THE MATHEMATICAL ASSOCIATION OF VICTORIA

PRIME NUMBER

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INSIDE

- Reasoning with 3D objects
- Enriching mathematical discussion
- Making sense of negative numbers
- Extended mathematical investigations

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ON THE COVER

The Wandjina are creator spirits from the Dreamtime. Tradition holds that these marks were made by the Wandjina when they left our world, and entered the spirit world. The Mowanjum Community keeps the images fresh at Raft Point, in the Kimberley region. Western Australia.

Image: iStockphoto

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CONTRIBUTE TO PRIME NUMBER

Prime Number welcomes contributions from classroom teachers, academics and mathematics educators. If you have content to contribute, email the editor, James Russo, via office@mav.vic.edu.au.

FROM THE EDITOR

James Russo

CELEBRATING THE CONTRIBUTIONS OF NEW TEACHERS TO MATHEMATICS EDUCATION

Teaching is a complex craft that takes many years to master, and experience is rightfully highly valued within the education profession. However, teaching is also a profession that benefits greatly from novel ideas, fresh perspectives, and the naïve energy brought in by 'outsiders', and those beginning their careers as teachers. For example, every year, my colleagues and I at Monash are blown away by the quality of the thinking demonstrated by some of our pre-service teachers. Despite their limited classroom experiences, these individuals are able to offer insights and suggest activities that would enrich any classroom. I personally get a buzz when I mark an assignment and think 'Wow, I'd never thought of doing that in a classroom. But I'd love to try it'. In response to such 'wow' moments, I have previously co-authored two articles in *Prime Number* developed from ideas embedded in assignments submitted by pre-service teachers (Russo & Contanoi, 2016; Russo, Swart, & Russo, 2016).

My article with Sarah Hopkins unpacking the Snakes and Ladders game was greatly enhanced by conversations we had in class with our university students (Russo & Hopkins, 2017; see photo).

In this spirit, I am going to dedicate a segment of *Prime Number* to contributions from pre-service teachers for 2019. Daniel Osment from La Trobe University, and his suggestions for teaching multiplication of negative numbers, is the article included in this issue. I would also like to draw your attention to an assignment completed by a student at Monash University, Scott Huddleston, which is available online in video format. As their final undergraduate assignment in mathematics education, students were encouraged to take off their 'teachers' hat', and undertake a mathematical investigation into a topic of personal interest to them. Scott chose his passion for running half-marathons. I think he did a remarkable job. It is definitely worth a watch: https://youtu.be/Y4vtCQGX1OU

I would encourage pre-service teachers who feel they have something to contribute to contact us with their ideas. I would also suggest that lecturers or tutors who come across outstanding assignments submitted by pre-service teachers to encourage these students to email the *Prime Number* team. Email us at office@mav.vic.edu.au or me personally at james.russo@monash.edu.

As an aside, it is also worth noting that students can join MAV for free (allowing them to, amongst other things, get access to *Prime Number*). I'd recommend university lecturers, tutors and mentor teachers to pass this information on to their pre-service teachers. I hope everyone has a terrific start to the year.

REFERENCES

Russo, J., & Contaoi, A. (2016). Get your game on: Hopping to 100. *Prime Number*, *31*(2), 6-7.

Russo, J., Swart, C., & Russo, T. (2016). Factor fiction. *Prime Number*, *31*(3), 14-15.

Russo, J., & Hopkins, S. (2017). Get your game on!: Snakes and ladders revisited. *Prime Number*, *32*(1), 6-10.

DEVELOPING FLUENCY: ROLL, MAKE AND COMPARE

Michael Nelson, Teaching and Learning Coordinator, Portarlington Primary School

Having games that can effectively allow students to practice multiple skills and that can be extended or condensed based on student need is a highly effective aspect of a teacher's approach. Games such as Roll, Make and Compare do not require students to learn a new game for each skill they are introduced to, and therefore allow students to solely focus on their understanding of the mathematical concepts.

This game is designed to allow students to develop a number of key concepts of numeration they will require before encountering formal place value. Students are required to make numbers up to 10, name the number of objects in a collection, record symbols and then match them to a collection and also to name the symbol.

Roll, Make and Compare also has a number of built in scaffolds that are embedded in the activity so students will not feel singled out if they still require them. It provides students with a safety net to allow them to feel successful and also give them ownership of their learning, as they can progress in the game when they feel comfortable enough to do so.

This game can also be extended into exploring place value concepts, as well as into operations including addition, multiplication and subtraction, without any major modifications to the game.

MATERIALS

Dice: Depending on the knowledge of the student, this can either be a 6 or 10 sided dice, with dots or symbols.

Counters: The amount of counters is equal to the number of sides on the dice being used by the students. Students will also need 10 yellow counters each.

Mini whiteboards: Paper can be used, but whiteboards are easier for students to practice their writing of symbols.

Tens frames: Used to support students understanding of the numbers to 10. Students can play without this if students are working on subitising. Each student requires two tens frames: one as their gameboard and one as their scoreboard.

