2025 Illinois Middle School Math Olympiad

Solutions — Sets A & B

Written by Ethan Zhang, Haihui Niu, Benny Wang

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Set A

Problem 1

Problem. $A(x)$ and $B(x)$ are two quadratic functions with real coefficients. If $A(A(x)) = B(B(x))$ for all real x, prove that $A(x)$ and $B(x)$ are identical functions.

Proof. Let $A(x) = a_1x^2 + b_1x + c_1 \quad (a_1 \neq 0)$, and $B(x) = a_2x^2 + b_2x + c_2 \quad (a_2 \neq 0)$. Since $A(x)$ is a quadratic, therefore $A(A(x))$ is a polynomial of degree of 4 with coefficient of x^4 term being a_1^3, and similarly, $B(B(x))$'s x^4 term has a coefficient of a_2^3.

Because of identity, we see $a_1^3 = a_2^3$ with real a_1, a_2, so $a_1 = a_2$.

Then by comparing the coefficients of $A(A(x))$ and $B(B(x))'s$ x^3 terms, we get $b_1 = b_2$. We continue on this path, and we also get $c_1 = c_2$.

Now this suggests $A(A(x))$ is identical to $B(B(x))$. □

Problem 2

Problem. Let n be a positive integer and k an odd positive integer. Prove that $1+2+\cdots+n$ always divides $1^k + 2^k + \cdots + n^k$.

Proof. Let $S_k = 1^k + 2^k + \cdots + n^k$, and it's easy to write $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

For each $1 \leq j \leq n$, pair the term j^k with the term $(n+1-j)^k$. Since k is odd, notice the sum of the pair is:

$$j^k + (n+1-j)^k \equiv j^k + (-j)^k \equiv 0 \pmod{n+1})$$

- If n is even: pair j with $n+1-j$ for $j = 1, 2, \ldots, n/2$. Each pair sums to a multiple of $n+1$, so $(n+1) \mid S_k$. Also, by pairing j with $n-j$ for $j = 1, 2, \ldots, \frac{n}{2} - 1$, we can similarly show each pair's sum is a multiple of n, and then consider the two extra terms of $(\frac{n}{2})^k$ and n^k, we can obviously get $\frac{n}{2} \mid S_k$. Since $gcd(\frac{n}{2}, n+1) \leq gcd(n, n+1) = 1$, we have $\frac{n(n+1)}{2} \mid S_k$.
- If n is odd: we copy the above process for n being even and can show both $n \mid S_k$ and $\frac{n+1}{2} \mid S_k$, thus reaching the same conclusion of $\frac{n(n+1)}{2} \mid S_k$.

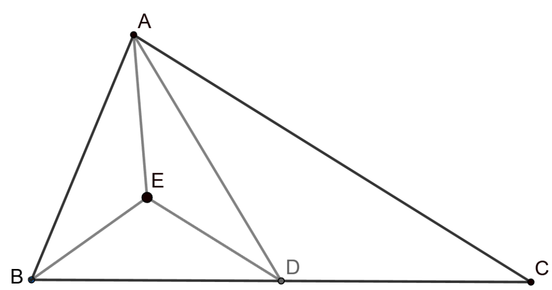
□

Problem 3

Problem. In $\triangle ABC$, point D is the midpoint of BC, and point E is inside $\triangle ABD$ such that $\triangle ABE \sim \triangle ACD$. Prove that $DE \parallel AC$.

Proof. Since $\triangle ABE \sim \triangle ACD$, there is a correspondence of vertices $A \leftrightarrow A$, $B \leftrightarrow C$, $E \leftrightarrow D$. This gives

$$AB \cdot AD = AC \cdot AE, \quad \angle BAE = \angle CAD, \quad \angle BAD = \angle CAE.$$



And it follows that $[BAD] = [CAE]$. On the other hand, since D is the midpoint of BC, $[BAD] = [CAD]$. Together, we have $[CAE] = [CAD]$, which proves $DE \parallel AC$ due to points D and E on the same side of AC.

□

Problem 4

Problem. John picks a positive real r; Joe picks the smallest positive integer n such that nr rounded to the nearest tenth is an integer. Once Joe chooses n, subject to this constraint, John gets n points. What is the greatest number of points that John can guarantee himself?

Answer. John can guarantee himself **19** points.

Proof. Let $\{x\}$ denote the fractional part of x, and obviously in this problem we only care about $\{r\}$. So WLOG, let $0 \le r < 1$. In this interval, John wants to pick a value for r such that he can guarantee as large an n as possible, so that $-0.05 \le nr - m < 0.05$ for some integer m.

