Intoxicated Mathematics

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I HAVE ACTED WITH HONESTY AND INTEGRITY IN PRODUCING THIS WORK AND AM UNAWARE OF ANYONE WHO HAS NOT.

Abstract

Many biological processes are characterized by their rates of change, which are well modeled by differential equations. The circulation of alcohol within the body is one such process that can be modeled by a system of differential equations. We solve a system of differential equations for the alcohol concentration in two connected compartments of the body: the stomach and blood. We implement linear algebra to solve this system. Specifically, we solve for the eigenvalues and eigenvectors of the corresponding vector-matrix system. The solution reveals exponential decay. We model blood alcohol content in two scenarios, such as what happens if one drink is consumed thirty minutes after the first drink. We also investigate the effects of varying the proportionality constants for the stomach and blood. Additionally, we calculate when the maximum level of blood alcohol content is reached, and how this changes according to how much alcohol is initially consumed. find the half-life of alcohol within the body. These examinations are particularly useful because varying the half-life of alcohol could have major impacts on how the effects of alcohol are experienced, or possibly even change the maximum blood alcohol content.

1 Overview

Blood Alcohol Content (BAC) refers to the concentration of alcohol within the blood. BAC depends on various factors, such as the amount of alcohol consumed, time between drinks, size of person, etc.[3] It has been proposed that the change in blood alcohol content over time can be modeled by the following system of differential equations:[2]

$$A'(t) = -k_1 A(t) \tag{1}$$

$$B'(t) = k_1 A(t) - k_2 B(t) \tag{2}$$

For a more detailed description of differential equations, please refer to section 5.1 of the appendix.

In Equation (1), $A(t)$ represents the concentration of alcohol in the stomach, $A'(t)$ represents the rate of change of concentration of alcohol in the stomach, and k_1 represents the proportionality constant of concentration of alcohol in the stomach. In Equation (2), $B(t)$ represents the concentration of alcohol in the blood, $B'(t)$ represents the rate of change of concentration of alcohol in the blood, and k_1 represents the proportionality constant of concentration of alcohol in the blood. This system of differential equations can be represented in the following vector-matrix system:

$$\begin{bmatrix} A'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} -k_1 & 0 \\ k_1 & -k_2 \end{bmatrix} \cdot \begin{bmatrix} A(t) \\ B(t) \end{bmatrix}$$

This vector-matrix system is a useful representation of our system of differential equations because we can use it to solve for the corresponding Eigensystem Equation. This Eigensystem Equation will allow us to find the eigenvalues and eigenvectors for this eigensystem. This will yield a solution that is a linear combination of these two eigenvectors, which will also be our solution to the system of differential equations.

2 Math Modeling

Solving the above vector-matrix system for the eigenvalues and eigenvectors yields the following solutions. The eigenvalues are $-k_1$ and $-k_2$, and the corresponding eigenvectors are $\begin{bmatrix} -\frac{k_1 - k_2}{k_1} \\ \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ \\ 1 \end{bmatrix}$. For a more detailed solution description, please refer to section 5.2 of the appendix. Thus, the solutions to the differential equations are the following:

$$A(t) = c_1 e^{-k_1 t} \tag{3}$$

$$B(t) = c_1 e^{-k_1 t} + c_2 e^{-k_2 t} \tag{4}$$

where c_1 and c_2 are the constants of integration that must be solved for using the initial conditions. In the case of a single drink being consumed, the initial condition for the stomach is $A'(t) = A_0$, where A_0 is the initial amount of alcohol consumed. The initial condition for the blood is $B'(t) = 0$, because all of the alcohol is in the stomach at the beginning. Once the initial conditions are solved for, we get the following solution set:

$$A(t) = A_0 e^{-k_1 t} \tag{5}$$

$$B(t) = A_0 (\frac{k_1}{k_2 - k_1})(e^{-k_1 t} - e^{-k_2 t}) \tag{6}$$

Next, we will model blood alcohol content for one standard drink consumed by a human. Below is a graph the blood alcohol concentration of the stomach and blood versus time.

