

# 2026 Illinois Middle School Math Olympiad

Solutions — Sets A & B

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## Contents

<b>Set A</b>	<b>2</b>
Problem 1 . . . . .	2
Problem 2 . . . . .	2
Problem 3 . . . . .	3
Problem 4 . . . . .	4
Problem 5 . . . . .	4
<b>Set B</b>	<b>6</b>
Problem 6 . . . . .	6
Problem 7 . . . . .	7
Problem 8 . . . . .	7
Problem 9 . . . . .	8
Problem 10 . . . . .	9

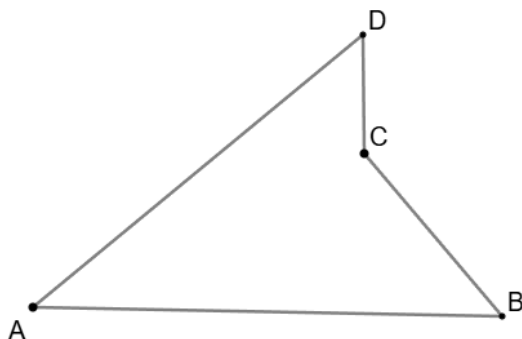
## Set A

### Problem 1

**Problem.** In concave quadrilateral  $ABCD$ ,  $\angle A = \angle B = \angle D = 45^\circ$ . Prove that  $AC = BD$ .

*Proof.* Because  $ABCD$  is a concave (non-convex) quadrilateral with  $\angle A = \angle B = \angle D = 45^\circ$ , the interior angle at  $C$  must make the angle sum equal  $360^\circ$ :

$$\angle A + \angle B + \angle C + \angle D = 360^\circ \implies \angle C = 360^\circ - 45^\circ - 45^\circ - 45^\circ = 225^\circ.$$



Extend  $DC$  to intersect  $AB$  at point  $E$ , then  $CE$  is perpendicular to  $AB$ . Right  $\triangle AEC$  is congruent to right  $\triangle DEB$  due to SAS. So  $AC = BD$ .

Using analytical geometry, we can assign coordinates to points  $A(0, 0)$ ,  $B(b, 0)$ ,  $C(c, b - c)$ ,  $D(c, c)$ , and it can be shown that  $AC^2 = BD^2 = c^2 + (b - c)^2$ .

□

### Problem 2

**Problem.** The function  $f(x)$  has domain and range  $[0, 1]$ . It satisfies  $f(0) = 0$ ,  $f(1) = 1$ , and

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$$

for all  $x, y \in [0, 1]$ . Prove that  $f(x) = x$  for all rational values of  $x \in [0, 1]$ .

*Proof.* With  $p$  and  $q$  being positive integers, considering three rational values  $\frac{p-1}{q}, \frac{p}{q}, \frac{p+1}{q} \in [0, 1]$ , we have  $f\left(\frac{p}{q}\right) = \frac{f\left(\frac{p-1}{q}\right) + f\left(\frac{p+1}{q}\right)}{2}$ , thus making  $f\left(\frac{p-1}{q}\right), f\left(\frac{p}{q}\right), f\left(\frac{p+1}{q}\right)$  an arithmetic sequence. Extending this to the function values of every rational with a denominator  $q$  (simplified or not), we see an arithmetic sequence of  $f(0), f\left(\frac{1}{q}\right), \dots, f\left(\frac{q-1}{q}\right), f(1)$  with starting value of 0 and ending value 1. Naturally, the common difference of the sequence is  $\frac{1}{q}$  and thus  $f\left(\frac{n}{q}\right) = \frac{n}{q}$  for any  $n$  such that  $0 \leq n \leq q$ . Since  $q$  can be any positive integer, we see  $f(x) = x$  for all rational  $x \in [0, 1]$ .

□

### Problem 3

**Problem.** Prove that every integer  $n \geq 2$  has a proper divisor  $d$  ( $d < n$ ) such that  $n + d$  has at least as many divisors as  $n$  has.

*Proof.* **Case 1:  $n$  is a multiple of 6. In other words,  $n$  has prime factors of 2 and 3.**

In this case, we set proper divisor  $d$  as  $\frac{n}{2}$  or  $\frac{n}{3}$  and we can show one of this would make  $n + d$  have at least as many divisors as  $n$ .