After a few attempts, we find that if $r = 0.05$, then Joe would pick the smallest possible value of 19 to be n. We then go ahead and show 19 is the greatest number that can be guaranteed no matter what value of r is picked.

Assume for a fixed r, that n is the smallest positive integer such that $\{nr\}$ rounded to the nearest tenth is an integer. Then consider $\{r\}, \{2r\}, ..., \{(n-1)r\}$, and all these $n-1$ values must be within the interval $[0.05, 0.95)$, simply because n is the first value to make

$\{nr\}$ not in this interval. If $n > 19$, then there must exist two distinct positive integers j and k (from 1 to $n-1$), such that

$$0 \leq \{jr\} - \{kr\} < \frac{0.95 - 0.05}{19 - 1} = 0.05,$$

This implies $0 \leq \{(j-k) \cdot r\} < 0.05$, suggesting $|j-k| \cdot r$ rounded to the nearest tenth is an integer. And since either $j-k$ or $k-j$ is a positive integer less than n, this simply can't happen. □

Problem 5

Problem. Inside a tour bus, there are several tourists. An interesting property holds: For every group of 10 tourists chosen from the bus, there is always exactly one person who is a mutual friend of all 10. What is the maximum number of tourists that could be on the bus? Notice friendship is mutual (if A is a friend of B, then B is a friend of A), and no one is his/her own friend.

Answer. The maximum number of tourists is **11**.

Proof. If every pair of 11 tourists are friends to each other, then obviously this group satisfies the requirement. We'll show this is the only scenario that satisfies the requirement.

First, we try to show no matter how many people such a satisfying group of tourists could have, there always exist 11 people as a subgroup such that inside the subgroup, everyone is friend to everyone else. Let's start with tourist #1, and obviously he(she) must have at least one friend, then call that friend #2. Now #1 and #2 must be involved in a group of 10 people, and they must have a common friend, so call this friend #3. Keep this process going, we'll have tourist #1, #2, ..., #10 all having a common friend #11, and of course, all 11 are friends to each other.

Second, we try to show any new tourist outside the existing 11 (call it #12) would not satisfy the requirement.

- Claim 1: #12 can't have two or more friends among tourists #1 to #11. If the opposite is true, then WLOG let two friends (of all possible friends of #12) be #1 and #2. Here, #1 becomes a common friend for group #3 to #12, and so is #2. Since this contradicts with the requirement of the uniqueness of the common friend, this simply would not be true.
- Claim 2: #12 can't have one friend or no friend among tourists #1 to #11. WLOG, assume #12 is not a friend to #2, #3, ..., #11. Now consider the group of 10 tourists #1, #4, #5, ..., #11 and #12, they must have a common friend, and call it #n. This tourist #n is certainly not #2, or #3, so #n must be a new tourist outside of #1 to #12. But the fact is #n has been made to have more than two friends among #1 to #11, which contradicts with Claim 1.

Notice we can generalize this problem. Replace 10 by variable m, then out answer is $m + 1$. □

Set B

Problem 6

Problem. Find all integer solutions to the 3-variable equation: $2025^x - 2025^y = z^2$.

Proof. Let $N = 2025 = 45^2 = 3^4 \cdot 5^2$.

Case 1: $x = y$**.** Then $2025^x - 2025^y = 0 = z^2$, so $z = 0$. This gives the family of solutions $(x, y, z) = (n, n, 0)$ for any integer n.

Case 2: $x > y$ **(and** $y \geq 0$**).** Factor: $2025^y(2025^{x-y} - 1) = z^2$. Since $2025 = 45^2$ is a perfect square, $2025^y = (45^y)^2$ is a perfect square. So $(45^y)^2(2025^{x-y} - 1) = z^2$, meaning $2025^{x-y} - 1$ must be a perfect square. Let $m = x - y \geq 1$ and write $2025^m - 1 = t^2$. Then $(45^m)^2 - t^2 = 1$, so $(45^m - t)(45^m + t) = 1$. Since $45^m > 0$ and $t \geq 0$: $45^m - t = 1$ and $45^m + t = 1$, giving $t = 0$ and $45^m = 1$, so $m = 0$. Contradiction with $m \geq 1$.

Hence no solutions with $x > y \geq 0$.

Case 3: $x > y$ **with** $y < 0$**.** If x, y are integers (not necessarily non-negative), then 2025^x and 2025^y are rational but not integers when negative. Since z must be an integer, z^2 is a non-negative integer, so $2025^x - 2025^y$ must be a non-negative integer. This restricts $x, y \geq 0$ if they are integers (as 2025^y is not an integer for $y < 0$). Hence we must have $x, y \geq 0$.

