2024 Illinois Middle School Math Olympiad

Official Solutions Manual

Benny Wang, Max Gurinas and Haihui Niu

This is the official solutions manual for the 2024 IMSMO, an in-person contest held on May 18th, 2024. The problems on the contest are also included in this document.

Contents

Problems

1. Suppose that a, b and c are nonzero real numbers satisfying the two inequalities:

$$\begin{cases} a < b < c, \\ ab > bc > ca. \end{cases}$$

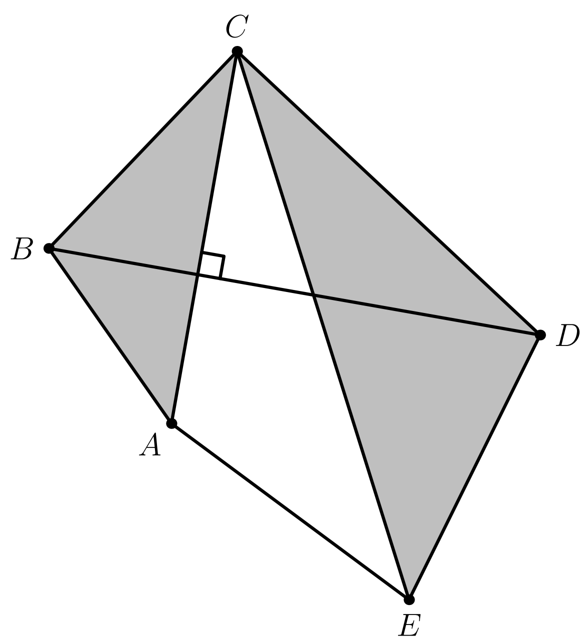
For each variable, determine whether it is positive, negative, or if there is not enough information to tell.

2. Find all positive integers n that satisfy the following property: there exist two (not necessarily distinct) positive divisors of n which sum to another positive divisor of n.

3. The Fibonacci sequence is defined as follows: $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all positive integers n. Prove that the sum of any three or more consecutive numbers in the Fibonacci sequence is not in the Fibonacci sequence.

   (For example, 5, 8, 13 and 21 are the fifth through eighth Fibonacci numbers. Indeed, their sum, $5 + 8 + 13 + 21 = 47$, is not a Fibonacci number.)

4. In convex pentagon $ABCDE$, assume that $\triangle ABC \sim \triangle EDC$ and that lines AC and BD are perpendicular. Prove that lines AE and BC are perpendicular.



5. Let n be a positive integer. Suppose we randomly roll a fair six-sided die until we roll n consecutive even numbers (e.g. $2, 4, 4, 6, 2, \ldots$) in a row. Find the expected number of rolls in terms of n.

6. Show that the product of any four consecutive positive integers plus one is a multiple of 25 if and only if none of the four integers is a multiple of 5.

7. In $\triangle ABC$, let G and O be the centroid and circumcenter respectively. Given that $\angle AGO = 90^\circ$, determine all possible values of $\frac{GA}{BC}$.

   (The centroid of a triangle is the point where its three medians meet. The circumcenter of a triangle is the center of the circle passing through its three vertices.)

8. Professional basketball player Jordan Smith wants to improve his free throw percentage (number made divided by number of attempts) from 75% to above 85% by the end of the season. Show that if this occurs, then there must exist a time during the season when his free throw percentage was exactly 80%.

9. Farley notices that 6^3 can be written as the sum of *three* positive perfect cubes, namely $3^3 + 4^3 + 5^3$, and that 7^3 can be written as the sum of *four* positive perfect cubes, namely $1^3 + 1^3 + 5^3 + 6^3$. Help Farley prove that, in fact, for *any* integer $i \geq 3$, there exists a perfect cube that can be written as the sum of i positive perfect cubes.

10. Six circles are drawn such that no circle's center is inside any other circle. Show that no point in the plane lies inside all six circles.

Solutions

Problem 1:

Suppose that a, b and c are nonzero real numbers satisfying the two inequalities:

$$\begin{cases} a < b < c, \\ ab > bc > ca. \end{cases}$$

For each variable, determine whether it is positive, negative, or if there is not enough information to tell.