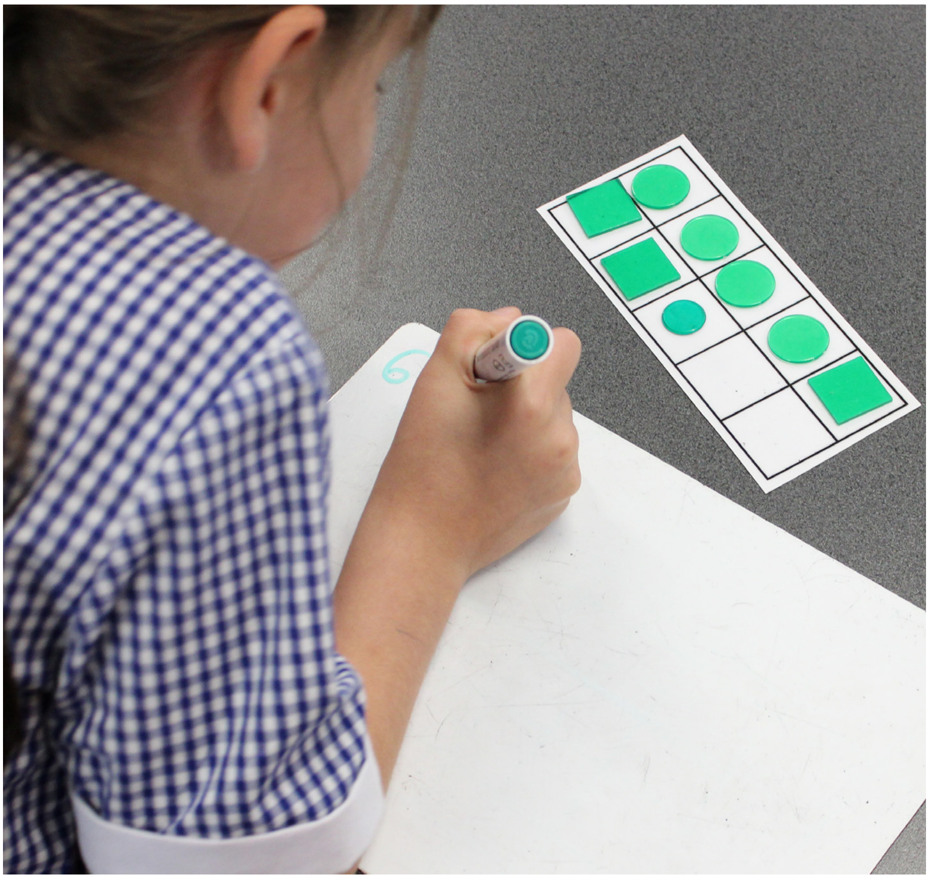

Figure 1. A student recording their thinking.

HOW TO PLAY

The game is suitable for two players, but can be extended to a third if students are focusing on being able to order rather than compare numbers. Students either sit with their back to their partner or if space is an issue, place a book up as a divider.

Player 1 rolls the dice and makes the number using counters on the tens frame that serves as their gameboard. Player 2 then rolls the dice and does the same. Once they have both made their numbers, they will compare the collections they have made on their tens frames. The student with the highest number wins and collects a piece of gold (a yellow counter) and places it on their second tens frame (their scoreboard).

Play continues until one student has collected ten pieces of gold.

BUILDING THE MATHS

Students will progress through Booker's stages of comparing. Level 1 involves compare materials-and-materials, Level 2 comparing materials-and-symbols and finally, Level 3 involves comparing symbols-and-symbols.

By beginning at the first level of the game, teachers can scaffold their students' new learning. For the second level, when comparing materials-and-symbols, one player is chosen to be the recorder. If the groups are mixed ability, this will generally be the more advanced student (at least to begin with), whilst for like-ability groups, the job can be selected randomly. The game is played as per before, however before turning and comparing, the recorder will write the symbol that matches the number in the

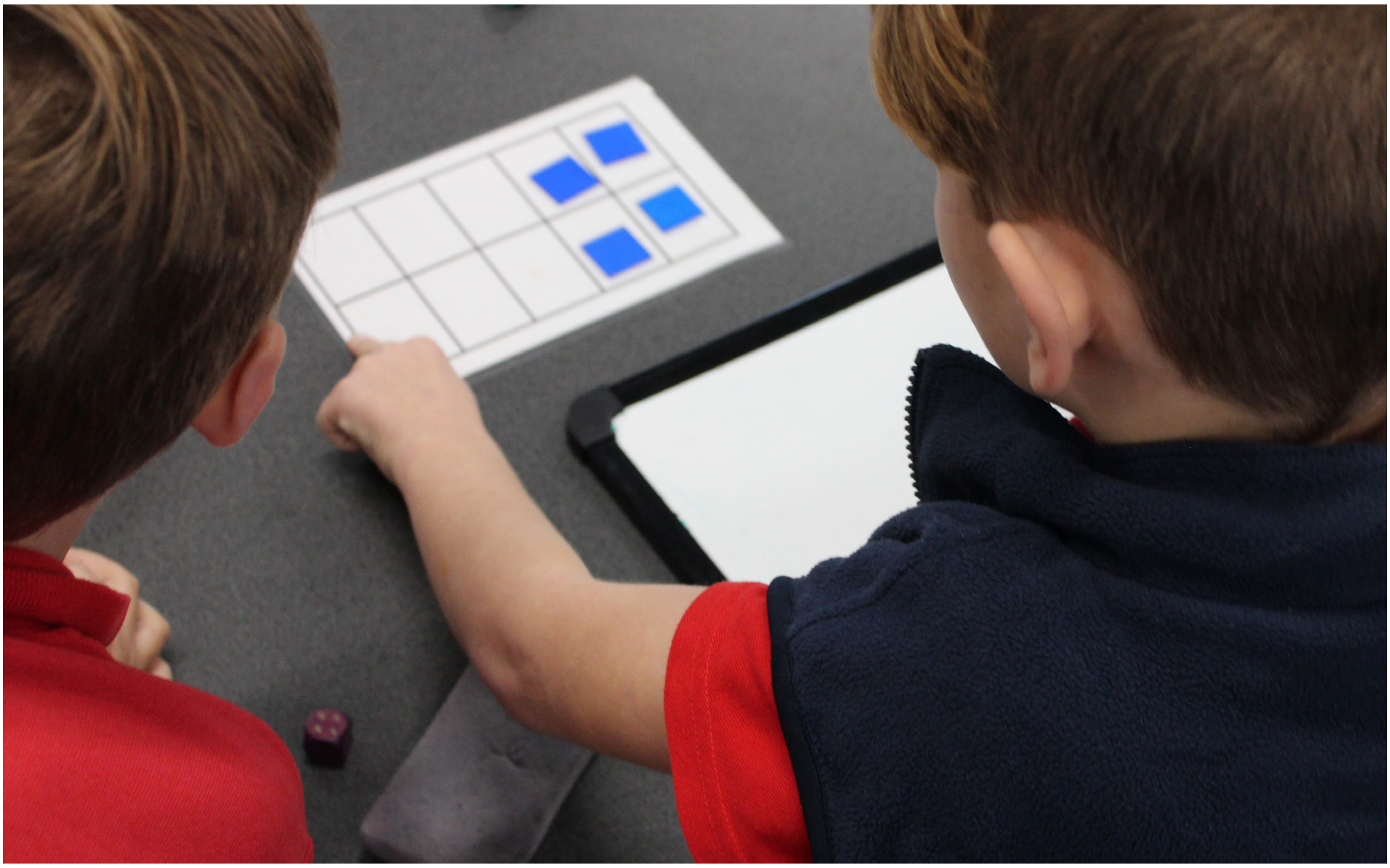

Figure 2. A student 'rolls' then 'makes' four.

collection on the whiteboard and cover the materials with it. Students will then compare. If the other student is unable to read the symbol, they can simply lift the whiteboard up and continue comparing materials.