Easy to see that we are only concerned about the part of the prime factorization of  $n$  that has factors of 2 and 3, and let it be  $2^a 3^b$  with  $a$  and  $b$  positive integers, which yields  $(a + 1)(b + 1)$  divisors. The number of divisors of  $\frac{3n}{2}$  is  $a(b + 2)$ , and the number of divisors of  $\frac{4n}{3}$  is  $(a + 3)b$ , then we know at least one of them is larger than or equal to  $(a + 1)(b + 1)$  by simple algebra.

**Case 2: Else**

For any such  $n$ , there must exist a smallest prime divisor, call it  $p$ , and write

$$n = p^a m,$$

where  $a \geq 1$  and  $p \nmid m$ .

Now choose  $d$  as a proper divisor of  $n$  in this way:

$$d = \frac{n}{p} = p^{a-1} m.$$

And it follows that

$$n + d = p^a m + p^{a-1} m = p^{a-1} m(p + 1).$$

Since  $p$  is the smallest prime divisor of  $n$ , every prime divisor of  $m$  is greater than  $p$ . On the other hand, every prime divisor of  $p + 1$  is at most  $p + 1$ , and in particular cannot equal any prime divisor of  $m$ . Hence

$$\gcd(m, p + 1) = 1.$$

Also,

$$\gcd(p^{a-1}, p + 1) = 1.$$

Therefore the factors  $p^{a-1}$ ,  $m$ , and  $p + 1$  are pairwise coprime, so the divisor-counting function is multiplicative:

$$\tau(n + d) = \tau(p^{a-1}) \tau(m) \tau(p + 1).$$

Since

$$\tau(p^{a-1}) = a,$$

we obtain

$$\tau(n + d) = a \tau(m) \tau(p + 1).$$

On the other hand,

$$\tau(n) = \tau(p^a) \tau(m) = (a + 1) \tau(m).$$

Because  $p + 1 \geq 3$ , we have

$$\tau(p + 1) \geq 2.$$

Thus

$$\tau(n + d) \geq 2a\tau(m).$$

Finally, since  $a \geq 1$ ,

$$2a \geq a + 1.$$

Therefore

$$\tau(n + d) \geq (a + 1)\tau(m) = \tau(n).$$

Hence there exists a proper divisor  $d < n$  such that

$$\tau(n + d) \geq \tau(n).$$

□

## Problem 4

**Problem.** An infinite sequence  $\{a_n\}$  is defined by  $a_1 = c + 1$  and  $a_{n+1} = c + \frac{1}{a_n}$  for all positive integers  $n$ , where  $c$  is a constant and  $0 < c < 1$ . Prove that every term of the sequence is greater than 1.

*Proof.* We claim  $1 < a_n \leq c + 1$  for all  $n \geq 1$ , and we prove by Induction.

**Base case:**  $a_1 = c + 1$ , so obviously true.

**Inductive step:** Suppose  $1 < a_n \leq c + 1$ , then we show  $1 < a_{n+1} \leq c + 1$ .

Since  $a_n > 1$ , we have  $\frac{1}{a_n} < 1$ , and so

$$a_{n+1} = c + \frac{1}{a_n} \leq c + 1.$$

And so  $a_{n+1} = c + \frac{1}{a_n} \geq c + \frac{1}{c+1} > c + \frac{1-c^2}{c+1} = c + (1 - c) = 1$ , i.e.,  $a_{n+1} > 1$

□

## Problem 5

**Problem.** A pile of 100 chips is placed on a table. Two players  $A$  and  $B$  take turns removing chips, with  $A$  going first. On each turn:

- At least one chip must be removed.
- On the first move,  $A$  may remove at most 99 chips.
- On each subsequent move, the number of chips removed is at most twice the number removed on the previous move.

The player who removes the last chip wins.

Determine whether player  $A$  has a winning strategy. (A winning strategy for a player is a complete plan of action that ensures victory from a given position against every possible sequence of legal moves by the opponent.)

