Intoxicated Mathematics

WITH SYSTEMS OF DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

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Southwestern University Math Capstone Fall 2020

Modeling Alcohol Content

- **Alcohol Content** in stomach and blood
- System of Differential Equations
- Linear Algebra
- Various Scenarios

Alcohol Content in Stomach

(1)

$$\frac{dA}{dt} = -k_1 A$$

Rate of change of alcohol in the stomach ($\frac{dA}{dt}$)

exponential decay constant of alcohol in stomach (k_1)

amount of alcohol in stomach (A)

Alcohol Content in Bloodstream

(2)

$$\frac{dB}{dt} = k_1 A - k_2 B$$

Rate of change of alcohol in the blood

exponential decay constant of alcohol in blood

Amount of alcohol in the blood

Alcohol Content in Bloodstream

$$(2)\quad \frac{dB}{dt} = k_1 A - k_2 B$$

Rate of change of alcohol in the blood

exponential decay constant of alcohol in blood

Amount of alcohol in the blood

Alcohol Content in Bloodstream

(2)

$$\frac{dB}{dt} = k_1 A - k_2 B$$

Rate of change of alcohol in the blood

exponential decay constant of alcohol in blood

Amount of alcohol in the blood

System of Differential Equations

(1) $$\frac{dA}{dt} = -k_1 A$$

(2) $$\frac{dB}{dt} = k_1 A - k_2 B$$

System of DEs Represented by Vector-Matrix System

(1) $$\frac{dA}{dt} = -k_1 A$$

(2) $$\frac{dB}{dt} = k_1 A - k_2 B$$

$$\begin{bmatrix} A'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} -k1 & 0 \\ k1 & -k2 \end{bmatrix} \begin{bmatrix} A(t) \\ B(t) \end{bmatrix}$$

System of DEs Represented by Vector-Matrix System

$$(1)\ (2)\ \begin{bmatrix} A(t) \\ B(t) \end{bmatrix}' = \begin{bmatrix} -k1 & 0 \\ k1 & -k2 \end{bmatrix} \begin{bmatrix} A(t) \\ B(t) \end{bmatrix}$$

- $\bar{x}' = M\bar{x}$ (3)

System of DEs Represented by Vector-Matrix System

$$\bar{x}' = \begin{bmatrix} A(t) \\ B(t) \end{bmatrix}'$$

$$\begin{matrix}(1)\\(2)\end{matrix} \quad \begin{bmatrix} A(t) \\ B(t) \end{bmatrix}' = \begin{bmatrix} -k1 & 0 \\ k1 & -k2 \end{bmatrix} \begin{bmatrix} A(t) \\ B(t) \end{bmatrix}$$

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- $\bar{x}' = M\bar{x}$ (3)

System of DEs Represented by Vector-Matrix System

$$\bar{x}' = \begin{bmatrix} A(t) \\ B(t) \end{bmatrix}' \quad \bar{x} = \begin{bmatrix} A(t) \\ B(t) \end{bmatrix}$$

$$M = \begin{bmatrix} -k1 & 0 \\ k1 & -k2 \end{bmatrix}$$

(1)
(2)

$$\begin{bmatrix} A(t) \\ B(t) \end{bmatrix}' = \begin{bmatrix} -k1 & 0 \\ k1 & -k2 \end{bmatrix} \begin{bmatrix} A(t) \\ B(t) \end{bmatrix}$$

- $\bar{x}' = M\bar{x}$ (3)

Linear Algebra

-
- $\bar{x}' = M\bar{x}$ (3)
- $y' = my$ where m is a scalar, and y is a function

Linear Algebra

-
- $\bar{x}' = M\bar{x}$ (3)
- $y' = my$ *where m is a scalar, and y is a function*
- $y = h\,e^{\lambda t}$ *solution form*

Linear Algebra

- $\bar{x}' = M\bar{x}$ (3)
- $y' = my$ where m is a scalar, and y is a function
- $y = h\,e^{\lambda t}$ where h and λ are unknowns

Linear Algebra

-
- $\bar{x}' = M\bar{x}$ (3)
- $\bar{y} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} e^{\lambda t}$

Linear Algebra

- $\bar{x}' = M\bar{x}$ (3)
  - $\bar{x} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} e^{\lambda t}$
  - $\bar{x}' = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \lambda e^{\lambda t}$

Linear Algebra

- $\bar{x}' = M\bar{x}$ (3)
  - $\bar{x} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} e^{\lambda t}$
  - $\bar{x}' = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \lambda e^{\lambda t}$

Linear Algebra

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \lambda\, e^{\lambda t} = M \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} e^{\lambda t}$$

Linear Algebra

- $$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \lambda\, e^{\lambda t} = M \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} e^{\lambda t}$$
- Find the Eigensystem Equation for M by finding the determinant.

General Solution

- $$\bar{x} = c_1 \begin{bmatrix} -\frac{(k_1 - k_2)}{k_1} \\ 1 \end{bmatrix} e^{-k_1 t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-k_2 t}$$

General Solution

- $\bar{x} = c_1 \begin{bmatrix} -\frac{(k_1 - k_2)}{k_1} \\ 1 \end{bmatrix} \mathrm{e}^{-\mathrm{k}_1 \mathrm{t}} + \mathrm{c}_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{e}^{-\mathrm{k}_2 \mathrm{t}}$
- $\begin{bmatrix} A(t) \\ B(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{(k_1 - k_2)}{k_1} \\ 1 \end{bmatrix} \mathrm{e}^{-\mathrm{k}_1 \mathrm{t}} + \mathrm{c}_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{e}^{-\mathrm{k}_2 \mathrm{t}}$

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Corresponding Solution Set

1. $A(t) = -c_1\left(\frac{k_1 - k_2}{k_1}\right)e^{-k_1 t}$
2. $B(t) = c_1\left(e^{-k_1 t}\right) + c_2\left(e^{-k_2 t}\right)$

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Corresponding Solution Set for One Drink Consumed