For the third level, play continues as per materials-and-materials, however this time both students are the recorder. Once students are comfortable, they can simply roll their dice and record the symbol without the need for creating and recording the collection.

USING BENCHMARKS

When using counters, students can develop their number sense by beginning to use benchmarks such as five and ten to compare, rather than simply counting by ones. For example, when comparing seven and three, rather than simply seeing seven as 'bigger' than three, students can begin to understand seven is more than five and three is less than five. When comparing seven and eight, students would see eight as being two less than ten and seven as three less than ten. This allows for those students who are developmentally ready to begin to use partitioning and make up to 10 to extend themselves.

EXTENSIONS

1. A third player can be added to extend the concept to ordering.
2. A second dice can be added, allowing students to explore 2 digit place value.
3. A second dice can be added, allowing students to explore addition, subtraction and multiplication by using the counters to model the operation.
4. Foundation students may use different types of materials (unifix block towers, toy figurines etc) to allow them to remain engaged, if the same activity is used for an entire week. For example, the tens frames might be labelled a paddock and students are charged with the task of working out which paddock has more sheep.

CONCLUDING REMARKS

Any task in which the focus is on the mathematics, rather than on students learning the activity, and that can be adjusted to suit multiple areas of mathematics is potentially powerful for supporting student learning.

Roll, Make, Compare, whilst appearing to be a simple game, allows students to work on a wide variety of numeration skills in an engaging manner with adequate support. I hope you enjoy this activity and I encourage you to add your own further modifications.

MAKE ME 5

Candice Rogers, Foundation Classroom Teacher, Learning and Teaching Leader St Mary Magdelen's Primary School, Clayton and Carmel Delahunty (Godfrey), Monash University

Students are fluent when they calculate answers efficiently, carrying out procedures flexibly, accurately, efficiently and appropriately, and recall factual knowledge and concepts readily (Australian Curriculum and Reporting Authority, 2011).

To this end, under the direction of their teacher, the Foundation students at St Mary Magdelen's School have adapted 'Ready Steady' (Godfrey, 2018). The students now include a 'narrator' in this fluency activity and have called it 'Make Me 5'. The focus of the activity is on the part-whole concept.

Students engaged in 'Make me 5'.

MAKE ME 5

In groups of three, two players use one hand each, in a fist, to beat three times on their knee, saying: 'Make me 5!'. On the fourth 'beat', the players hold up any amount of fingers, for example: Player 1, might show 3 fingers and Player 2, 1 finger. The third player is the narrator whose task it is to add the amount of fingers displayed aloud, and make the appropriate observation, in this example: 'That's not 5, that's less than 5'.

Depending on the focus, the narrator can also state how many less than (or how many greater than) five; and how many more are needed to make 10. Players decide if they will take turns as 'narrator', or change roles when the narrator declares: 'That makes 5!'

Listening to the conversation during this quick activity gives insight into fundamental mathematical skills and concepts including estimation, strategies, vocabulary, reasoning and equivalence.

FLUENCY IS IMPORTANT!

In his paper, Teaching Mathematics Using Research-informed Strategies, Sullivan (2011) reminds us of Skemp's (1986) thoughts about mechanical and automatic skills practice: with automatic practice, built on understanding, students can be procedurally fluent while having conceptual understanding.

Recent thinking about cognitive load and working memory supports the types of understanding or knowledge needed to be employed by students in problem-solving situations. We know that our working memory has a limited capacity. Teachers need to assist students to develop strategies for fluency in calculations as a way of reducing cognitive load - thus allowing more capacity for other mathematical reasoning and actions.

CONCLUDING THOUGHTS

Askew (2009) notes: 'practice enables students to develop fluency and strategies. It is not a case of practice or problem-solving, understanding or rapid recall, but practice and problem-solving, understanding and (appropriate) rapid recall' (p. 2-3).

Often, it is simple activities that work best. The activity in this article needs no preparation or resources and can involve everyone. Do you think automatic practice activities such as 'Make Me 5' could assist in developing fluency and strategies in your classroom?

REFERENCES

Askew, M. (2009) Do it again. *Maths Advice*. Retrieved from mikeaskew.net/ page 3/page 2/files/Mathematicalfluency

Australian Curriculum, Assessment and Reporting Authority (2011) Fluency. Retrieved from https://www.australiancurriculum.edu.au/resources/ mahematics.../portfolios/fluency

Godfrey, C (2018) Three Fluency Activities: Ready, steady, add. *Prime Number*, *33*(1), 6-7.

Sullivan, P. (2011) *Teaching Mathematics: Using research-informed strategies*. ACER Press, Camberwell: Victoria.

ENRICHING MATHEMATICAL DISCUSSION

Jacqui Crane, St Augustine's Primary School

QUIT PING PONG AND JOIN BASKETBALL

Encouraging dialogue and rich discussion in the mathematics classroom has become a focus of current key visions of effective teaching in mathematics around the world. Researchers have highlighted and justified the importance of mathematical discourse with evidence that a classroom that encourages rich dialogue - in which the students, discuss, justify, conjecture and learn from one another - facilitates conceptual understanding and provides opportunity to learn (Teuscher, Switzer & Morwood, 2016; Gervasoni, Hunter, Bicknell & Sexton, 2012).

Mathematical discourse is described as a whole-class discussion where students have opportunities to discuss concepts and ideas, debate amongst themselves for clarity, and reason about their mathematical understanding. Effective questioning is the pinnacle component of this discourse as it has the ability to make students thinking visible and enable the teacher to make connections. However, most teachers usually place themselves in the center with the dialogue 'ping ponging' between teacher and student. Ping Pong questioning or what is more formally referred to as the IRE (Inquiry, Response, Evaluation) framework is described by Cazden (2001) as our 'default' questioning method. A question is asked, one child (usually with hand up) responds and the teacher evaluates the answer ('that's right', 'interesting answer'). In the act of evaluating the answer, the teacher has unknowingly overlooked an opportunity to build on student thinking and has sent the message to the students that the teacher is the authority and knowledge builder. The ideal of active whole class discussion is lost.