**Answer.** *Player  $A$  has a winning strategy.*

*Proof.* Let total number of chips be  $n$ . By experimenting on small values of  $n$ , we can find only when  $n = 2, 3, 5, 8, 13, \dots$  chips, player  $A$  has no winning strategy. Let these numbers (for which  $A$  has no winning strategy) form an increasing sequence  $n_i$ , we want to show the below claims by Induction:

- $n_{i+2} = n_{i+1} + n_i$  for all positive integer  $i$ , and  $n_1 = 2, n_2 = 3$ ;
- For a pile of  $n_i$  chips, player  $B$  can always remove the last chip, and the final action of  $B$  removes less than or equal to  $n_{i-1}$  chips ( $i > 1$ );
- If  $n_i < n < n_{i+1}$ , then player  $A$  can always remove the last chip, and the final action of  $A$  removes less than or equal to  $n_i$  chips.

**Induction Proof. (Base case can be easily verified)**

- When  $n = n_{i+1}$ , then 1) if  $A$  removes greater than or equal to  $n_{i-1}$  chips, there would be fewer than or equal to  $n_i$  chips left, and since  $n_i < 2n_{i-1}$ ,  $B$  simply removes all the rest of the chips; 2) if  $A$  removes less than  $n_{i-1}$  chips, then based on induction hypothesis,  $B$  can remove the  $n_{i-1}$ th chip, leaving exactly  $n_i$  chips on the table, with the final action of removing fewer than  $n_{i-2}$  chips, so now facing  $n_i$  chips,  $A$  can't take them all at one time because  $n_i > 2n_{i-2}$ , by induction hypothesis  $A$  will lose. Therefore  $A$  always loses.
- When  $n_i < n < n_{i+1}$ , then let  $k = n - n_i$ . If  $k < \frac{n_i}{2}$ , then  $A$  can directly remove  $k$  chips, leaving exactly  $n_i$  chips on the table, and since  $n_i > 2k$ ,  $B$  loses by induction hypothesis. If  $n_{i-2} < \frac{n_i}{2} \leq k < n_{i-1}$ , then by induction hypothesis,  $A$  can somehow remove  $k$ th chip with the final action of removing less than or equal to  $n_{i-2}$  chips, thus  $B$  loses.

Sequence  $n_i$  is the Fibonacci sequence without the first two terms. Since 100 is not in Fibonacci sequence, therefore, **player  $A$  has a winning strategy.**  $\square$

## Set B

### Problem 6

**Problem.** For positive integers  $a, b, c, d$  with  $a + b = cd$  and  $ab = c + d$ , find all possible values of  $a + b + c + d$ .

**Answer.** The possible values are  $\boxed{8, 11}$ .

*Proof.* Subtract the two equations from each other. Then we get the new equation

$$a + b - ab = cd - c - d.$$

This rearranges into the equation

$$ab - a - b + 1 + cd - c - d + 1 = 2,$$

which further simplifies into

$$(c - 1)(d - 1) + (a - 1)(b - 1) = 2.$$

We are given they are positive integers, so

$$a, b, c, d \geq 1.$$

Then this means that  $a - 1, b - 1, c - 1,$  and  $d - 1$  are all nonnegative integers, so pairwise products will also be nonnegative. Due to symmetry, we consider only two cases.

**Case 1:**

$$(a - 1)(b - 1) = (c - 1)(d - 1) = 1.$$

Then it must be true:

$$a - 1 = b - 1 = c - 1 = d - 1 = 1,$$

so

$$a = b = c = d = 2.$$

After checking original equations, we know this indeed works, so this case contribute a value of

$$a + b + c + d = 8.$$

**Case 2:**

WLOG, set  $(a - 1)(b - 1) = 0 \implies a = 1$  or  $b = 1$   
and  $(c - 1)(d - 1) = 2$ .