1. $A(t) = -c_1 \left(\frac{k_1 - k_2}{k_1}\right) e^{-k_1 t}$

2. $B(t) = c_1 \left(e^{-k_1 t}\right) + c_2 \left(e^{-k_2 t}\right)$

<u>*Initial Conditions for One Drink Consumed*</u>:

- $A(0) = A_d$
- $B(0) = 0$

Corresponding Solution Set for One Drink Consumed

1. $A(t) = A_d e^{-k_1 t}$

2. $B(t) = A_d \left(\frac{k_1}{k_2 - k_1}\right)\left(e^{-k_1 t} - e^{-k_2 t}\right)$

Corresponding Solution Set for Two Drinks Consumed

1. $A(t) = -c_1\left(\frac{k_1 - k_2}{k_1}\right)e^{-k_1 t}$

2. $B(t) = c_1\left(e^{-k_1 t}\right) + c_2\left(e^{-k_2 t}\right)$

<u>Initial Conditions:</u>

- $A(0) = A_s + A_d$
- $B(0) = B_s$

Corresponding Solution Set for Two Drinks Consumed

1. $A(t) = (A_s + A_d)e^{-k_1 t}$

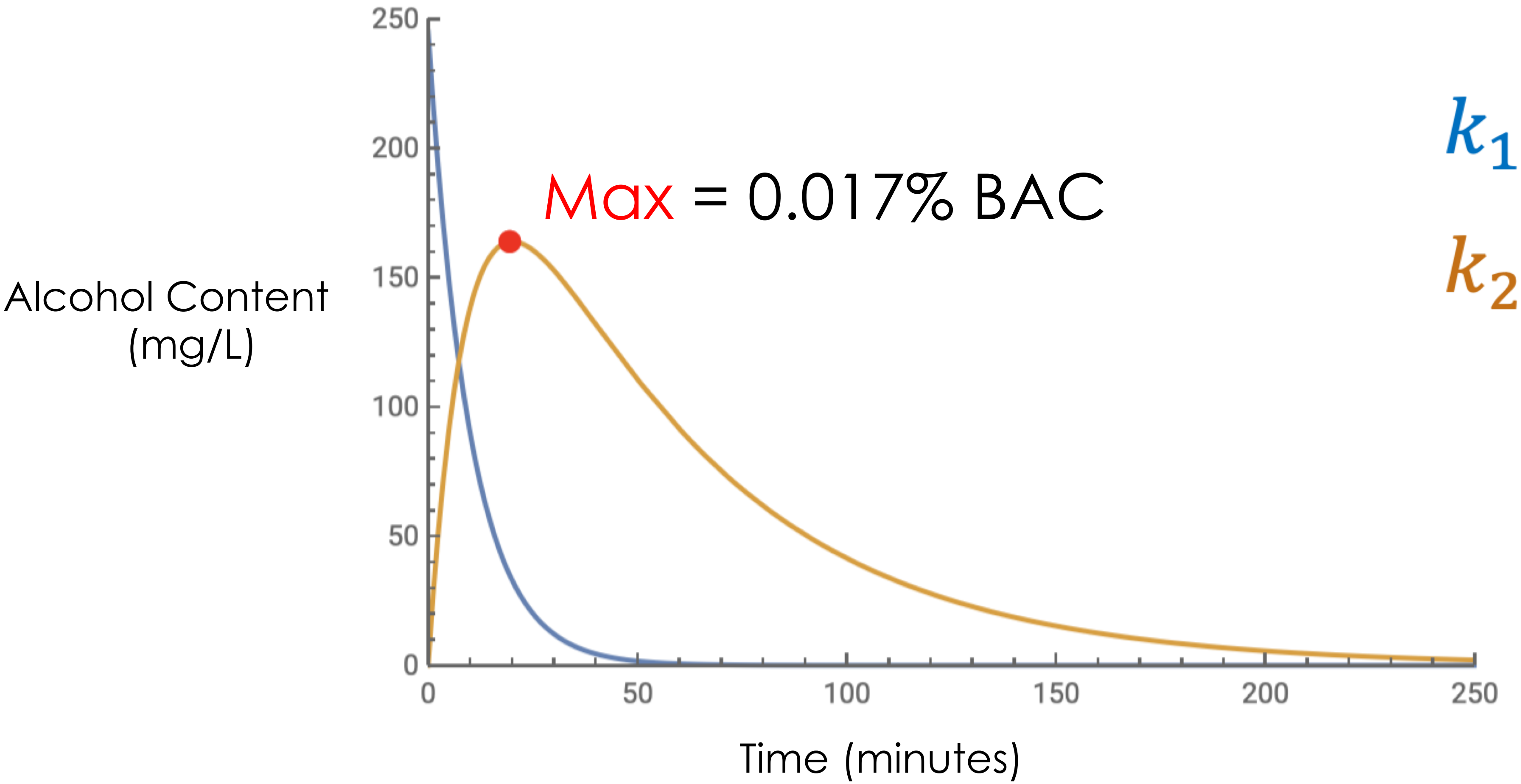
2. $B(t) = (A_s + A_d)\left(\frac{k_1}{k_2 - k_1}\right)e^{-k_1 t} +$

$$(B_s - (A_s + A_d)\left(\frac{k_1}{k_2 - k_1}\right))e^{-k_{21} t}$$

Assumptions

1. Subject is 75kg
2. One std. drink = 245 mg/L alcohol
3. All alcohol is consumed at once
4. Alcohol enters the stomach immediately after is consumed
5. Subject is not drinking water or eating food