As a classroom teacher, I am aware that I revert to the IRE framework as my main pattern of classroom questioning. Ping Pong questioning is a classroom practice that we revert to 'unless deliberate action is taken to achieve some alternatives' (Cazden, 2001, Pg 53). Unless teachers are aware of their reliance on the IRE model, and the fact that it can actually stagnate our opportunities to encourage mathematical discussion, it will continue to be our default teacher-pupil interaction. The simple act of indicating whether a student's answer is correct or incorrect eliminates the opportunity to build upon student thinking and leads the student to the conclusion 'that thought is not required in mathematics class and the way to be successful is to watch the teachers carefully and to copy what they do' (Boaler, as cited by Henning, McKeny, Foley & Balong, 2012, Pg. 458). However, small changes to our questioning practice can allow classroom discourse to become more of a basketball-type interaction, with the dialogue bouncing from one student to another. The remainder of this article describes the beginning of our journey to embracing a richer approach to questioning in our mathematics classrooms.

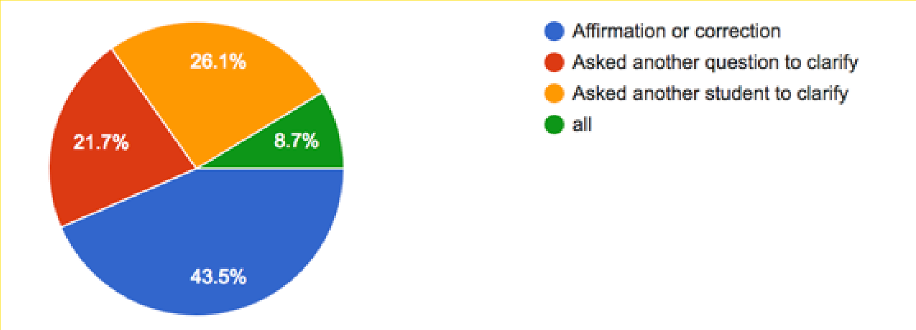

Figure 1. Initial results of survey into how our teachers followed-up student responses.

PLAN

As numeracy leader, I had the opportunity to discuss the use of Ping Pong questioning in mathematics, and highlight how a small change in practice could lead to enhanced mathematical discussions. To begin, I conducted an initial survey of teacher questioning, questioning purpose, and teacher responses after the student had answered. The results showed that teachers were in fact using effective questions and, as suspected, it was the follow up to the student responses that was a reason the majority of the teachers were finding mathematical discussion difficult (see Figure 1).

Using the survey data, I planned a Professional Learning Team (PLT) meeting to discuss the benefits of using alternatives to evaluation within the IRE framework. After a collaborative discussion of research findings and our survey results, teachers were presented with simple alternatives to evaluating student responses to their questions. Teams then conducted their own research into one alternative to present to their peers, highlighting the process and benefits. As a follow up to the meeting, teachers were asked to choose one alternative to trial in their classroom and video for viewing and reflection at the following meeting. At the next PLT meeting, teachers viewed each other's videos and were given time to discuss and reflect on their experience.

ALTERNATIVES TO IRE

ALTERNATIVE #1: WAIT TIME

According to studies, teachers tend to wait about 0.7-1.4 seconds after they ask a question to a student. Studies further suggest teachers tend to give less waiting time to students whom they consider low level (Pearsall, 2012). Increasing the average wait time to 3 seconds has been found to improve the quality of student responses, alter student and teacher expectations and have a positive effect on student attitudes (Rowe, 1986; as cited in Cazden, 2001).

ENRICHING MATHEMATICAL DISCUSSION (CONT.)

A Foundation teacher chose to focus on wait time as a tool to enhance her classroom discussion during mathematics. She reported back to her peers, 'I was surprised that when given wait time to respond 'Student X' was able to answer and was correct. I wasn't expecting him to say anything back.' Over the given period of exploration a number of teachers reported that the implementation of wait time had surprised them in various ways; from receiving a correct response from a child they perceived as low level, to students adding previously absent evidence to their answers without prompting.

ALTERNATIVE #2: NO HANDS QUESTIONING

Ping Pong questioning usually relies on the voluntary participation of students who indicate that they are willing and able to answer the question by placing their hand in the air. Wiliam (2014) argues that this is fundamentally flawed as it is usually the smart children raising their hands and participating in the teacher-pupil dialogue. This has a number of detrimental effects, such as the teacher using the answers gathered from the minority to evaluate collective understanding and a lack of engagement of all students in the discussion. If a student does not voluntarily raise their hand, they know they will not be called upon and disengage from the discussion entirely.

The no hands up rule ensures that all students are required to respond and participate. If not relying on the usual suspects to respond, the teacher has the opportunity to evaluate the collective understanding of students from a wider range and the students who previously disengaged are forced to pay attention. The practice of no hands up has the opportunity to remove the stigma of 'wrong answers' (Pearsall, 2012). By changing the mindset that wrong answers are viewed as an evaluation of student ability to the view that they are opportunities to learn, the teacher is ultimately encouraging a growth mindset. Wiliam (2009) argues that if the responses a teacher receives are always correct then the teacher is negligent in their teaching, as they are not requiring students to struggle and think.

ALTERNATIVE #3: POSE, PAUSE, POUNCE, BOUNCE

Wiliam's (2009) process of Pose, Pause, Pounce, Bounce, also known as basketball questioning, involves posing a question, pausing to allow for students to formulate an answer, pouncing on a student to answer the question and bouncing to another student who evaluates. Another student is then called upon to explain how or why the response is right or wrong. Hands up are discouraged during this process. During Ping Pong questioning, very few students are engaged in the dialogue. The Pose, Pause, Pounce, Bounce alternative ensures a larger percentage of the class respond and engage. It removes the teacher as authority as it is the students who are evaluating each other's answer and providing the justification.

ALTERNATIVE #4: PHONE A FRIEND

A typical scenario in any class is when the teacher calls upon a student and their response is 'I don't know' or 'I forgot'. This student clearly wants to be viewed by their peers as knowing as much as them so has placed their hand in the air to fit in. Phone a friend or question relay (Pearsall, 2012) allows a student who has been called upon to admit they don't know but it is then their responsibility to find out the answer. They simply ask three other class members to explain what they are thinking and then put these explanations into their own words.

Teachers identified the benefits for using 'Phone a friend' to be that it promotes mathematical discussion between students and also helps other students explain their thinking and understanding. It limits reliance on the teacher for support and it emphasises the importance of mathematical thinking. A Year 1 teacher reported that her students enjoyed using phone a friend, and a larger percentage of children were involved in the classroom discussion due to the fact they knew they could ask for assistance from their peers when they were unsure of an answer.