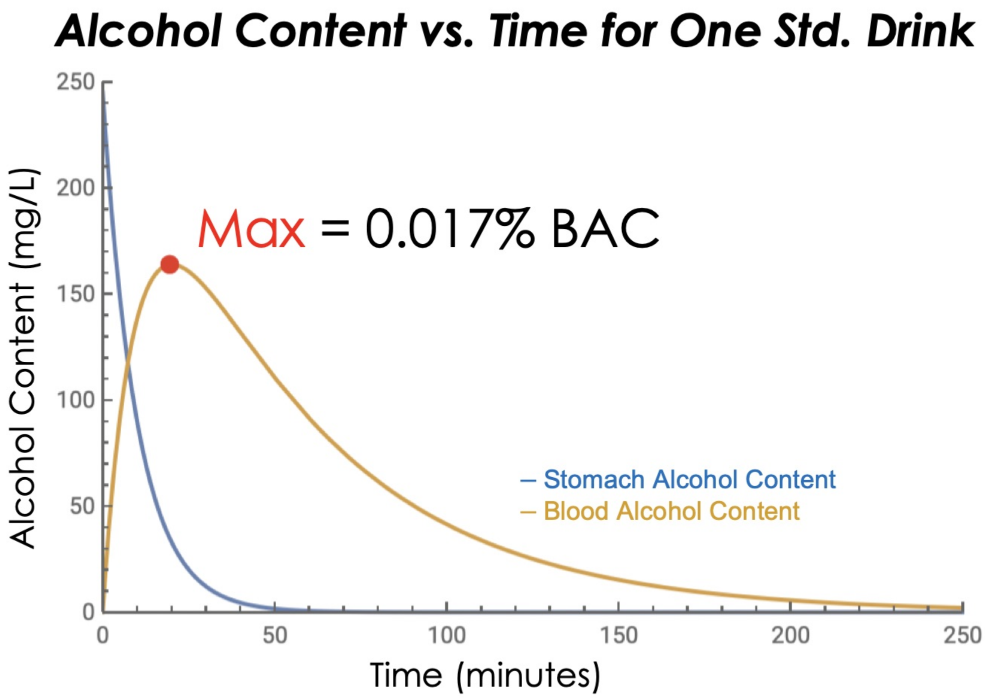

Figure 1: Blood Alcohol Content for One Standard Drink

The blood alcohol content quickly reaches its maximum of $0.172mg/L$, which corresponds to a blood alcohl content percentage of 0.017%, twenty minutes after the alcohol was consumed, and then exponentially decays at a slower rate. We calculated the time it takes to reach the maximum blood alcohol content in Mathematica by calculating when the derivative of $B(t)$ equals zero, and solving for t. This point will correspond to the time (t) it takes to reach the maximum blood alcohol content. Then, we plugged that time (in this case twenty minutes) back into $B(t)$ to solve for the blood alcohol content at that time, which is the maximum blood alcohol content. In order to convert the blood alcohol concentration to a blood alcohol content percentage, we simply divided by 100. It is also evident that the alcohol leaves the stomach at a much faster rate than it leaves the bloodstream.

Next, we will model the blood alcohol content for two standard drinks spaced thirty minutes apart. For the case of two standard drinks being consumed, the solution set of differential equations (3) and (4) remains unchanged, but the initial conditions with which we use to solve for the integration constants, c_1 and c_2 will be different. At time thirty minutes after the consumption of the first drink, the concentration of alcohol in the stomach is $0.009 mg/L$, and the concentration for alcohol in the bloodstream is $0.161 mg/L$. We will let $A_1 = 0.009 mg/L$, the concentration of alcohol in the stomach after thirty minutes, and $B_1 = 0.161 mg/L$, the concentration of alcohol in the bloodstream after thirty minutes. $A(t) = A_1 + A_0$ and $B(t) = B_1$ will serve as our initial conditions for the second drink consumed. When we solve for the integration constants, we arrive at the following solution set:

$$A(t) = (A_1 + A_0)e^{-k_1 t} \tag{7}$$

$$B(t) = (A_1 + A_0)(\frac{k_1}{k_2 - k_1})e^{-k_1 t} + (B_0 - (A_1 + A_0)(\frac{k_1}{k_2 - k_1}))e^{-k_2 t} \tag{8}$$

Below is we will model the case of two drinks consumed thirty minutes apart.