Focus on the second equation for now.  $(c - 1, d - 1)$  must be  $(1, 2)$  or  $(2, 1)$ . With symmetry, just set

$$(c - 1, d - 1) = (1, 2) \implies (c, d) = (2, 3).$$

This gives  $c + d = 5, cd = 6,$  and also back to the original equations we can easily get  $a + b = cd = 6,$  thus

$$a + b + c + d = 6 + 5 = 11.$$

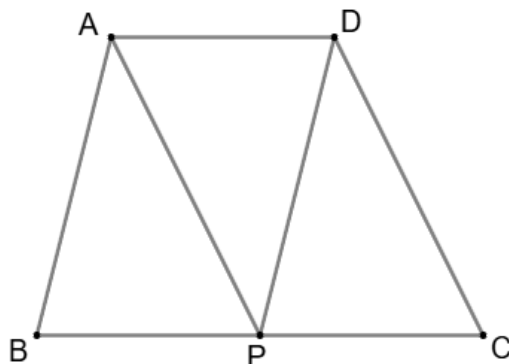
Therefore, we conclude that  $a + b + c + d$  can equal 8 or 11.

□

## Problem 7

**Problem.** Let  $ABCD$  be a trapezoid with  $AD \parallel BC$  and  $AD < BC$ . Suppose there exists a point  $P$  on segment  $BC$  such that triangles  $APB$ ,  $APD$ , and  $DPC$  all have equal perimeters. Prove that  $BC = 2AD$ .

*Proof.* First we draw a diagram.



Since triangles  $APB$  and  $APD$  have equal perimeters,  $AB + BP = AD + DP$ . Let  $BP > AD$ , so there exists a point  $P'$  on  $BP$  such that  $PP' = AD$ . By SAS, we see  $\triangle APP' \cong \triangle PAD$ , while  $AB + BP = AB + BP' + P'P > AP' + P'P = AD + DP$ . Contradiction leads to  $BP > AD$  not true, and likewise  $BP < AD$  not true. Then we can only have  $BP = AD$ . Similarly,  $CP = AD$ . Therefore,  $BC = 2AD$ . □

## Problem 8

**Problem.** Let  $n$  be a fixed positive integer. Suppose  $a, b, c$  are positive integers satisfying

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Determine, in terms of  $n$ , the largest possible value among  $a, b, c$ .

**Answer.** The largest possible value is  $\boxed{n^4 + 2n^3 + 2n^2 + n = n(n+1)(n^2+n+1)}$ .

*Proof.* WLOG, let  $a \leq b \leq c$ , and we just want to maximize  $c$ . First, we claim that  $n < a \leq 3n$ , because  $\frac{1}{a} \geq \frac{1}{b} \geq \frac{1}{c}$ , and  $\frac{1}{a} \geq \frac{1}{3n}$ . Then

$$\frac{1}{n} - \frac{1}{a} = \frac{1}{b} + \frac{1}{c}. \implies \frac{a-n}{an} = \frac{b+c}{bc}$$

Applying the same algebraic approach as in Simon's Favorite Factoring Trick, we get

$$((a-n)b - an)((a-n)c - an) = (an)^2$$

To maximize  $c$ , we have to let  $(a - n)c - an = (an)^2$ , with the right hand side being the largest factor of  $(an)^2$ , and this gives

$$c = \frac{a^2n^2 + an}{a - n} = n^2(a - n) + \frac{n^2(n^2 + 1)}{a - n} + 2n^3 + n$$

Since  $n$  is fixed, and  $1 \leq a - n \leq 2n$ , the only two possible values of  $a - n$  that could maximize  $c$  are just 1 or  $2n$ . Comparing the two resulting polynomials about  $n$ , we can conclude  $c$  has a greater (or equal) value when  $a - n = 1$  than  $2n$ . (Also notice when  $a - n = 2n$ , and now  $\frac{1}{3n} = \frac{1}{a} \leq \frac{1}{b} \leq \frac{1}{c}$ , we should simply have  $b = c = 3n$ , which is much smaller.)