We determine the signs of the three variables, step-by-step:

- To determine the sign of c: since $a < b$, we have that $b - a > 0$; in other words, $b - a$ is positive. Also, since $bc > ca$, we have $c(b - a) > 0$. Since we already know $b - a$ is positive, it follows that $\boxed{c \text{ is positive}}$.
- To determine the sign of b: since $a < c$, we have that $c - a > 0$; in other words, $c - a$ is positive. Also, since $ab > bc$, we have that $b(c - a) < 0$. Since we already know that $c - a$ is positive, it follows that $\boxed{b \text{ is negative}}$.
- To determine the sign of a: we already know b is negative, and since $a < b$, it follows that $\boxed{a \text{ is negative}}$. (Alternatively, we could repeat similar reasoning as we used to determine the sign of b and c.)

Remark: An example of a triple (a, b, c) that satisfies the conditions in the problems is $(-2, -1, 1)$. For a correct solution, including a valid example is *not required*.

Problem 2:

Find all positive integers n that satisfy the following property: there exist two (not necessarily distinct) positive divisors of n which sum to another positive divisor of n.

The answer is $\boxed{\text{even integers } n}$. As with any “find all” problem, there are two parts to this problem: showing that any even number satisfies the property and showing that any odd number fails the property.

- Proving that evens work: if n is even, then 2 divides n by definition. Obviously, 1 divides n, and since $1 + 1 = 2$, n satisfies the property.
- Proving that odds fail: if n is odd, all of its divisors are odd. So if we take two divisors of n and add them, we will always get an even number. This cannot be a divisor of n, so we are guaranteed that the condition fails for all odd n.

Problem 3:

The Fibonacci sequence is defined as follows: $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all positive integers n. Prove that the sum of any three or more consecutive numbers in the Fibonacci sequence is not in the Fibonacci sequence.

Lemma: for any integer $i \geq 3$, we have

$$F_1 + F_2 + \cdots + F_{i-2} = F_i - 1.$$

Proof: We will prove this with induction on i.

- Base case: when $i = 3$, our lemma is indeed true: $F_1 = F_3 - 1$ since $F_1 = 1$ and $F_3 = 2$.
- Inductive step: we assume that our lemma is true for some value $k - 1 \geq 3$, and we will use this to show that our lemma is also true for k. Our assumption is that

  $$F_1 + F_2 + \cdots + F_{k-3} = F_{k-1} - 1.$$

  If the above is true, we can add F_{k-2} to both sides:

  $$F_1 + F_2 + \cdots + F_{k-3} + F_{k-2} = F_{k-1} + F_{k-2} - 1 = F_k - 1.$$

  The above equation is what we wanted to show in our inductive step, so the proof of the lemma is complete. $\square$

Now, we will show what the problem asks us: if i and j are positive integers and $i < j - 1$, then the sum of the consecutive Fibonacci numbers

$$F_i + F_{i+1} + \cdots + F_j$$

is not a Fibonacci number. (The $i < j - 1$ condition is equivalent to ensuring our sum contains at least 3 numbers, as the problem states.)

This is true since

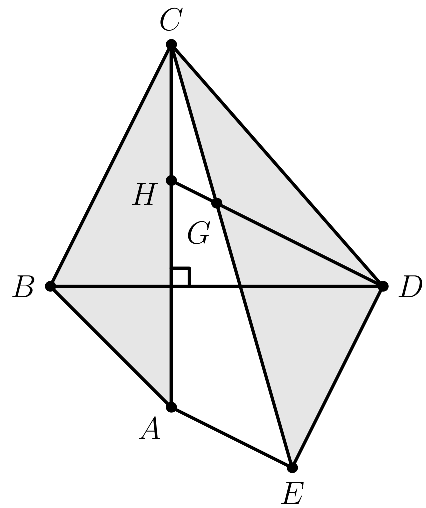
$$\begin{gathered} F_{j+1} = F_{j-1} + F_j \\ < F_i + F_{i+1} + \cdots + F_j \leq \\ F_1 + F_2 + \cdots + F_j = F_{j+2} - 1. \end{gathered}$$

In other words, the sum $F_i + F_{i+1} + \cdots + F_j$ lies strictly between the consecutive Fibonacci numbers F_{j+1} and F_{j+2}, so it is certainly not a Fibonacci number.