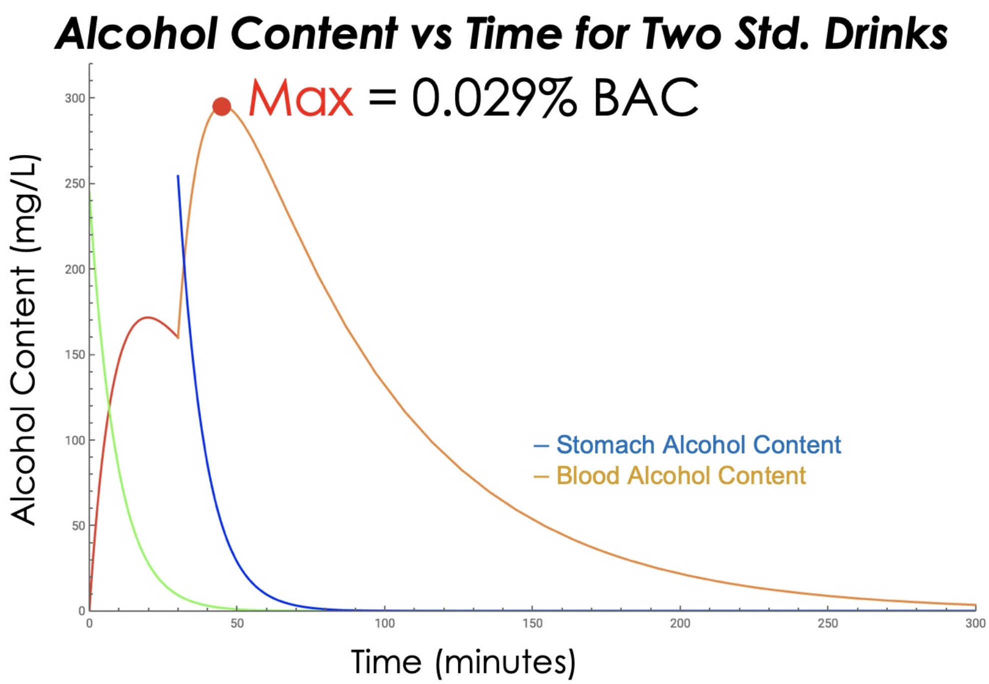

Figure 2: Blood Alcohol Content for Two Standard Drinks Spaced Thirty Minutes Apart

The blood alcohol content peaks at twenty minutes after the first drink is consumed, and again twenty minutes after the second drink is consumed (fifty minutes after consumption of first drink). The maximum blood alcohol content that is reached is $0.292 mg/L$, which corresponds to a blood alcohol content percentage of 0.029%. Again, We calculated these times in Mathematica by calculating when the derivative of $B(t)$ equals zero, and solving for t. These points will correspond to the time (t) it takes to reach the maximum blood alcohol content. Then, we plugged those times back into $B(t)$ to solve for the blood alcohol content at those times, which is the maximum blood alcohol content. In order to convert the blood alcohol concentration to a blood alcohol content percentage, we simply divided by 100. Like Figure 1, the blood alcohol content decays exponentially after the maximum BAC is reached.