ALTERNATIVE #5: THUMBS UP

The thumbs up alternative has become an important component of Number Talks, a short daily routine that helps students build computational fluency. The hand signals are used to indicate children's thinking progress. A fist placed on the chest to show the student is thinking of a response, a thumbs up to reveal the student has an answer and raising more fingers signifying the student has more than one strategy. The hand signals give children the opportunity to think quietly without the pressure of seeing hands waving in the air around them but still challenges those children who arrive at the answer quickly.

The junior teachers in particular loved the idea of this alternative. They thought thumbs up would alleviate students putting their hand in the air just to look good to peers as their willingness to share is kept more private, and you as the teacher know from their hand signs who has an answer. Teachers noticed from viewing this alternative in action via the video examples that the classroom was calmer and all students were visibly thinking about the answer.

REFLECTION

On many occasions when I have had the privilege of hearing Peter Sullivan speak he has repeatedly said, 'Teachers are too nice.' Not many teachers will admit that they enjoy seeing a student struggle and it is for this reason I believe we rely on this Ping Pong style of questioning during classroom discourse. We think we are providing the child with timely feedback and are acknowledging their mathematical ability with an affirmation. However research has proven this practice is fundamentally flawed as it has very low pupil participation rate, ignores mathematical thinking, is reflective of only a small portion of the classes' responses and places the teacher in the role of knowledge builder and authority (Wiliam 2014).

Although our trial of alternatives to IRE has just begun, the teachers have already acknowledged significant positive changes in their classroom discussion. Due to the use of the thumbs up strategy, discussion time is calmer and not all about waving your hand around in the air for attention; being able to phone a friend has alleviated the stress of a teacher calling upon you when you are not prepared; wait time has given student thinking the much needed focus; and basketball questioning has elicited greater participation from students and is allowing them to build upon each others ideas.

So what about you and your classroom, are you ready to quit Ping Pong and try basketball?

REFERENCES

Cazden, C. B. (2001). Variations in lesson structure. In *Classroom discourse: The language of teaching and learning* (pp. 53-79). Portsmouth, NH: Heinemann.

Gervasoni, A., Hunter, R., Bicknell, B., & Sexton, M. (2012). Powerful pedagogical actions in mathematics education. In B. Perry, T. Lowrie, T. Logan, A. MacDonald, & J. Greenlees (Eds.), *Research in Mathematics Education in Australasia 2008-2011* (pp. 193-218). Rotterdam, The Netherlands: Sense Publishers.

Henning, J. E., McKeny, T., Foley, G. D., & Balong, M. (2012). Mathematics discussions by design: creating opportunities for purposeful participation. *Journal of Mathematics Teacher Education*, *15*(6), 453-479. doi:10.1007/s10857-012-9224-1

Pearsall, G. (2012). *The top 10 strategic questions for teachers*. Retrieved from http://tta.edu.au/free-ebook

Solution tree. (2009, October 9). *Solution tree: Dylan Wiliam, content then process* [Video file]. Retrieved from https://www.youtube.com/watch?time_continue=135&v=029fSeOaGio

Teuscher, D., Switzer, J. M., & Morwood, T. J. (2016). Unpacking the practice of probing student thinking. *Mathematics Teacher Educator*, *5*(1), 47-64. doi:10.5951/mathteaceduc.5.1.0047

Wiliam, D. (2014). The right question, the right way. *Using Assessments Thoughtfully*, *71*(6), 16-19. Retrieved from http://www.ascd.org/publications/educational-leadership/mar14/vol71/num06/The-Right-Questions,-The-Right-Way.aspx

BLIND NOUGHTS AND CROSSES

James Russo, Monash University and Toby Russo, Bell Primary School

Promote visualisation, spatial reasoning and strategic thinking with this twist on the traditional game, Noughts and Crosses (Tic-Tac-Toe) by having one of the players play blind! Blind Noughts and Crosses is inspired by a problem outlined in an article we read recently (Roeder, 2018). It is suitable for students in Year 1 onwards (so far, we have played it with Year 2 and Year 5/6 students).

To begin, get children to pair up and play several rounds of regular Noughts and Crosses to (re)familiarise themselves with game-play (and potentially game-strategy).

Next, model blind Noughts and Crosses with the class. Begin by creating a Noughts and Crosses board, and include the digits 1 to 9 (see picture).

1	2	3
4	5	6
7	8	9

The blind player turns their back to the board, and plays first. They choose to place an X in one of the squares by stating the corresponding number. For example, they might say '5', whereby their opponent (the seeing player) would record an X in the centre square. The seeing player would then place a O somewhere on the board, and say 'Your turn'. If, at any point, the blind player chooses a square that is already taken, they are told by the seeing player to 'Try again'.

With practice, we found that many children can learn to visualise the board whilst playing blind, which improves their chances of drawing (and even winning) the game. Older children can contemplate whether there is an optimal strategy that can allow the blind player to (at least) draw every game they play (the original problem in the article we came across). We love this activity, and hope your class has fun with it!

REFERENCE

Roeder, O. (2018, December 14). Can You Win Tic-Tac-Toe Blindfolded? Retrieved from https://fivethirtyeight.com/features/can-you-win-tic-tac-toe-blindfolded/

PRE-SERVICE TEACHER CONTRIBUTION: MULTIPLYING NEGATIVE NUMBERS

Daniel Osment
La Trobe University

HOW SHOULD WE TEACH STUDENTS ABOUT MULTIPLYING NEGATIVE NUMBERS?

The concept of multiplying negative numbers has long been an issue for students and teachers. From the moment negative numbers are introduced to students, questions arise as to why two negatives make a positive, yet the answers are not always adequate (Crowley & Dunn, 1985). Negative numbers are probably the first non-intuitive notion that students will face (Arcavi & Bruckheimer, 1981), which means that conceptual explanations need to be clear, simple and relatable to real-life situations.

Booker, Bond, Sparrow, and Swan (2014, p. 22) explain that the idea of negative numbers has been problematic for mathematicians for centuries. The concept of a negative number appeared meaningless and was often ignored. Negative numbers are, therefore, a relatively recent addition into mathematics instruction, however there is not a clear account of the concept's historical development (Crowley & Dunn, 1985).

Patterns have been used based on known mathematics to explain multiplying a positive number by a negative number using repeated addition, however, the multiplication of two negative numbers still remains difficult to explain (Booker et al., 2014, p. 22). Students understand that $4 \times 3 = 12$ and that $4 \times (-3)$ signifies a repeated addition toward the negative, resulting in (-12).