Problem 4:

In convex pentagon $ABCDE$, assume that $\triangle ABC \sim \triangle EDC$ and that lines AC and BD are perpendicular. Prove that lines AE and BC are perpendicular.

Solution A:



Let H be the orthocenter of $\triangle BCD$, and let G be the intersection of lines DH and CE.

Claim: We have $\triangle CBH \sim \triangle CDG$.

Proof: We have

$$\angle CBH = 90^\circ - \angle BCD = \angle CDG,$$

since H is the orthocenter. (More specifically, if we extend line BH to meet line CD at X, $\triangle BXC$ is a right triangle, implying that $\angle CBH$ and $\angle BCD$ are complementary. Similar reasoning shows that $\angle BCD$ and $\angle CDG$ are complementary.)

We also have

$$\angle HCB = \angle GCD,$$

since the problem gives us $\triangle ABC \sim \triangle EDC$.

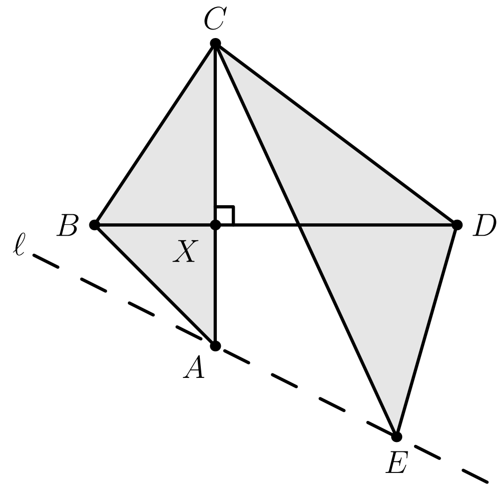
Therefore, triangles CBH and CDG are similar by angle-angle similarity. $\square$

From $\triangle CBH \sim \triangle CDG$ and $\triangle ABC \sim \triangle EDC$, it follows that

$$\left(\frac{CH}{CB}\right) = \left(\frac{CG}{CD}\right)$$
$$\left(\frac{CH}{CB}\right)\left(\frac{CB}{CA}\right) = \left(\frac{CG}{CD}\right)\left(\frac{CD}{CE}\right)$$
$$\left(\frac{CH}{CA}\right) = \left(\frac{CG}{CE}\right),$$

which implies that $\overline{HG} \parallel \overline{AE}$. Since H is the orthocenter of $\triangle BCD$, by definition, $\overline{HG}$ is perpendicular to $\overline{BC}$. Therefore, $\overline{AE}$ is perpendicular to $\overline{BC}$ as well.

Solution B:



Let X be the intersection of $\overline{AC}$ and $\overline{BD}$. We will show this problem statement is true even when $ABCDE$ is concave.

Let $\mathcal{T}$ represent the following two-step transformation:

1. A counterclockwise rotation about C by $\angle BCA$, followed by
2. A dilation about C by a factor of $\frac{CA}{CB}$.

Because of the similarity condition, it follows that $\mathcal{T}(D) = E$.

Note that $\mathcal{T}$ is only defined only based on A, B, C. Suppose we fix points A, B, C while varying D along line BX; define E as $\mathcal{T}(D)$. Then, E will also vary along a fixed line ℓ; in particular, E always lies on $\mathcal{T}(\overleftrightarrow{BX})$.

To determine exactly what the line ℓ is, we just need to find two possible locations for E, since those two points will determine line ℓ:

- If we set D as the reflection of B over X, then by symmetry, $E = A$. Therefore, A lies on ℓ.
- If we set D to coincide with point B, then E is the reflection of A over line BC.

Since ℓ passes through A and the reflection of A over line BC, it follows that ℓ is perpendicular to line BC. Since E and A both lie on ℓ, it follows that lines EA and BC are perpendicular.

