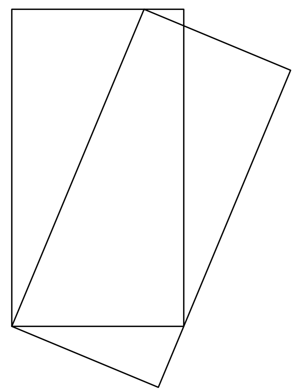
Sample Set

Illinois Middle School Math Olympiad

1. Timothy is taking a ten problem math contest where each problem is worth an integer score out of 10 points. On each of the first nine problems, Timothy earns a higher score on that problem than anybody else. Find, with proof, the lowest score he could get on the last problem which would still guarantee that his overall score is higher than anybody else's score.

2. In the diagram below, two rectangles share a vertex, and each rectangle has another vertex lying on the other rectangle's perimeter. Prove that the two rectangles have equal area.



3. [1]Alice chooses a real constant C, possibly negative. Given that

$$x^2 + Cxy + y^2 \geq 0$$

for all (possibly negative) real numbers x and y, find all possible constants C that Alice could have chosen.

4. A positive integer is *semi-prime* if it can be written as the product of two prime numbers. Prove that no semi-prime can be written as the sum of two consecutive prime numbers.

5. [2]A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

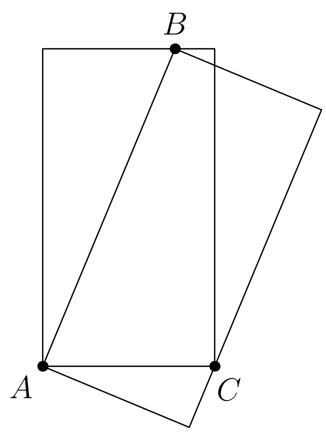
[1]Due to Jason Lee.
[2]2017 IMO Shortlist, C1.

Solutions

1. The answer is $\boxed{2}$.

   Timothy scores at least 1 point higher than the next competitor for each of the first nine problems, so overall, he is at least 9 points ahead of the second-best competitor before the last question. If Timothy scores a 1 out of 10 on the last question, the second-best competitor could score a 10 out of 10, finishing in a tie. However, if Timothy scores a 2 out of 10, his competitor would gain at most 8 more points than Timothy, not enough to close the 9-point gap.

2. Label points as follows:

   Clearly, $\triangle ABC$ is half the area of each rectangle, so the two rectangles' areas are the same.

3. The answer is that $-2 \leq C \leq 2$.

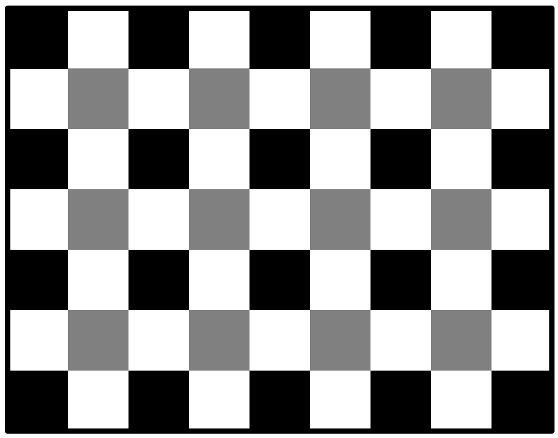
   Solution 1: Firstly, if C was greater than 2, we could select $x = 1$ and $y = -1$; then, we would have
   $$x^2 + Cxy + y^2 = 1 - C + 1 = 2 - C$$
   which would be less than 0. Similarly, if C was less than -2, we could select $x = y = 1$; then, we would have
   $$x^2 + Cxy + y^2 = 1 + C + 1 = 2 + C < 0.$$

   On the other hand, if $-2 \leq C \leq 2$, we know that $x^2 + Cxy + y^2$ must lie somewhere between $x^2 - 2xy + y^2$ and $x^2 + 2xy + y^2$. The former expression factors as $(x - y)^2$ and the latter expression factors as $(x+y)^2$. Since both expressions are perfect squares, and hence nonnegative for all x and y, it follows that $x^2 + Cxy + y^2$ is also nonnegative for all x and y.

   Solution 2: Plugging in $y = 0$ gives us that $x^2 \geq 0$ which is always true regardless of C. If $y \neq 0$, it's safe to divide the inequality by y^2:
   $$\left(\frac{x}{y}\right)^2 + C\left(\frac{x}{y}\right) + 1 \geq 0.$$
   In other words, the quadratic $(\frac{x}{y})^2 + C(\frac{x}{y}) + 1$ has either 0 or 1 roots. This is equivalent to the discriminant of the quadratic being nonpositive:
   $$C^2 - 4 \leq 0 \implies -2 \leq C \leq 2.$$

4. For the sake of contradiction, assume that there exists two consecutive prime numbers p and q which sum to a semi-prime. If $(p, q) = (2, 3)$, then $p + q = 5$ which is not semi-prime. Otherwise, if we select a greater pair of consecutive primes, p and q must both be odd, so $p+q$ is even. So, 2 is one of the prime divisors of the semi-prime $p + q$. Let r be the other prime divisor; then, we have
$$p + q = 2r \implies r = \frac{p+q}{2},$$
so r lies in between p and q. But since p and q are supposedly consecutive primes, this is a contradiction. Therefore, no such p and q exist.

5. Color the unit squares of $\mathcal{R}$ like below:



If we partition $\mathcal{R}$ into smaller rectangles, each small rectangle's corners are either (a) all white, in which case *more than half* of the squares inside the rectangle are white, (b) all black, in which case *less than half* of the squares inside the rectangles are white, or (c) all gray, in which case *less than half* of the squares inside the rectangles are white.

Since the corners of $\mathcal{R}$ are black, less than half of the squares in $\mathcal{R}$ are white. So, in our partition, at least one rectangle will be of type (b) or type (c).

If a rectangle's corners are all black, then the distance from each of its sides to the sides of $\mathcal{R}$ are all even; if a rectangle's corners are all gray, the distance from each of its sides to the sides of $\mathcal{R}$ are all odd.

So, a type (b) or type (c) rectangle, which must exist in our partition, satisfies the condition in the problem.