Another scenario we modeled is what happens when we vary the concentration of the alcohol for the initial drink that is consumed:

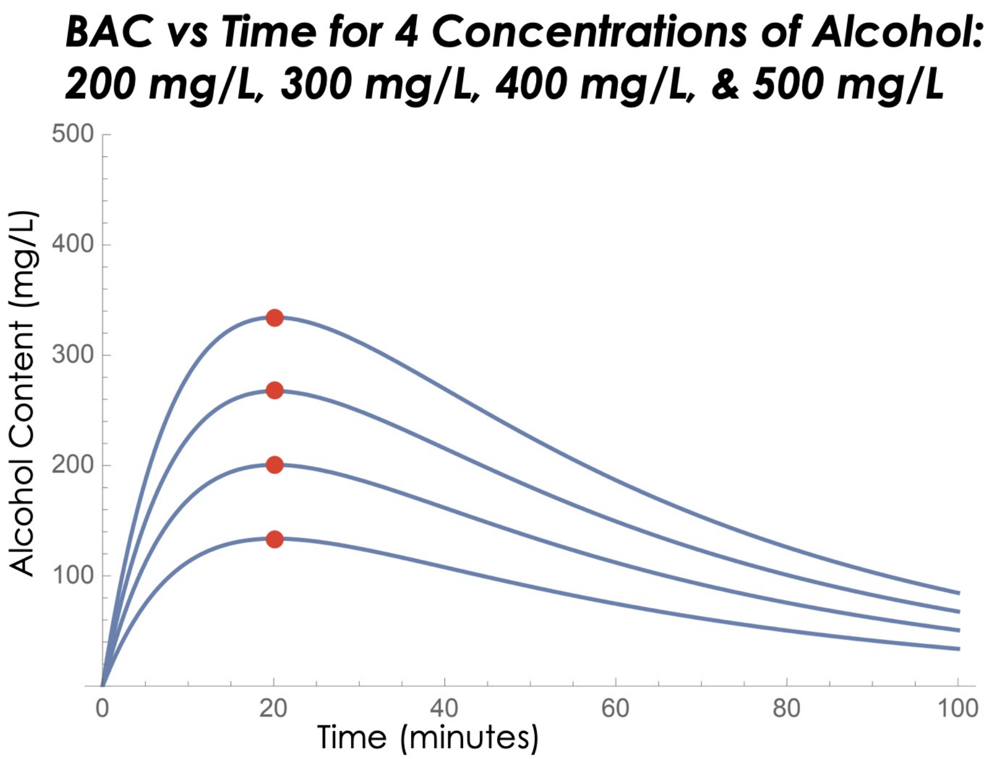

Figure 3: Blood Alcohol Content for Four Different Concentrations of Alcohol, consumed separately as opposed to in succession.

Figure 3 shows the blood alcohol content over time for four different volumes of alcohol: 200mg/L, 300mg/L, 400mg/L, and 500 mg/L. The higher the concentration that is consumed, the higher the maximum that is reached. However, there is no difference in the time it takes to reach each maximum because the proportionality constants are kept constant.

Next, we investigated what happens when we vary the proportionality constant for the blood:

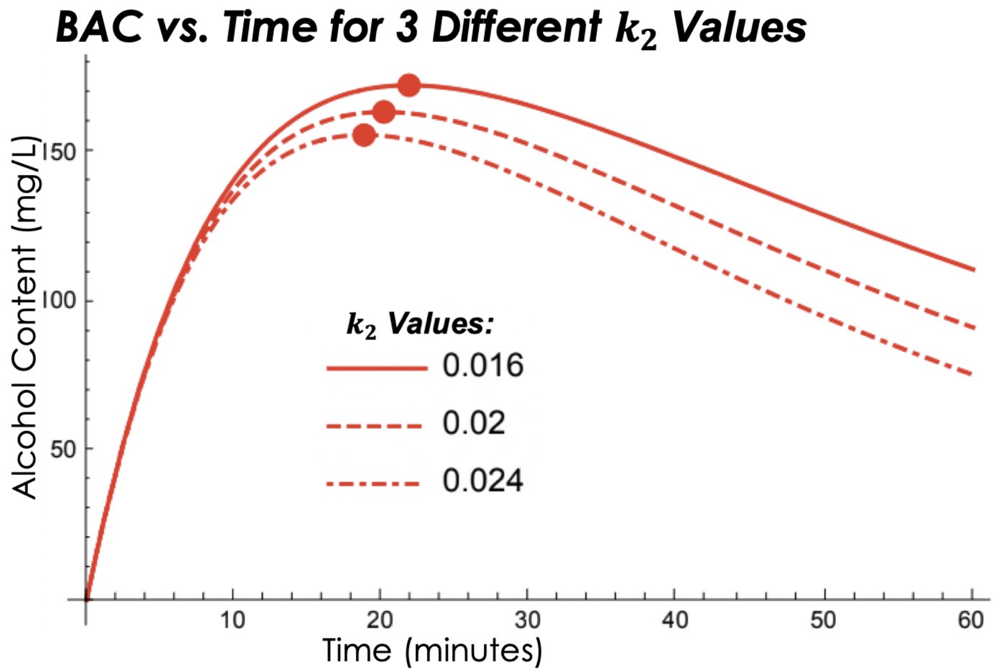

Figure 4: Blood Alcohol Content for Three Different Values of k_2.

0.016 is a 20% decrease from the original $k_2 = 0.02$, and 0.024 is a 20% increase. Neither variance results in a significant change in max BAC. As the proportionality constant for the blood decreases, the time it takes to reach the maximum blood alcohol content increases. Likewise, the maximum blood alcohol content reached increases as well.

Finally, we investigated the effect of varying the proportionality constant k_1 for the stomach:

Max BAC for 5 Different k_1 Values

% Change in k_1	k_1	Max B(t) (mg/L)	Time (minutes)	% Change In Max
-10%	0.0985	167.861	21.2305	2.60%
-5%	0.1040	170.248	20.5103	1.21%
0%	0.1095	172.337	19.8458	0%
5%	0.1149	174.162	19.2303	-1.06%
10%	0.1204	175.752	18.6584	-1.98%

Figure 5: Blood Alcohol Content for Five Different Values of k_1.

A 10% in the proportionality constant in the blood does not result in a significant change in the maximum blood alcohol content reached. As the proportionality constant for the stomach decreases, the time it takes to reach the maximum blood alcohol content increases. However, unlike Figure 4, the maximum reached actually decreases ask1 decreases.

The half-life h of alcohol in the bloodstream can be represented by the equation $h = \frac{ln(0.5)}{k_2}$. Thus, if the proportionality constant (k_1) increases, then the half-life decreases, and vice versa, so the half-life depends on the proportionality constant. This is important to note because a change in the proportionality constant of the concentration of alcohol in the blood will change the maximum concentration of alcohol in the blood, and the time over which it leaves the bloodstream. Thus, if alcohol had a shorter half-life, a higher maximum would be reached. For example, when $k_2 = 0.0168$ (five percent less than the original k_1 value, the maximum blood alcohol content reached after consumption of two standard drinks is $0.298 mg/L$, which is a higher maximum than our previous calculations. Likewise, when $k_1 = 0.0186$ (five percent higher than the original k_1 value, the maximum blood alcohol content reached is 0.285 mg/L, which is lower than our previous calculations. More detailed analysis on proportionality constants could be done in future work

3 Conclusions

One weakness of this model is that each time a new drink is consumed, we must solve for a new set of initial conditions. Additionally, it is assumed that each drink is consumed immediately as opposed to over a period of time, which would be more accurate. This model also does not account for the consumption of food and/or water, which are also factors that affect blood alcohol content. Also, we do not account for gender and/or weight of the human, which are factors that also affect blood alcohol content.

Strengths of this model include the ease with which the eigenvectors and eigenvalues can be solved for using the eigensystem equation. The solving method for this problem is very efficient, although as mentioned above, its accuracy has yet to be tested. Another strength is that this model can be easily generalized to other biological processes. Finally, this model can be easily extended to apply to other organs within the body.

4 Further Work

There are various possibilities for further work. Firstly, we can model multiple drinks consumed over various periods of time. We could also figure out how to model a single drink consumed over a period of time, and possibly combine this model with the model of multiple drinks. Additionally, a model could be developed for how blood alcohol content changes over time while taking into account water and/or food consumption, as well of size or gender of human. Another opportunity for further work is modeling what happens to blood alcohol content when the half-life of alcohol is increased and/or decreased significantly and comparing these results to other known results. Lastly, a mathematical model could be developed for other organs in the body, such as concentration of alcohol in the brain, liver, etc.