Problem 5:

Let n be a positive integer. Suppose we randomly roll a fair six-sided die until we roll n consecutive even numbers (e.g. $2, 4, 4, 6, 2, \ldots$) in a row. Find the expected number of rolls in terms of n.

Rolling an even number happens with probability $\frac{1}{2}$, so we will replace the die in the problem with a coin that lands heads or tails – we want to know how many flips it would take (on average) until we got n heads in a row.

Firstly, we must show that this average value is finite. Consider an alternate situation, where we will only stop flipping if the previous n coins were all heads and if the number of total flips we've made is a multiple of n. The probability that any chunk of n coins is all heads is $\frac{1}{2^n}$, and so by a well known formula, on average, it takes $n \cdot 2^n$ flips to stop flipping. The expected value in the problem should be less than the expected value in this alternate situation, so it is certainly finite.

Now, let e be the expected number of flips it would take us. We consider n scenarios:

- When we start flipping, there is a $\frac{1}{2}$ chance that our first flip is a tails. After that one tail, we start fresh again, and on average it will take e more flips to get a string of n heads, for a total of $1 + e$ flips on average.
- There is a $\frac{1}{4}$ chance that our first two flips are HT. After these two flips, our streak is ruined – we start fresh and it will take e more flips to get a string of n heads, for a total of $2 + e$ flips on average.
- There is a $\frac{1}{8}$ chance that our first three flips are HHT. After these three flips, our streak is ruined – we start fresh and it will take e more flips to get a string of n heads, for a total of $3 + e$ flips on average.
- The pattern continues...
- There is a $\frac{1}{2^n}$ chance that our first n flips are $n - 1$ heads followed by a tail. After these flips, our streak is ruined – we start fresh and it will take e more flips to get a string of n heads, for a total of $n + e$ flips on average.
- There is a $\frac{1}{2^n}$ chance that our first n flips are all heads, in which case we're done and we took n flips.

Taking the weighted average of the items in this list should give us e by definition. So, we have

$$\begin{aligned}
e &= \frac{e+1}{2} + \frac{e+2}{4} + \cdots + \frac{e+n}{2^n} + \frac{n}{2^n} \\
\frac{e}{2^n} &= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n} + \frac{n}{2^n} \\
e &= 1 \cdot 2^{n-1} + 2 \cdot 2^{n-2} + \cdots + n \cdot 2^0 + n \\
&= \left(2^0\right) + \left(2^0 + 2^1\right) + \cdots + \left(2^0 + 2^1 + \cdots + 2^{n-1}\right) + n \\
&= \left(2^1 - 1\right) + \left(2^2 - 1\right) + \cdots + \left(2^n - 1\right) + n \\
&= 2^1 + 2^2 + \cdots + 2^n \\
&= \boxed{2^{n+1} - 2}.
\end{aligned}$$

Problem 6:

Show that the product of any four consecutive positive integers plus one is a multiple of 25 if and only if none of the four integers is a multiple of 5.

Let n be an odd integer. We can represent our four consecutive integers as

$$\frac{n-3}{2}, \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}.$$

These integers are all not multiples of 5 if and only if n is a multiple of 5.

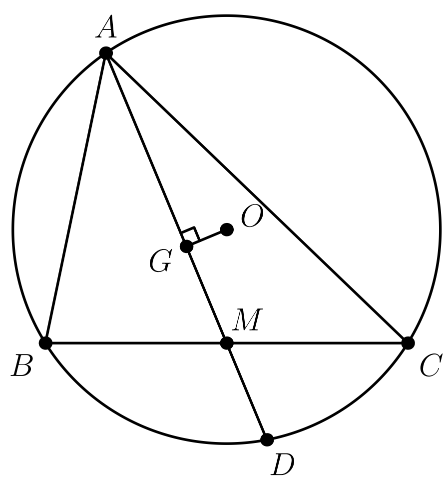
Their product plus one is equal to

$$\frac{(n-3)(n-1)(n+1)(n+3)}{16} + 1 = \frac{(n^2-1)(n^2-9)+16}{16} = \frac{n^4-10n^2+25}{16} = \left(\frac{n^2-5}{4}\right)^2,$$

which is also a multiple of 5 if and only if n is a multiple of 5. Therefore, the two assertions in the problem are equivalent.