Case 4: $x < y$**.** Then $2025^x - 2025^y < 0$, but $z^2 \geq 0$. No solutions.

Conclusion. All integer solutions are

$$\boxed{(x, y, z) = (n, n, 0) \text{ for any integer } n \geq 0.}$$

□

Problem 7

Problem. a, b, c, d are positive integers such that $ab = cd$. Prove that the sum $a + b + c + d$ cannot be a prime number.

Proof. Since $ab = cd$, we have $\frac{a}{c} = \frac{d}{b}$. Write this ratio in lowest terms as $\frac{p}{q}$ where $\gcd(p, q) = 1$ and p, q are positive integers. Then:

$$a = pk, \quad c = qk, \quad d = pm, \quad b = qm$$

for some positive integers k, m. Now compute:

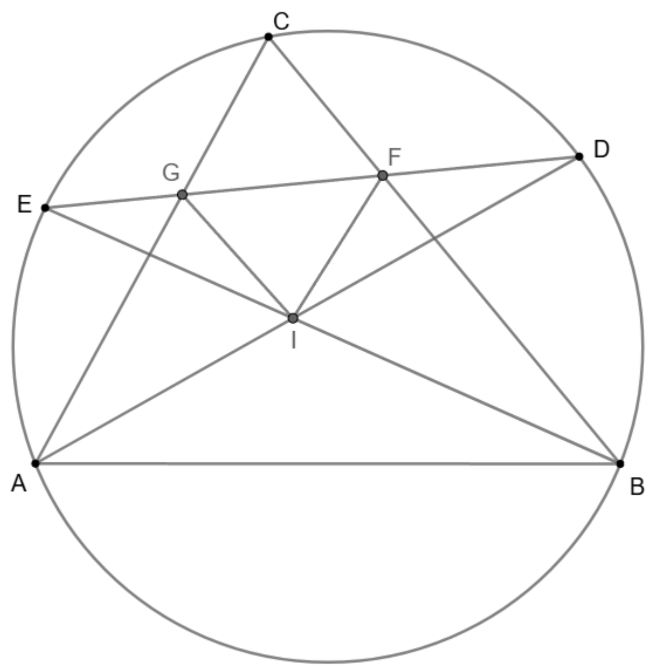
$$a + b + c + d = pk + qm + qk + pm = (p + q)(k + m).$$

Since a, b, c, d are positive integers, we have $p, q, k, m \geq 1$, so $p + q \geq 2$ and $k + m \geq 2$. Thus $a + b + c + d = (p + q)(k + m)$ is a product of two integers each at least 2, hence composite. Therefore $a + b + c + d$ cannot be prime. □

Problem 8

Problem. Let point I be the incenter of $\triangle ABC$. Lines AI and BI meet the circumcircle of $\triangle ABC$ again at points D and E, respectively. Suppose DE meets BC and AC at points F and G, respectively. Prove that quadrilateral $IGCF$ is a rhombus.

Proof. Draw the diagram first.



Since $\angle DIC = \angle ACI + \angle CAI = \frac{1}{2}(\angle ACB + \angle CAB)$, and $\angle DCI = \angle BCD + \angle BCI = \frac{1}{2}(\angle CAB + \angle ACB)$, we have $\angle DIC = \angle DCI$, and thus $DC = DI$. Similarly $EC = EI$. This implies DE perpendicularly bisects CI. And because points F and G are on DE, we see $FC = FI$ and $GC = GI$.

On the other hand, $\angle CGF = \angle CAD + \angle ADE = \frac{1}{2}(\angle CAB + \angle ABC)$, and $\angle CFG = \angle DEB + \angle CBE = \frac{1}{2}(\angle CAB + \angle ABC)$, so $\angle CGF = \angle CFG$, $GC = FC$.

Combining everything, we get $FC = FI = GC = GI$. Therefore, $IGCF$ is a rhombus. $\square$

Problem 9

Problem. Let $n > 3$ be a positive integer, and let $a_0 < a_1 < \cdots < a_n$ be positive integers with $a_n \leq 2n - 3$. Prove there exist five distinct indices $p, q, r, s, t \in \{0, 1, \ldots, n\}$ such that $a_p + a_q = a_r + a_s = a_t$.

Proof. We have $n+1$ strictly increasing positive integers $a_0 < a_1 < \cdots < a_n$ with $a_n \leq 2n-3$. We'll first show when $a_n = 2n - 3$, the statement is true. And after that, we'll show when $a_n < 2n - 3$, which makes the sequence more dense, the statement is also true.