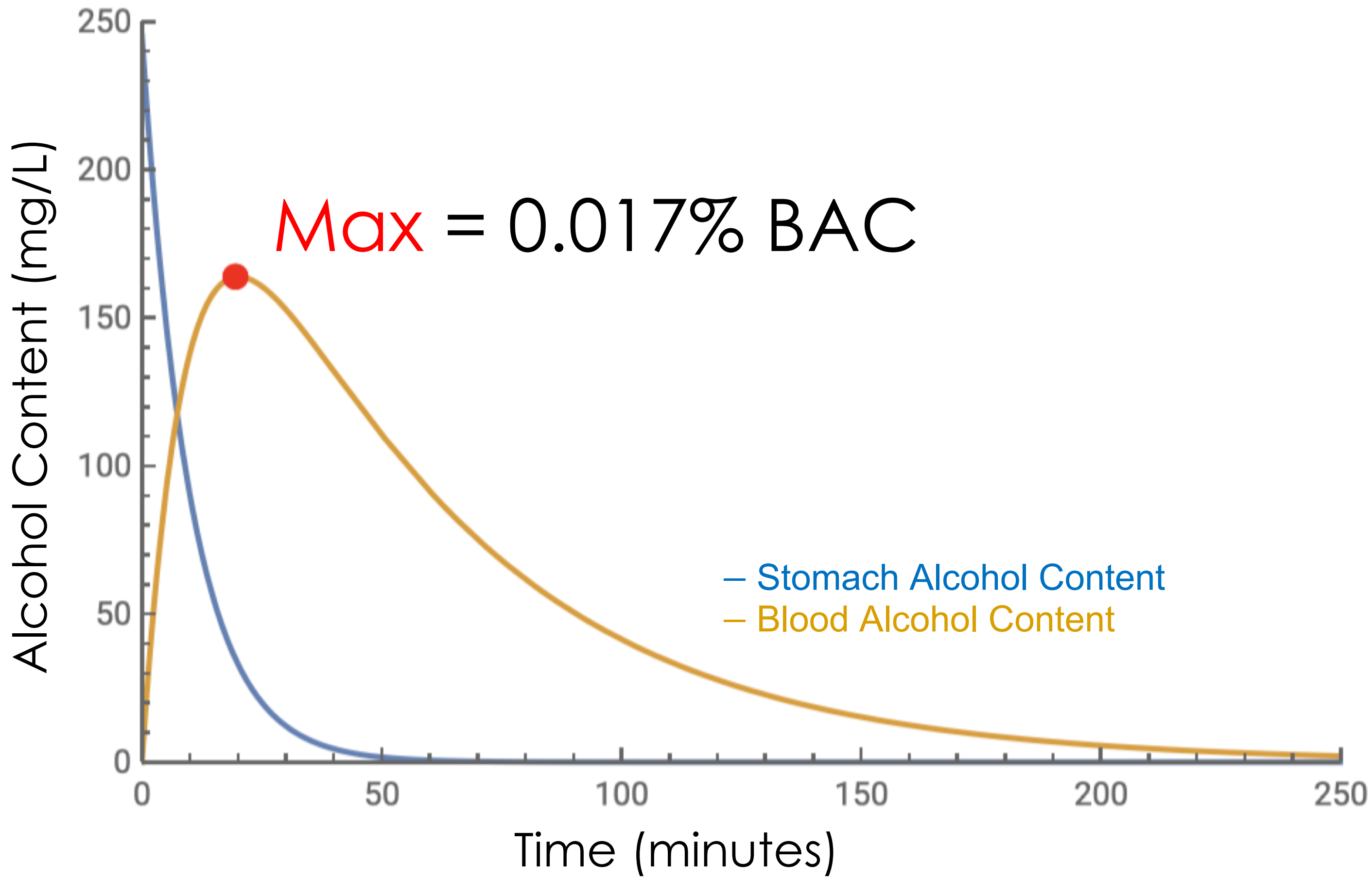
Alcohol Content vs Time for One Std. Drink



