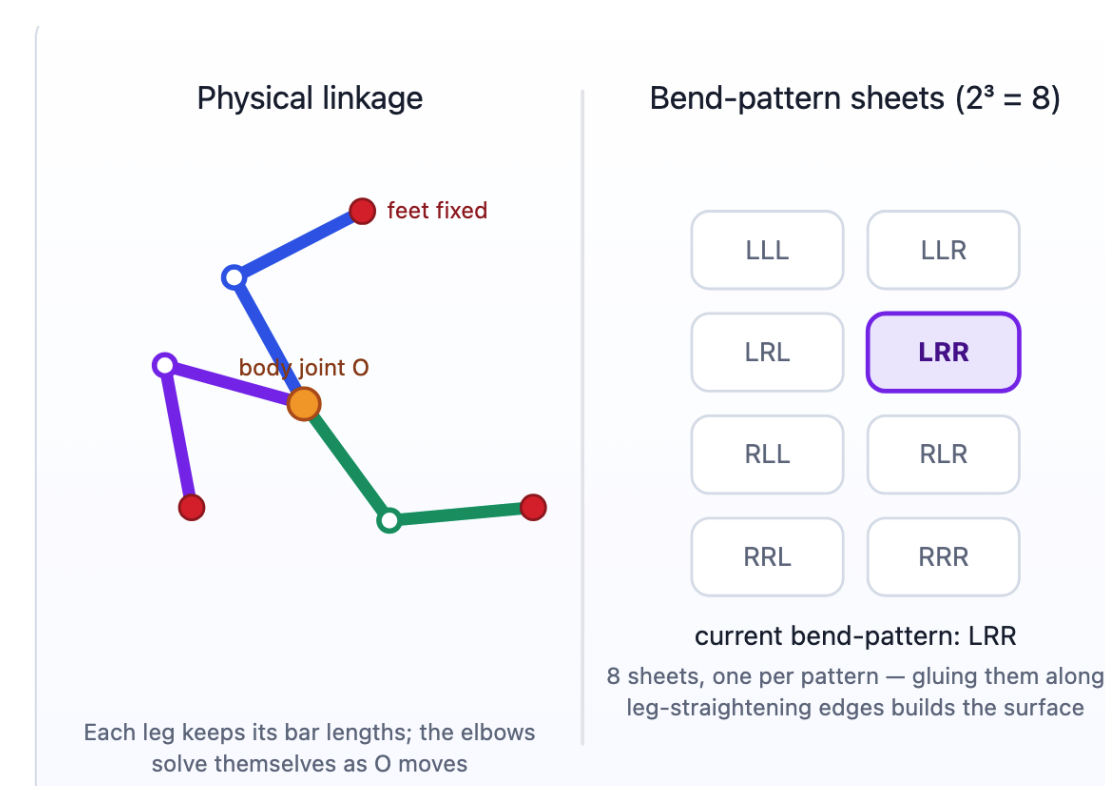
- **Configuration space: Torus**



- **Configuration space: Sphere.** Now, let's assume the two bars are equal length and can move freely. This would produce a configuration space in the form of a sphere, since the equal bar lengths would cover every point in the circle of a circles, which wouldn't be a torus, but a sphere.

Three-legged spider linkage -> Glued Surface

- **Three two-bar legs** meet at a shared, freely moving body joint O , while each leg's far end is pinned to a fixed foot. $2^3 = 8$ bend-patterns.

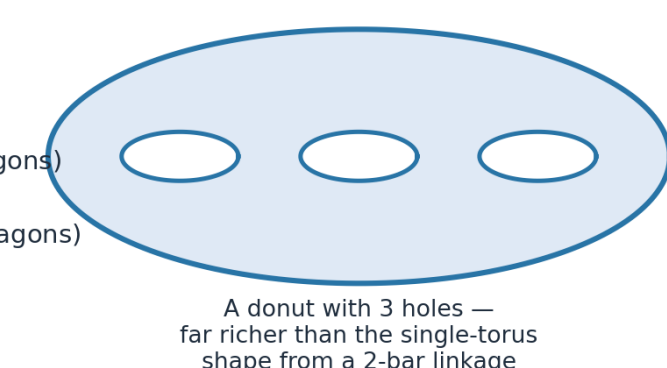


- **Configuration space:** Euler's formula

Euler characteristic of the spider's configuration space

$$\begin{aligned}\chi &= F - E + V \\ F &= 8 \text{ (the 8 glued hexagons)} \\ E &= \frac{8 \times 6}{2} = 24 \text{ (each edge shared by 2 hexagons)} \\ V &= \frac{8 \times 6}{4} = 12 \text{ (each vertex shared by 4 hexagons)} \\ \chi &= 8 - 24 + 12 = -4 \\ \text{Genus formula: } -2 - 2g = \chi &\rightarrow -g = \frac{2 - (-4)}{2} = 3\end{aligned}$$

Genus-3 surface



A donut with 3 holes — far richer than the single-torus shape from a 2-bar linkage

Acknowledgments:



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