It is not so simple to teach and demonstrate how $(-4) \times (-3) = (+12)$. Although there are inductive methods that involve adopting the patterns we already know to help us understand what we do not know, this does not equate to a mathematical proof that a negative multiplied by a negative is equal to a positive number (Arcavi & Bruckheimer, 1981; Booker et al., 2014).

One can offer a procedural explanation for multiplying negative numbers. Martin Gardner puts it plainly, 'Minus times minus equals plus – the reason for this we need not discuss' (Gardner, 1977, p. 132). A negative number multiplied by a positive number is always negative. A negative number multiplied by a negative number is always positive. This explanation has come under heavy scrutiny because it does not have any connection with personal experience or conceptual understanding (Arcavi & Bruckheimer, 1981; Blume & Schrock, 2017; Crowley & Dunn, 1985).

Booker et al. (2014, p. 22) explains that in the tenth century, Arab mathematicians proposed the idea that numbers can be viewed as a distance from zero on a number line. Consequently, 1 on a number line represents a distance of one unit length from zero. This explanation of positive numbers can be extended to represent negative numbers on the number line preceding zero.

REPEATED ADDITION AND NUMBER LINES

This starting point also provides one explanation of the multiplication of negative numbers by considering multiplication as repeated addition. For example, $3 \times (-2)$ can be written as $(-2) + (-2) + (-2) = (-6)$. I like to describe this type of algorithm as three people with a debt of $2 each. If the total debt were to be counted there would be $6 debt in total. At this point in learning, some students may begin to have confidence in multiplying negative numbers (Booker et al., 2014). See Figure 1.

This use of repeated addition is a common introduction to the multiplication of negative numbers, as it begins with a procedure that the learner is familiar with. Blume and Schrock (2017) describe $3 \times (-2)$ as a repeated addition having three addends, each of which is (-2). However, this idea of addends is difficult to understand when the product is a negative number, for example $(-3) \times (-2) = ?$.

PATTERNS AND SEQUENTIAL NUMBER SENTENCES

Another approach is to consider patterns generated through sequential number sentences. Patterns exist throughout mathematics and can be helpful in answering questions where some information is missing (Thornton, 1977). Booker et al. (2014, p. 22) show how using patterns that begin with what we already know (multiplying positives with positives) can be helpful to demonstrate what happens when the product falls below zero and becomes a negative.

$3 \times 1 = 3$

$2 \times 1 = 2$

$1 \times 1 = 1$

$0 \times 1 = 0$

$(-1) \times 1 = -1$

$(-2) \times 1 = -2$

$(-3) \times 1 = -3$

These examples are clear demonstrations that a positive number multiplied by a negative number is always a negative number. In order to move forward to explain the product of two negative numbers we can simply extend the pattern out to include two negatives.

The following example demonstrates how multiplying any number by (-1) results in a change in direction on the number line, or a change in its positive or negative symbol.

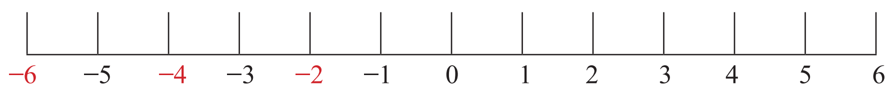

Figure 1. 3 × (-2) as repeated addition towards the negative

$(-1) \times 3 = (-3)$

$(-1) \times 2 = (-2)$

$(-1) \times 1 = (-1)$

$(-1) \times 0 = 0$

$(-1) \times (-1) = 1$

$(-1) \times (-2) = 2$

$(-1) \times (-3) = 3$

REAL-LIFE CONTEXTS

Alternatively, Mills (2013) uses a real-life situation to conceptually describe multiplying negative numbers by referring to cheques and bills. 'Giving' represents a positive and 'taking' represents a negative, while 'bills' are negative and 'cheques' are positive. For example, if I give you two cheques for \$10, you are \$20 better off ($2 \times 10 = 20$). If I take from you two cheques for \$20, then you are \$40 worse off ($-2 \times 20 = -40$). However, if I take from you two bills of \$5 each, you will essentially have \$10 more. Another way to describe this could be to say 'I twice take from you negative 5', which can be written as $(-2) \times (-5) = 10$. Arcavi and Bruckheimer (1981) offer a similar analogy of good and bad people moving into and leaving town.

CONCLUDING THOUGHTS

How should we teach students about multiplying negative numbers?

- Always begin with students' prior knowledge – this may begin with the addition of a negative and a positive number.
- Number lines are a suggested tool for conceptual development (Beswick, 2011).
- Beswick (2011) suggests a range of contexts in students' lives that can be helpful in initiating thinking processes of numbers below zero other than debt, such as temperature, sea level and underground levels in mining or in building elevators.
- Do not provide the procedure until students have understood the topic conceptually.
- Encourage students to engage in non-algorithmic thinking and model their thinking strategies.
- Encourage students to provide reasoning and proof for their thinking (Reys et al., 2012).

ACKNOWLEDGEMENTS

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REFERENCES

Arcavi, A., & Bruckheimer, M. (1981). How shall we teach the multiplication of negative numbers? *Mathematics in School*, *10*(5), 31-33. Retrieved from http://www.jstor.org/stable/30214312

Beswick, K. (2011). Positive experiences with negative numbers: Building on students in and out of school experiences. *The Australian Mathematics Teacher*, *67*(2), 31-40. Retrieved from https://search.informit.com.au/documentSummary;dn=178499160077442;res=IELHSS

Blume, G., & Schrock, C. (2017). Facilitator's guide for the product of two negative numbers. In R. Zbiek, G. Blume & M. Heid (Eds.), *Facilitator's guidebook for use of mathematics situations in professional learning* (pp. 29-58). Charlotte, NC: Information Age Publishing Inc.

Booker, G., Bond, D., Sparrow, L., & Swan, P. (2014). *Teaching primary mathematics* (5 ed.). Frenchs Forest, N.S.W.: Pearson Australia.

Crowley, M., & Dunn, K. (1985). On multiplying negative numbers. *The Mathematics Teacher*, *78*(4), 252-256. Retrieved from http://www.jstor.org/stable/27964483

Gardner, M. (1977). Mathematical games. *Scientific American*, *236*(6), 131-134. doi:10.1038/scientificamerican0177-110

Mills, T. (2013). Negative times negative. *Vinculum*, *50*(3), 13. Retrieved from https://search.informit.com.au/documentSummary;dn=540172392790094;res=IELAPA

Reys, R., Lindquist, M., Lambdin, D., Smith, N., Rogers, A., Falle, J., Frid, S., & Bennett, S. (2012). *Helping children learn mathematics* (1st Australian ed.). Milton, Queensland: John Wiley & Sons Australia.