Exponential decay functions can be used to describe a plethora of biological processes,[1] so the mathematical model outlined in this paper might be applicable to other biological processes. It might be especially valuable to develop a mathematical model similar to this for substances metabolized by the human body that could be generalized to all such metabolic processes and would be somewhat accurate.

5 Appendix

5.1 Differential Equations

A differential equation represents the relationship between a rate(s) of change of a function(s), and the function(s) itself. Differential Equations are particularly useful in instances where the rate of change of a function is directly proportional to the function itself. Here is one example of a differential equation where the rate of change of a function is directly proportional to the function itself.:

$$f'(x) = c \cdot f(x) \tag{9}$$

In (9), $f'(x)$ is the function, $f'(x)$ is the rate of change of the function (otherwise known as the derivative), and c is the proportionality constant. It is suggested by Ludwin that (9) can be used to describe blood alcohol content within the stomach. To do this, we will simply rename the function

$(f(x))$ A, the derivative $(f'(x))$ $\frac{dA}{dt}$, and the proportionality constant (c) k_1. Thus we obtain:

$$\frac{dA}{dt} = -k_1 \cdot A \tag{10}$$

$$\tag{11}$$

In (10), A represents the concentration of alcohol in the stomach, $\frac{dA}{dt}$ represents the rate of change of alcohol concentration in the stomach, and k_1 is the proportionality constant of alcohol concentration in the stomach, which is equal to the exponential decay constant of alcohol in the stomach. k_1 is negative because alcohol is leaving the stomach, so the rate of change is always decreasing. We are making the assumption that at time $t = 0$, the concentration of alcohol in the stomach is at its maximum. This implies the assumptions (1) that an entire drink is consumed immediately as opposed to the more likely scenario where it is consumed over a period of time, and (2) that immediately upon the time of consumption, all of the alcohol has entered the stomach, as opposed to reaching the stomach over a period of time versus all at once. (10) can be rearranged to:

$$\frac{dA}{dt} + k_1 \cdot A = 0 \tag{12}$$

Thus, we can classify (10) as a first order linear homogeneous differential equation. (10) is first order because the highest order of the derivative in this equation is the first derivative. Additionally, (10) is linear because the orders of the function and the derivative are the same. Lastly, (10) is homogeneous because it can be arranged in the form of the function plus the derivative multiplied by a constant equals zero.

Secondly, it is proposed by Ludwin that the concentration of alcohol in the blood can likewise be modeled by a differential equation:

$$\frac{dB}{dt} = k_1 \cdot A - k_2 \cdot B \tag{13}$$

$$\tag{14}$$

In (13), B represents the concentration of alcohol in the bloodstream, $\frac{dB}{dt}$ represents the rate of change of concentration of alcohol in the bloodstream, and k_2 is the proportionality constant for concentration of alcohol in the bloodstream. As seen above, $\frac{dB}{dt}$ has two components. First, alcohol comes into the bloodstream from the stomach. This component is identical to the rate of change of alcohol in the stomach because the blood enters the bloodstream as it leaves the stomach, so in this equation this component is positive. Second, alcohol leaves the bloodstream through various ways, such as absorption in the tissues or elimination by the liver. (13) can be classified as a first order linear nonhomogeneous differential equation. Together, (10) and (13) are a system of differential equations. In this paper, we are interested in solving for B, the amount of alcohol in the bloodstream. We will do so by solving this system of differential equations through linear algebra. This solution method is outlined in the next section.

5.2 Linear Algebra

Linear Algebra is often used to solve differential equations, and we will employ linear algebra solving methods to solve for blood alcohol content. Our system of differential equations can be represented by the following vector-matrix system:

$$\begin{bmatrix} A'(t) \\ \\ B'(t) \end{bmatrix} = \begin{bmatrix} -k1 & 0 \\ \\ k1 & -k2 \end{bmatrix} \cdot \begin{bmatrix} A(t) \\ \\ B(t) \end{bmatrix}.$$

The first matrix is the derivative matrix, representing the rates of change of the concentration of alcohol in the stomach ($A'(t)$) and blood ($B'(t)$). The second matrix is the coefficient matrix (which is a square matrix). Lastly, the third matrix is the function matrix. In order to solve for the concentrations of alcohol in the stomach ($A(t)$) and blood ($B(t)$), we must solve for the Eigensystem for this vector-matrix system because the eigenvectors and eigenvalues will correspond to our solution for our system of differential equations.