If $a_n = 2n - 3$, then there are n distinct positive integers in $a_0, a_1, ..., a_{n-1}$ ranging from 1 to $2n - 4$. Meanwhile, it's easy to see there are possibly $n - 2$ positive integer pairs (two distinct numbers/not ordered) that sum to $2n - 3$. With Pigeonhole Principle, we know $a_0, a_1, ..., a_{n-1}$ would form at least two pairs of positive integers that sum to $2n - 3$, which makes $a_p + a_q = a_r + a_s = 2n - 3 = a_t = a_n$ true.

If $a_n = 2n-4$, then there are n distinct positive integers in $a_0, a_1, ..., a_{n-1}$ ranging from 1 to $2n - 5$. Meanwhile, comparing to the previous case, there are $n - 3$ positive integer pairs that sum to $2n - 4$, with $(n - 2) + (n - 2) = 2n - 4$ not counted in due to two identical integers in the pair. Even though one of the n distinct integers in the sequence could be $n - 2$, that'd still give us at least $n - 1$ integers to be distributed into $n - 3$ pairs, which guarantees at least two pairs being formed to make $a_p + a_q = a_r + a_s = 2n - 4 = a_t = a_n$ true.

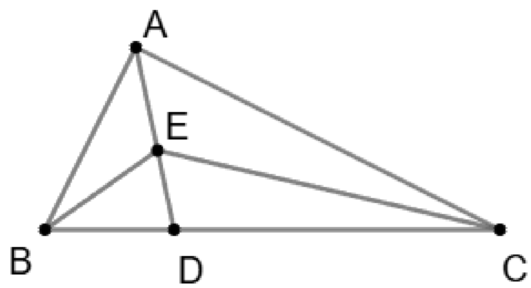
a_n can only be either even or odd, and if we decrease a_n further, it would make the sequence denser. Hence, we say that Pigeonhole Principle could still be used to show the statement true.

Let's check the initial case of $n = 4$ (which we probably should do at the beginning). The sequence would be just 1,2,3,4,5 and obviously we have $1 + 4 = 2 + 3 = 5$

□

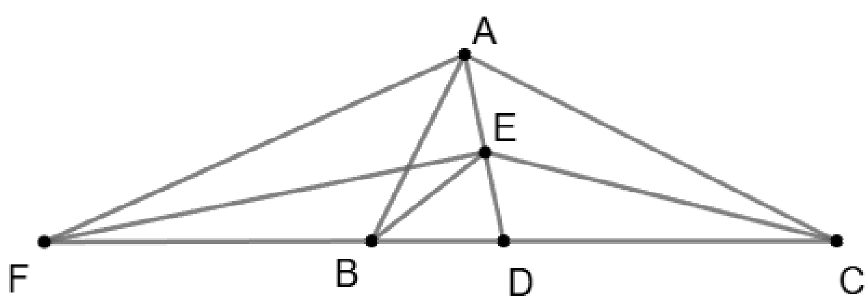
Problem 10

Problem. Let $\triangle ABC$ be right-angled at A. Point D lies on BC, and point E is the midpoint of AD. Suppose $\angle BED = \angle CED$. Prove $\angle BDA = 2\angle BAD$.



Proof. See below diagram. Extend ray DB to point F so that $\angle FAB = \angle BAD$. Now we only need to show $\angle FDA = \angle FAD$, which is the same thing as showing $\triangle FAD$ is isosceles. Since E is the midpoint of AD, as long as we show $FE \perp AD$, we'll be good. And we use the condition of DE bisecting $\angle BEC$, we know it'd be true that FE is the exterior angle bisector, which we focus on how to prove.

We start with AB bisecting $\angle FAD$, and it follows that $\frac{BF}{BD} = \frac{AF}{AD}$. On the other hand, since $\angle BAC = 90°$, we have CA bisecting the exterior angle of $\angle FAD$, then $\frac{AF}{AD} = \frac{CF}{CD}$. Therefore, $\frac{BF}{BD} = \frac{CF}{CD}$, the same thing as $\frac{BD}{CD} = \frac{BF}{CF}$. In addition, DE bisects $\angle BEC$, which gives $\frac{BD}{CD} = \frac{BE}{CE}$. The two previous equalities imply $\frac{BF}{CF} = \frac{BE}{CE}$, which suggests FE bisects the exterior angle of $\angle BEC$. Thus $FE \perp AD$, and with $AE = DE$, easy to see $\angle FDA = \angle FAD = 2\angle BAD$.



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