Thornton, C. (1977). A glance at the power of patterns. *The Arithmetic Teacher*, *24*(2), 154-157. Retrieved from http://www.jstor.org/stable/41191349

ULURU

Toby Russo, Bell Primary School and James Russo, Monash University

USE THIS PICTURE AS A STIMULUS FOR CLASSROOM TASKS

Each task is linked to the relevant content descriptor from the Victorian Curriculum.

Use this image of Uluru to explore a series of challenging mathematical problems.

FOUNDATION - YEAR 2

1 There are many different ways to get from Melbourne to Uluru and back again: fly, drive or by train. Plan a trip from Melbourne to Uluru, and back again. What are some of the different ways you might travel? To get to Uluru from Melbourne, it takes 3 hours to fly, 25 hours to drive or 35 hours by train (via Adelaide). Order the different travel combinations from shortest to longest return trip.

YEARS 3 AND 4

2 Each year 300,000 people visit Uluru. On average, approximately how many people visit per day? (hint: Is it more or less than 1000? 100? 500?). April to October is the most popular time to visit Uluru, as it is not too hot. Create a table showing how many people might visit Uluru each month (hint: make sure the total number of visitors adds up to 300,000). Uluru has a circumference of 9.4 km. If, on the busiest day of the year, all the visitors to Uluru spread out and held hands, could they make it all the way around? Explain your reasoning.

VICTORIAN CURRICULUM REFERENCE →

Recognise, model, read, write and order numbers to at least 100. Locate these numbers on a number line (ACMNA013)

Solve simple addition and subtraction problems using a range of efficient mental and written strategies (ACMNA030)

Apply place value to partition, rearrange and regroup numbers to at least tens of thousands to assist calculations and solve problems (ACMNA073)

Develop efficient mental and written strategies and use appropriate digital technologies for multiplication and for division where there is no remainder (ACMNA076)

YEARS 5 AND 6

3 Uluru has a circumference of 9.4km. Estimate how long it would take you to walk around Uluru. Measure a distance in your school and time how long it takes you to walk that distance. Using this information, can you make a more accurate estimate for how long it might take to walk around Uluru? In the blistering heat of central Australia, you can only walk two-thirds your normal pace. How long would it take you to walk around Uluru at that rate?

Solve problems involving division by a one digit number, including those that result in a remainder (ACMNA101)

Solve problems involving the comparison of lengths and areas using appropriate units (ACMMG137)

EXTRA CHALLENGE

4 Uluru is thought to be 600 million years old. The Anangu People are the Aboriginal traditional land owners of Uluru and have considered it a sacred sight for at least 60,000 years. Uluru has been a major tourist attraction for around 60 years, since the 1950s. Can you create a proportional timeline that represents the history of Uluru?

Uluru is about 350m high, 3.6km long and 2.4km wide. How much do you think it weighs? Hint: 1 cubic meter of rock weighs about 3 tonnes (3,000kg).

The Prime Number team are always on the lookout for mathematically stimulating images. Contact the editor if you have a photo or a suggestion, email office@mav.vic.edu.au.

RESOLVE: REASONING WITH 3D OBJECTS

Andrew Noordhoff, Year 3 teacher,
Jells Park Primary School

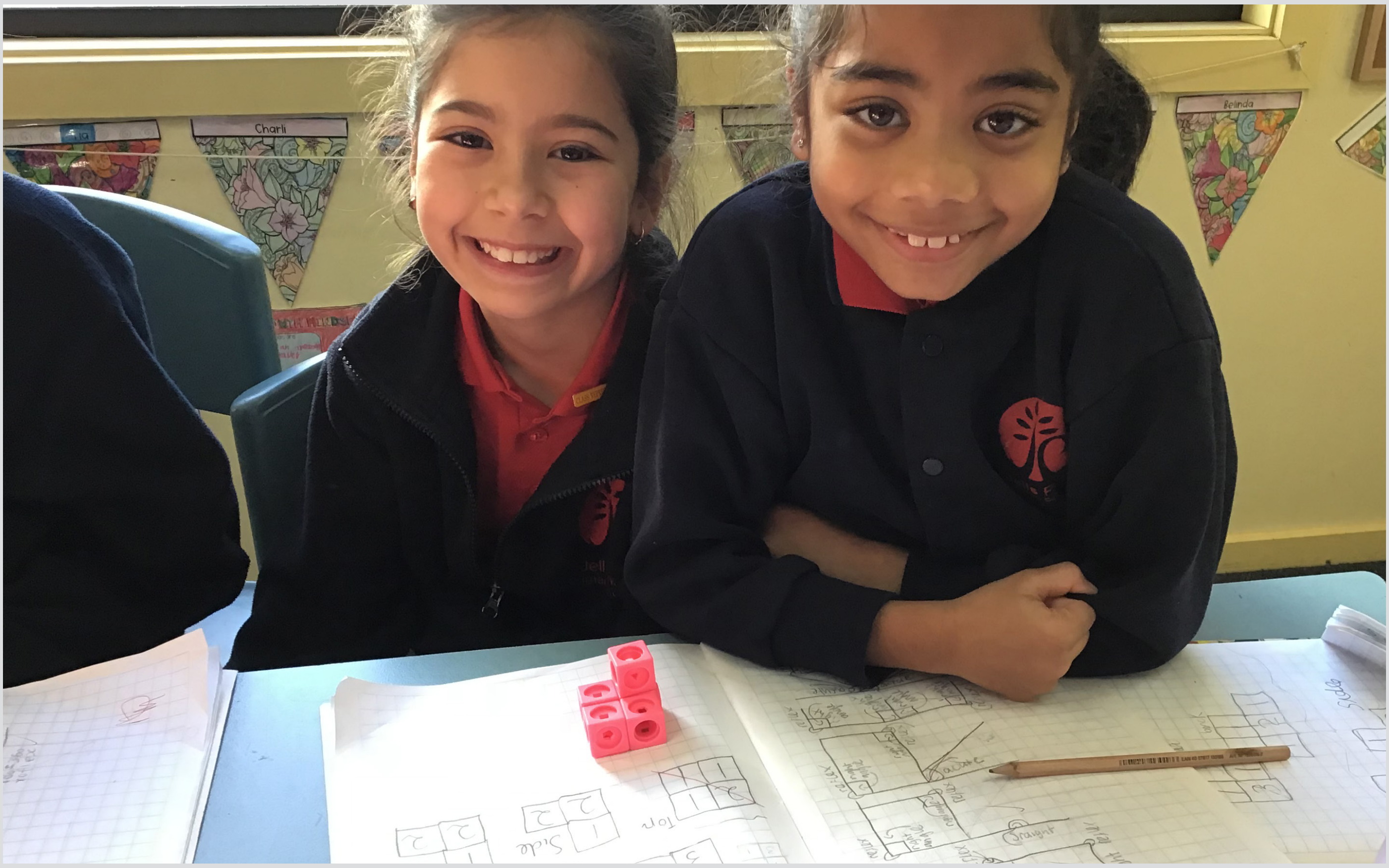


Figure 1. Students find a solution to the Building Symmetry task.