This vector-matrix system is in the form $\bar{x}' = M\bar{x}$, where $\bar{x}'$ is the vector equivalent to the derivative matrix, $\bar{x}$ is the vector equivalent to the function matrix, and M is equivalent to the coefficient matrix. This equation is quite similar to the differential equation $y' = my$ where m is a scalar, and y is a function. This is a linear homogeneous differential equation and the assumed solution form for this particular differential equation is $y = he^{\lambda t}$, where h and λ are constants and unknowns, and correspond to the constant of integration and the proportionality constant, respectively. We can substitute the assumed solution for y in for $\bar{x}$. However, since $\bar{x}$ is a vector, we will substitute in the vector $(h_1, h_2)e^{\lambda t}$ in for $\bar{x}$, and the vector $(h_1, h_2)\lambda e^{\lambda t}$ in for $\bar{x}'$. Then we have $(h_1, h_2)\lambda e^{\lambda t} = M(h_1, h_2)e^{\lambda t}$. In order to solve for the Eigensystem, we must first rearrange our equation to equal the zero vector, so now we have $(h_1, h_2)\lambda e^{\lambda t} - M(h_1, h_2)e^{\lambda t} = \bar{0}$. We can factor out $(h_1, h_2)e^{\lambda t}$ to obtain $(M - \lambda I)(h_1, h_2)e^{\lambda t} = \bar{0}$ where I represents the identity matrix. Since $e^{\lambda t}$ is a nonzero element, we can cancel it out from the equation to obtain $(M - \lambda I)(h_1, h_2) = \bar{0}$. This is our Eigensystem Equation for M.

Next, in order to find the eigenvalues and eigenvectors that correspond with this Eigensystem, we must find the determinant for this Eigensystem. We will do this by solving the following the equation: $det(M - \lambda I) = \bar{0}$. This equation yields the resulting eigenvalues: $\lambda_1 = -k_1$ and $\lambda_2 = -k_2$. Additionally, these are the resulting eigenvectors: $(-\frac{k_1 - k_2}{k_1}, 1)$ and $(0, 1)$. Since we have obtained the eigevalues and eigenvectors, we can now substitute them back into our original equation to obtain a linear combination of two vectors, which will be the general solution to our system of differential equations:

$$\begin{bmatrix} A(t) \\ \\ B(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{k_1 - k_2}{k_1} \\ \\ 1 \end{bmatrix} e^{-k_1 t} + c_2 \begin{bmatrix} 0 \\ \\ 1 \end{bmatrix} e^{-k_2 t}$$

This solution to our vector-matrix system corresponds with the following solutions to our system of differential equations:

$$A(t) = c_1 e^{-k_1 t}$$
$$B(t) = c_1 e^{-k_1 t} + c_2 e^{-k_2 t}$$

5.3 Math Autobiography

I am currently a senior at Southwestern University and my most recent excursions in Mathematics include: Calculus I, II, III, Linear Algebra, Discrete Math, Probability Math Stats, and Differential Equations. I am currently taking Topology and Algebraic Structures. I also did two weeks of research with Dr. Marr in May of 2020 on a specific application of graph theory. I also have taken

Computer Science I and II. Consequently, I was able to use some of my coding skills to conduct my graph theory research. I am continuing my research on graph theory this semester (Fall 2020).

5.4 Acknowledgements

I express my utmost thanks to Dr. Shelton for her patient guidance, unceasing encouragement, and the numerous extra hours she has dedicated to helping me complete this project.

I also extend my deepest gratitude to Southwestern University for providing me with this opportunity, and to my friends and family for their unwavering support.

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