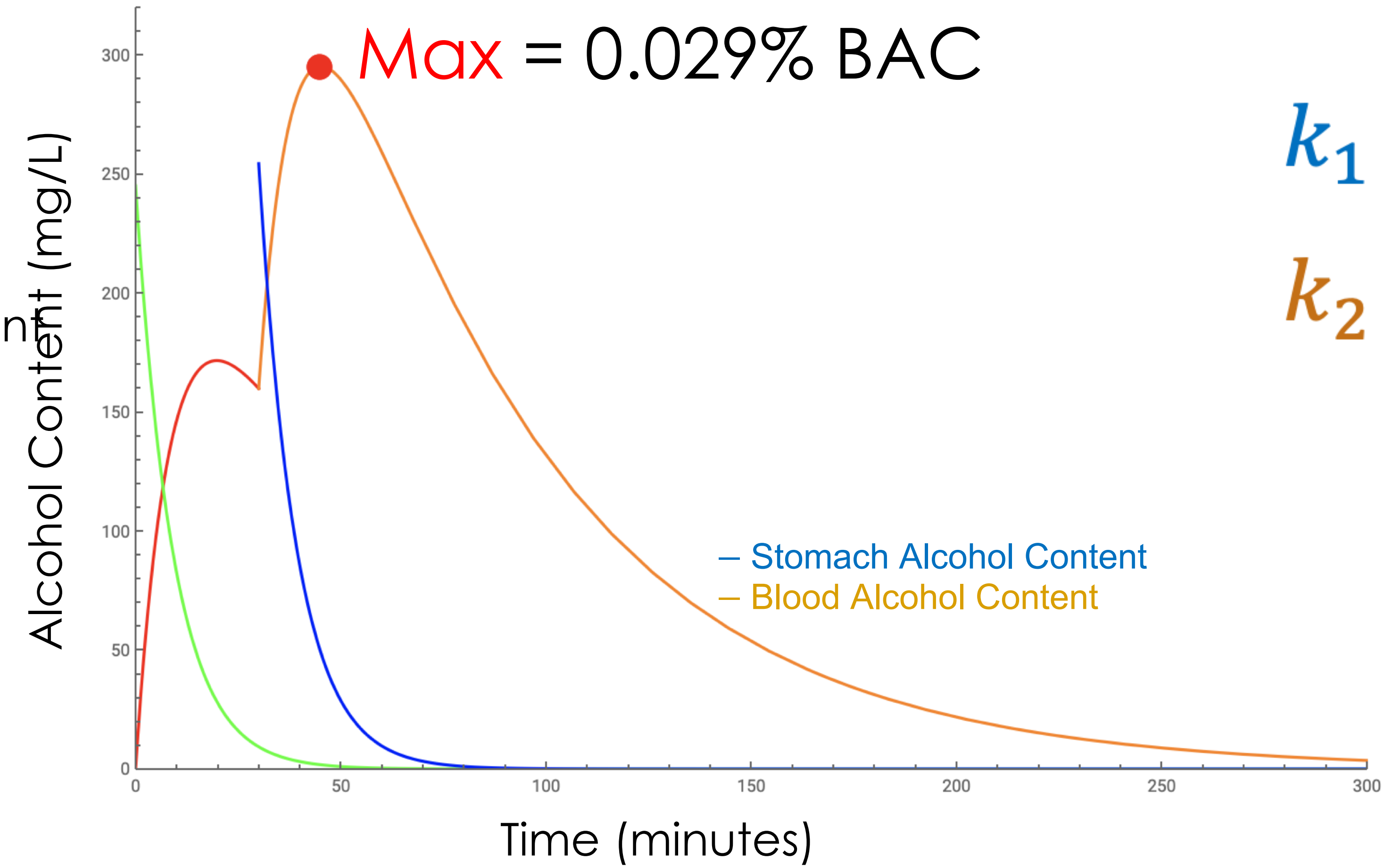
$k_1 = 0.109$

$k_2 = 0.018$

Alcohol Content vs. Time for One Std. Drink



Alcohol Content vs Time for Two Std. Drinks

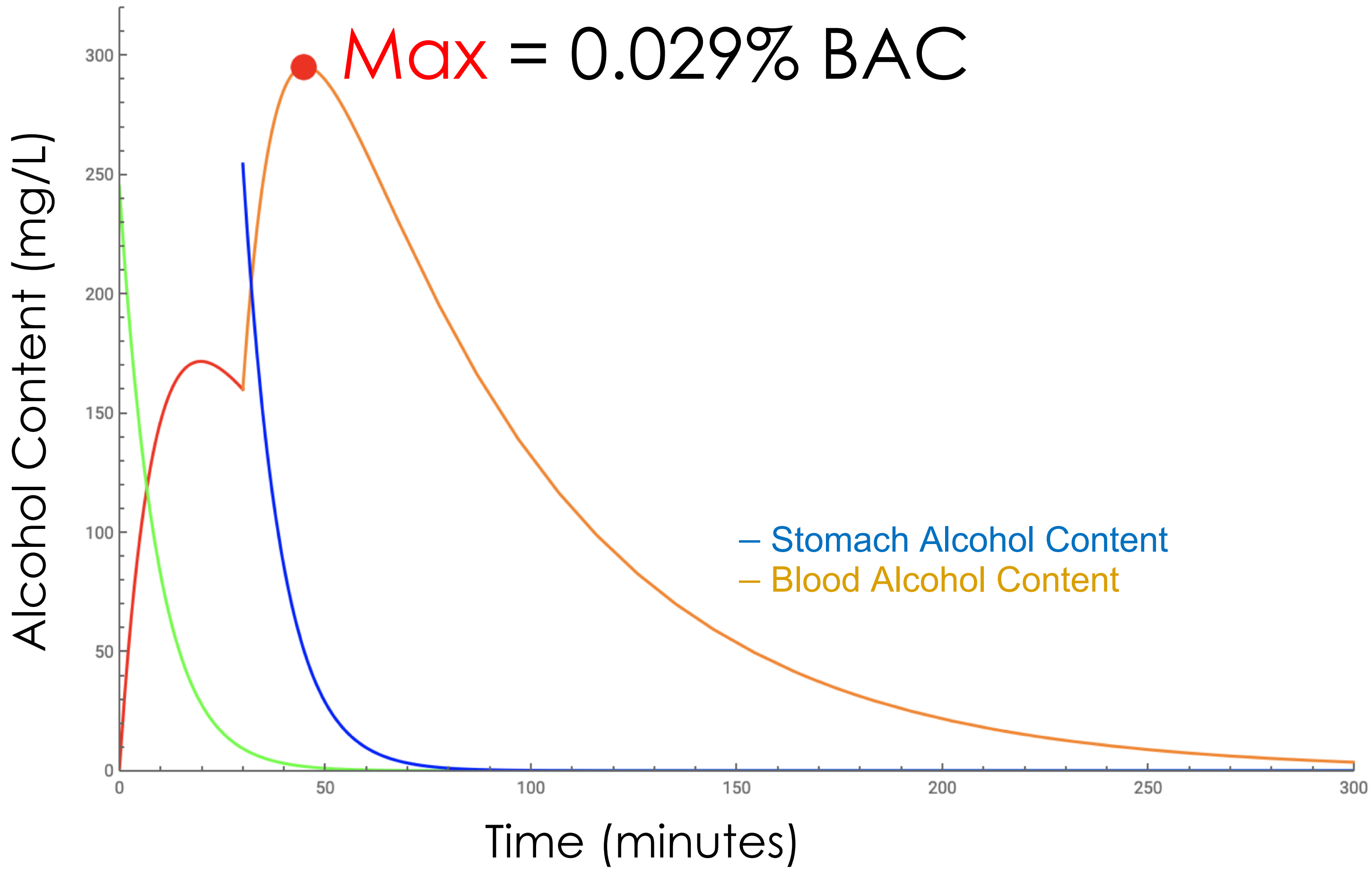


Alcohol Content (mg/L)