Therefore, the largest possible value among  $a, b, c$  is

$$\boxed{n^4 + 2n^3 + 2n^2 + n = n(n + 1)(n^2 + n + 1)}.$$

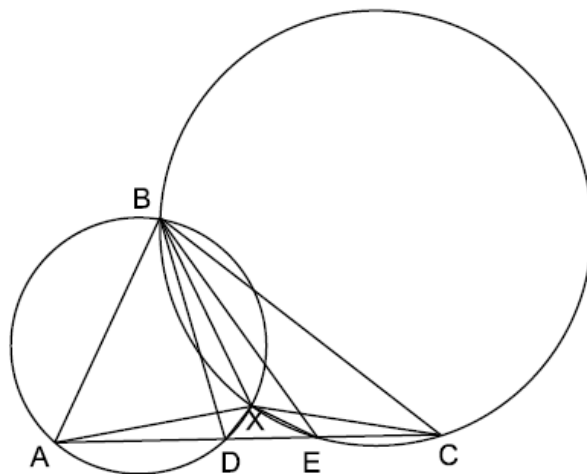
For  $n = 2$ :  $2 \cdot 3 \cdot 7 = 42$ .  $\checkmark$

Sidenote: Using Greedy algorithm, that is, first take  $\frac{1}{a}$  as close to  $\frac{1}{n}$  as possible, and then take  $\frac{1}{b}$  as close to  $\frac{1}{\frac{n}{n+1}}$  as possible, would give us the same final answer. But this approach needs to have a rigorous proof.  $\square$

## Problem 9

**Problem.** Let  $\triangle ABC$  be acute with  $AB < BC < AC$ . Points  $D$  and  $E$  lie on  $AC$  such that  $BD = DC$  and  $AE = EB$ . Suppose the circumcircles of  $\triangle ADB$  and  $\triangle BCE$  intersect again at a point  $X$ . Prove that  $AX = XC$ .

*Proof.* First we draw a diagram.



Let  $X'$  be the circumcenter of  $\triangle ABC$ , and let the circumcircles of  $\triangle ABX'$  and  $\triangle BX'C$  meet  $AC$  at  $D'$  and  $E'$ , respectively. Focus on  $ABX'D'$  first. By property of circumcenter, we know that  $\angle ABX' = \angle BAX'$ . Then  $\angle AX'B = 180 - 2\angle BAX'$ , so  $\angle AD'B = 180 - 2\angle BAX'$  as well. Thus  $\angle BD'C = 2\angle BAX'$ . But we also have that  $\angle BAX' = \angle BD'X'$  in cyclic

quadrilateral  $ABX'D'$ , so substituting gives  $2\angle BD'X' = \angle BD'C = \angle BD'X' + \angle CD'X'$ . It follows that  $\angle BD'X' = \angle CD'X'$ . Additionally,  $\angle D'BX' = \angle D'AX' = \angle D'CX'$ , so by AAS Congruence we find  $\triangle BD'X' \cong \triangle CD'X'$ . It thus follows that  $BD' = CD'$ , showing that  $D$  is the same point as  $D'$ . Similarly, we may show that  $\triangle AX'E' \cong \triangle BX'E'$ , so  $E$  is also the same point as  $E'$ . The two circles are given to intersect again at  $X$ , so  $X'$  is the same point as  $X$ . Therefore, because  $X$  is the circumcenter, then we conclude that  $AX = XC$ . □

## Problem 10

**Problem.** Let  $k$  be a positive integer. A  $2k \times 2k$  grid has exactly  $3k$  black squares. Prove there exist  $k$  rows and  $k$  columns such that every black square lies in at least one of these rows or columns.

*Proof.* We prove the result by a greedy selection.

**Step 1: Select  $k$  rows greedily.**

Pick the  $k$  rows containing the most black squares. We show the top  $k$  rows contain at least  $2k$  black squares.

Proof by Contradiction. If the top  $k$  rows contain at most  $2k - 1$  black squares, then the rest of  $k$  rows must contain at least  $k + 1$  black squares, which suggests by Pigeonhole Principle there exists at least one row that has at least two black squares. So the top  $k$  rows must have at least two black squares each, which gives a contradiction.

**Step 2: Cover remaining black squares with  $k$  columns.**

If we can find  $k$  columns that hold at least one black squares each, then that must mean we are having a total of at least  $3k$  black squares covered. If we can't find such  $k$  columns, that simply means all  $3k$  black squares have been covered by the new addition of even less than  $k$  columns.

**Now we are done!** □