Set A

2024 Illinois Middle School Math Olympiad
May 18th, 2024

1. Suppose that a, b and c are nonzero real numbers satisfying the two inequalities:

$$\begin{cases} a < b < c, \\ ab > bc > ca. \end{cases}$$

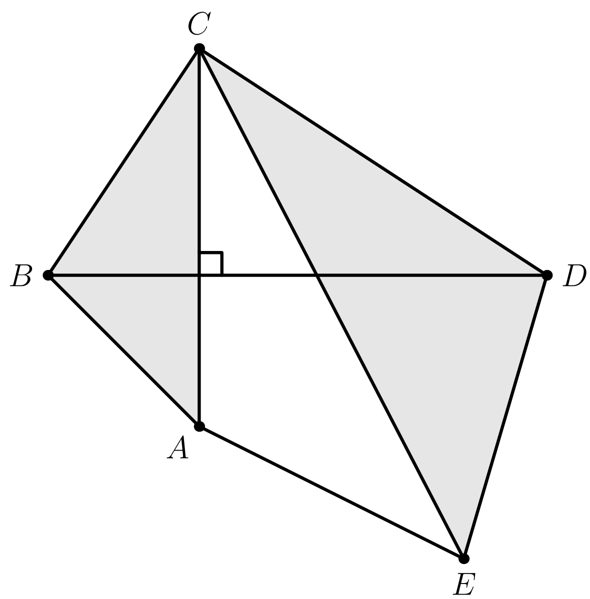
   For each variable, determine whether it is positive, negative, or if there is not enough information to tell.

2. Find all positive integers n that satisfy the following property: there exist two (not necessarily distinct) positive divisors of n which sum to another positive divisor of n.

3. The Fibonacci sequence is defined as follows: $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all positive integers n. Prove that the sum of any three or more consecutive numbers in the Fibonacci sequence is not in the Fibonacci sequence.

   (For example, 5, 8, 13 and 21 are the fifth through eighth Fibonacci numbers. Indeed, their sum, $5 + 8 + 13 + 21 = 47$, is not a Fibonacci number.)

4. In convex pentagon $ABCDE$, assume that $\triangle ABC \sim \triangle EDC$ and that lines AC and BD are perpendicular. Prove that lines AE and BC are perpendicular.



5. Let n be a positive integer. Suppose we randomly roll a fair six-sided die until we roll n consecutive even numbers (e.g. $2, 4, 4, 6, 2, \dots$) in a row. Find the expected number of rolls in terms of n.

Set B

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6. Show that the product of any four consecutive positive integers plus one is a multiple of 25 if and only if none of the four integers is a multiple of 5.

7. In $\triangle ABC$, let G and O be the centroid and circumcenter respectively. Given that $\angle AGO = 90^\circ$, determine all possible values of $\frac{GA}{BC}$.

   (The centroid of a triangle is the point where its three medians meet. The circumcenter of a triangle is the center of the circle passing through its three vertices.)

8. Professional basketball player Jordan Smith wants to improve his free throw percentage (number made divided by number of attempts) from 75% to above 85% by the end of the season. Show that if this occurs, then there must exist a time during the season when his free throw percentage was exactly 80%.

9. Farley notices that 6^3 can be written as the sum of *three* positive perfect cubes, namely $3^3 + 4^3 + 5^3$, and that 7^3 can be written as the sum of *four* positive perfect cubes, namely $1^3 + 1^3 + 5^3 + 6^3$. Help Farley prove that, in fact, for *any* integer $i \geq 3$, there exists a perfect cube that can be written as the sum of i positive perfect cubes.

10. Six circles are drawn such that no circle's center is inside any other circle. Show that no point in the plane lies inside all six circles.