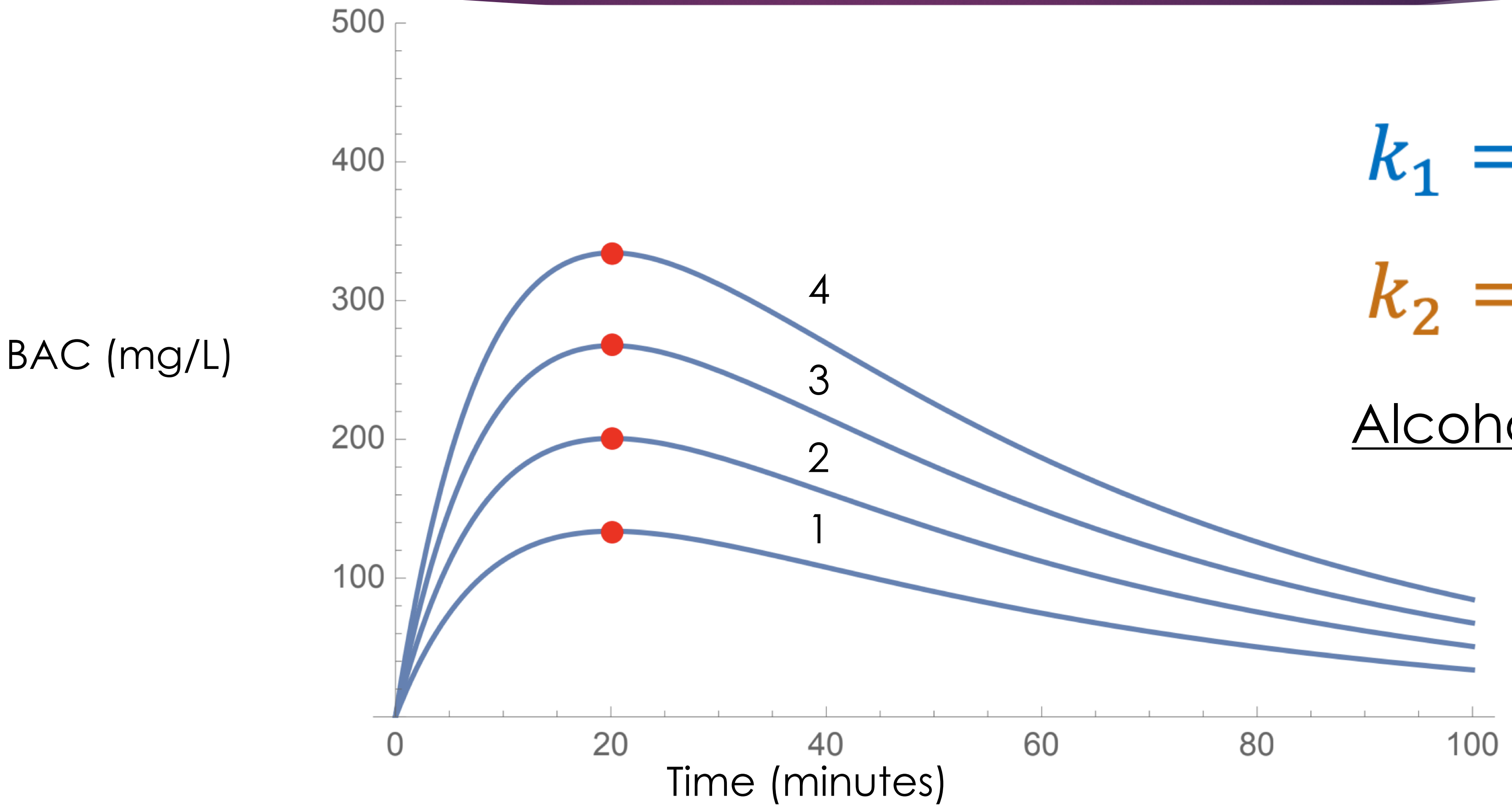
$$k_1 = 0.109$$

$$k_2 = 0.018$$

Alcohol Content vs Time for Two Std. Drinks



BAC vs Time for 4 Concentrations of Alcohol: 200 mg/L, 300 mg/L, 400 mg/L, & 500 mg/L