Problem 7:

In $\triangle ABC$, let G and O be the centroid and circumcenter respectively. Given that $\angle AGO = 90^\circ$, determine all possible values of $\frac{GA}{BC}$.



Let M be the midpoint of $\overline{BC}$ and let D be the intersection of ray AM with the circumcircle of $\triangle ABC$. Since G is the foot from O to chord $\overline{AD}$, it follows that $AG = GD$. It's a well known property of the centroid that $AG : GM = 2 : 1$. Since $GD = GM + MD$, it follows that

$$AG : GM : MD = 2 : 1 : 1.$$

We have from power of a point that

$$\frac{BC^2}{4} = BM \cdot MC = AM \cdot MD = \left(\frac{3}{2} \cdot AG\right)\left(\frac{1}{2} \cdot AG\right) = \frac{3AG^2}{4},$$

so $\frac{GA^2}{BC^2} = \frac{1}{3}$. Square rooting both sides of this, it follows that

$$\boxed{\frac{GA}{BC} = \frac{\sqrt{3}}{3}}.$$

Problem 8:

Professional basketball player Jordan Smith wants to improve his free throw percentage (number made divided by number of attempts) from 75% to above 85% by the end of the season. Show that if this occurs, then there must exist a time during the season when his free throw percentage was exactly 80%.

Consider the integer quantity

$$T = (\#\text{ hits}) - 4 \cdot (\#\text{ misses}).$$

When Jordan has a 75% accuracy rate, T is less than 0; when Jordan has an 85% accuracy rate, T is greater than 0. Consider the shot Jordan scores which makes T nonnegative for the first time. Since this shot increases T by exactly 1, the nonnegative number that T becomes must be 0. When $T = 0$, Jordan has exactly an 80% hit rate, so we are done.

Problem 9:

Farley notices that 6^3 can be written as the sum of *three* positive perfect cubes, namely $3^3 + 4^3 + 5^3$, and that 7^3 can be written as the sum of *four* positive perfect cubes, namely $1^3 + 1^3 + 5^3 + 6^3$. Help Farley prove that, in fact, for *any* integer $i \geq 3$, there exists a perfect cube that can be written as the sum of i positive perfect cubes.

We prove this problem with induction on i.

- Base cases: $i = 3, 4$. These are given to us in the problem: we have $6^3 = 3^3 + 4^3 + 5^3$ and that $7^3 = 1^3 + 1^3 + 5^3 + 6^3$.
- Inductive step: $i \to i + 2$. Suppose n^3 can be written as the sum of i perfect cubes. Then, $27n^3$ can also be written as the sum of i perfect cubes since we multiply each perfect cube from before by 27. Now, we have that

  $$(6n)^3 = 27n^3 + (4n)^3 + (5n)^3,$$

  and since $27n^3$ can be written as the sum of i perfect cubes, $(6n)^3$ can be written as the sum of $i + 2$ perfect cubes. This completes our inductive step.

Problem 10:

Six circles are drawn such that no circle's center is inside any other circle. Show that no point in the plane lies inside all six circles.

Assume, for the sake of contradiction, that there is a point P that lies inside of all six circles. Let the centers of the circles be labelled A, B, C, D, E and F in clockwise order. One of the six angles $\angle AOB$, $\angle BOC$, ..., $\angle FOA$ must be less than or equal to $60°$ (since the six angles sum to $360°$); without loss of generality, assume $\angle AOB \leq 60°$ and that $OA \geq OB$. We will show that the circle centered at A contains point B.

Let r_a be the radius of the circle centered at A; since this circle contains O, it follows that $r_a \geq OA$. In a triangle, the longest side is opposite the largest angle – since in $\triangle AOB$, $\angle AOB \leq 60°$, it follows that $\overline{AB}$ cannot be the strictly longest side of the triangle. So, $AB \leq OA$. So, $r_a \geq OA \geq AB$, implying that the circle centered at A contains point B; this is a contradiction, so it follows that no point can lie in all six circles.