$$k_1 = 0.109$$

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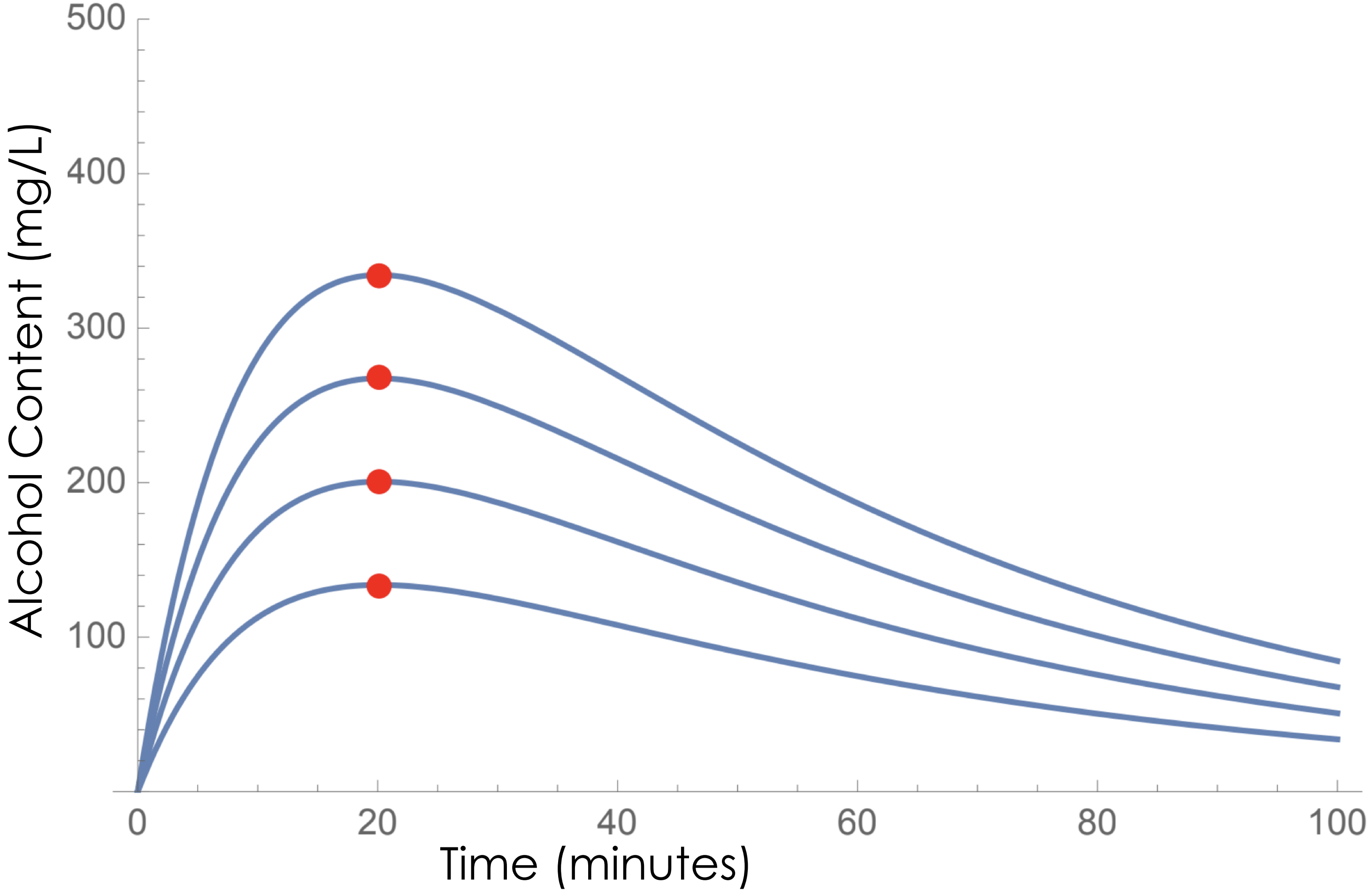
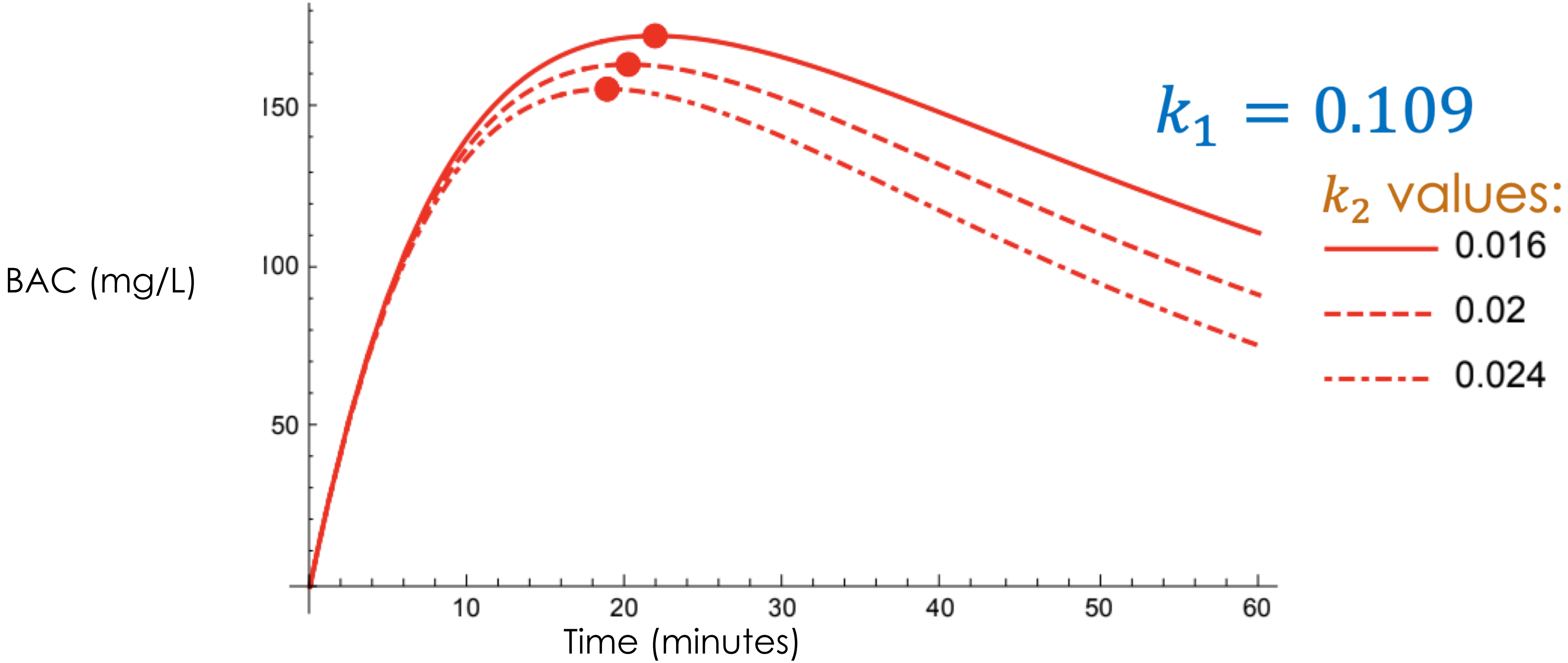
<u>Alcohol Conc. (mg/L)</u>

1. 200
2. 300
3. 400
4. 500

BAC vs Time for 4 Concentrations of Alcohol: 200 mg/L, 300 mg/L, 400 mg/L, & 500 mg/L



BAC vs Time for 3 Different k_2 Values



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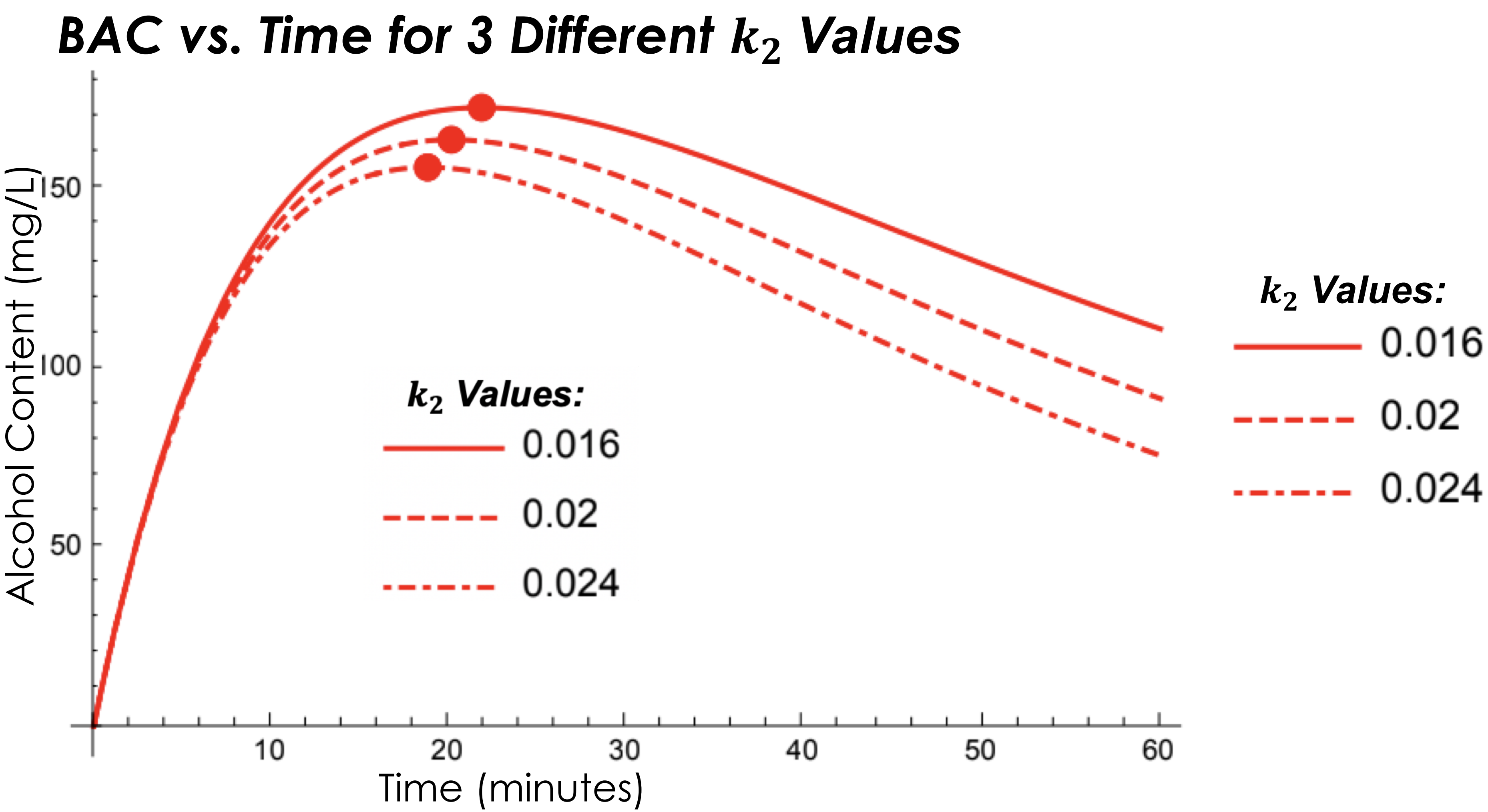

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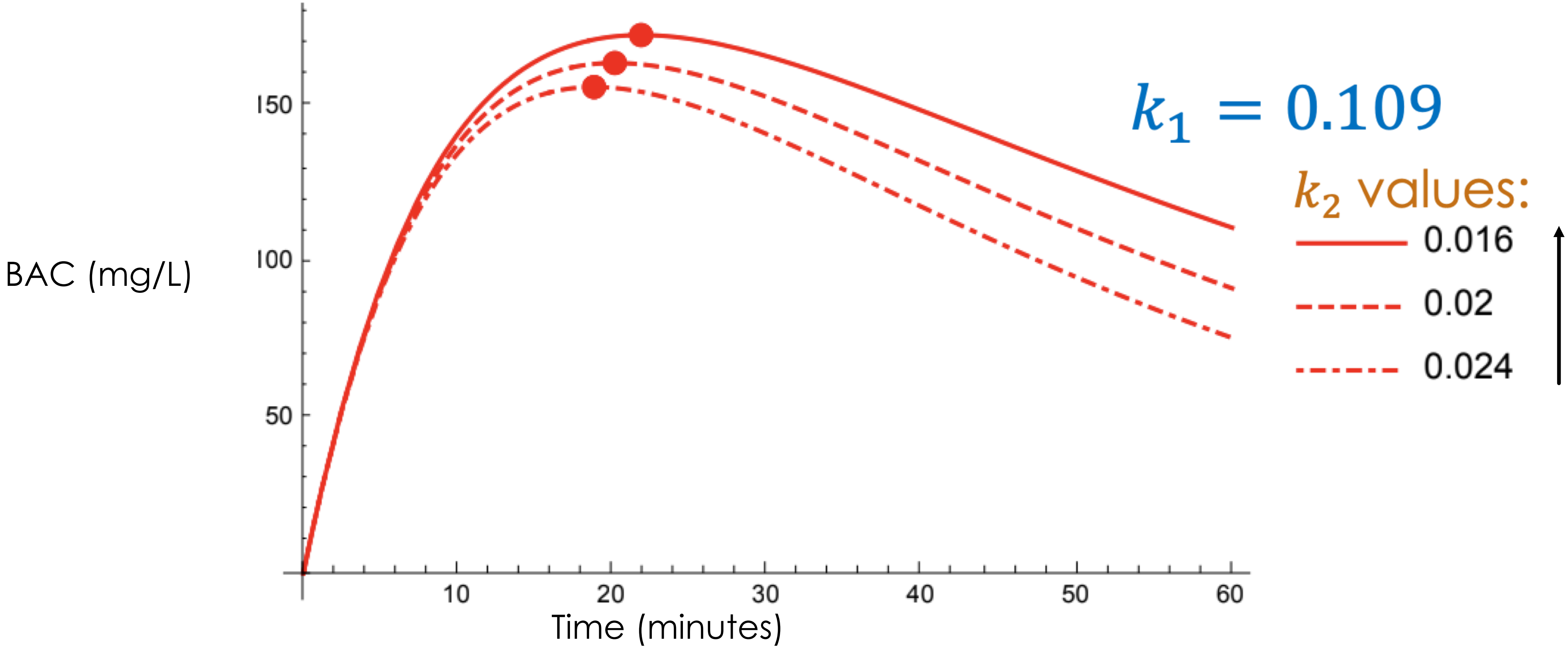

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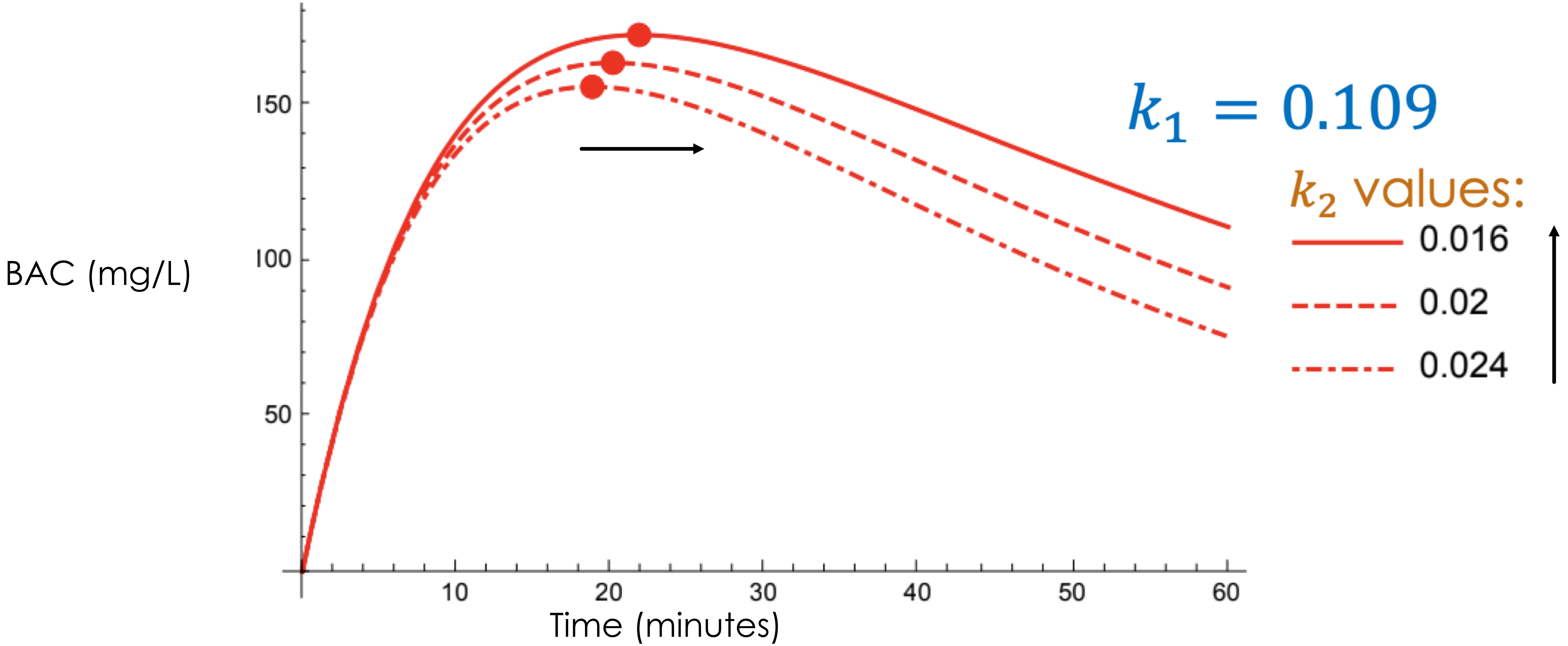

Max BAC for 5 Different k_1 Values

% Change in k_1	k_1	Max B(t) (mg/L)	Time (minutes)	% Change In Max
-10%	0.0985	167.861	21.2305	2.60%
-5%	0.1040	170.248	20.5103	1.21%
0%	0.1095	172.337	19.8458	0%
5%	0.1149	174.162	19.2303	-1.06%
10%	0.1204	175.752	18.6584	-1.98%

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Conclusions

- Strengths
  - simple
  - accurate
- Weaknesses
  - Is this modeling realistic situations?

Further Work

- Calculate BAC
  - For various weights
  - With consumption of food
  - Without assuming ingestion is immediate
  - And comparing to other methods, such as fractional calculus
- Explore how these equations and this method of solving can be generalized

Mind-Blowing Fact of the Day

A sheet of paper is about 0.1mm thick. If you folded such a sheet of paper in half fifty times, the resulting stack would reach ¾ of the way to the sun! It would take light 6.3 minutes to travel this distance.

- Garfinkel, Alan, et al.

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$$P(t) = P_0 e^{k_1 t}$$

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$$A(t) = A_0 e^{-k_1 t}$$

Resources

- Ludwin, Chris. *Blood Alcohol Content*. Undergraduate Journal of Mathematical Modeling: One + Two, 2011, scholarcommons.usf.edu/cgi/viewcontent.cgi?article=4819&context=ujmm.
- Garfinkel, Alan, et al. "Modeling, Change, and Simulation." *Modeling Life: The Mathematics of Biological Systems*, Springer International Publishing, 2017.
- Dawkins, Paul. "Paul's Online Notes." *Differential Equations - Solutions to Systems*, 2018, tutorial.math.lamar.edu/Classes/DE/SolutionsToSystems.aspx.

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Thank you!