In 2016, at the MAV annual conference, one of the workshops I attended was an introduction into the 'ReSolve: Maths by Inquiry' program. We worked on a couple of tasks in the session and they were highly engaging so I was determined to find out more.

The ReSolve program was looking for 'Champions' to trial and embed their lessons into schools, so I jumped at the chance and was accepted as part of the 'Champions' program. Since then, I've been working hard to empower my colleagues to incorporate ReSolve lessons into their classrooms and the feedback and results have been excellent.

The resources are great because they are incredibly well thought out, are purposeful for the students, accessible to all learners and finally promote the spirit of inquiry and curiosity in mathematics. The feedback from staff are that lessons plans are very detailed, so staff who are not as confident in their ability to teach mathematics can pick them up easily and deliver an engaging lesson. In most cases, all of the additional resources required to teach the lesson are included.

In our Year 3 classrooms we have been working through the 'Reasoning with 3D objects' sequence. It presented as a great opportunity to do some team teaching and, as we have streamed Numeracy groups, it allowed us to work between ability groups with students sharing their knowledge with other students not from their normal maths group.

The first lesson of the sequence is called 'Building Symmetry' and, as the name suggests, it involves building symmetry. The students were given 5 interlocking blocks and asked to recreate a pattern with the blocks from a picture on the board (from the accompanying PowerPoint).

Then, working with partners, they tried to move one block at a time to make the model of blocks that was not symmetrical become symmetrical (see Figure 1).

Many of the students quickly found one or two possibilities but were encouraged to continue to go back and see if they could find more. By keeping all of their previous models together, it allowed them to see what they had already done and understand how some of their models, when rotated, were actually the same - therefore they hadn't found a new solution and went back to thinking through other possibilities.

Once they had made their 3D model, the students drew a top view, side view and front view of their 3D model in their books. They used numbers to show the depth of the blocks and drew a line of symmetry each time it was present in each of the views (which was not always the case). Importantly the students worked together to discuss what they were seeing and how to transfer this to their books (see Figure 2).

Figure 2. Students draw their four unique solutions in their workbook.

The second lesson of the sequence is called '4 Cubes' and centres on using 4 interlocking blocks to make a unique model. There are 8 possible unique models that can be made (try it for yourself!).

To enhance the task, we made a flip booklet for the students. Working with their partner, they created a unique 3D model using 4 blocks and brought it to the teacher. If the model was unique, they would receive a photo of that model to stick on the front of their booklet. Then they would again draw a top view, a front view and finally, using isometric paper, they would try to draw it as a 3D model, which was a new (and tricky) skill for many of the students.

The importance of students working together and discussing their models and their drawings was an integral part of the sessions and required patience from some, but overall working in partners enhanced the learning of all students.

Once the students had completed their drawings for the first 3D model, they presented back to the teacher, and went to work on a new 3D model. Again, students were encouraged to keep their original 3D models so they could see what they had already done and discussions around similarities and differences could occur. At the end of the hour, most partners had at least three 3D models built and drawn up.

Figure 3. Students find all eight solutions to the 4 Cubes task.

The next day, we completed our second hour of the '4 Cubes' lesson. The students were set the challenge to see if they could build and draw the remaining 3D models before the hour was over. As students started working towards their final few 3D models, things got a little trickier.

As students continued to move their 4 blocks around trying to create new unique models, they often realised it was a flipped or rotated version of one of their previous solutions. This was often met with a wry smile, and a determination to continue to persevere to make the remaining models, followed by a sense of achievement when a new model was found. This was particularly the case when the last elusive model was discovered (see Figure 3).

From a teacher perspective, our Year 3 team noted it was great to not only discuss making 3D models and symmetry, but as suggested above, the concepts of flipping and rotating an object, not to mention all the skills involved in drawing their models!

The hands on nature of the lesson and students manipulating the blocks and discussing their findings with a partner worked really well and the students were highly engaged throughout.

The lesson sequence is not something we would ever have envisioned on our own and there are many more like these on the ReSolve website. We intend to teach more of these terrific lessons over the course of the year. I'd encourage teachers and educators to visit the ReSolve website and trial a lesson in your classroom!

For more information, and for free downloads of lesson materials, visit www.resolve.edu.au

THE FOOD TRUCK PROJECT
AN EXTENDED INVESTIGATION

Joshua Teo, Dandenong West Primary School

When I ask my students which class activities they enjoyed this term, the most common answers will be the most fun and engaging activities. When I follow up the question with what they were learning in the activity, the answers tend to be slower in response. What is at the forefront of students' minds are activities and projects that grab their curiosity, allowing them to be creative and have a vested interest. Mathematics education experts have argued that:

'… it is critical that teachers use 'worthwhile tasks' which is interpreted to mean they are meaningful and relevant to the students' (Anthony and Walshaw, 2009, as cited in Sullivan, 2011).

At Dandenong West Primary School, we use challenging tasks, open-ended tasks, and problem-solving activities to engage kids in mathematics, but these tasks tend to be one or two lessons long. If students value these activities, then how can we evolve a singular investigation into a multi-purpose project? The 5/6 team explored applied mathematical projects that ranged from 5 to 10 weeks. Students worked in pairs or threes and engaged in the project once or twice a week. Our most recent project was to create a food truck for Queen Victoria Market and we believe it was made up of many worthwhile tasks.

Design Brief - Stage 1

Congratulations! You and your business partner have been successful in your application for a grant to establish a food truck business at Queen Victoria Market.

Please see attached your Food Vendors License.

Queen Victoria Market has been a Melbourne icon for 140 years. There is a range of fresh food, cafes shops and general merchandise. Food trucks bring more diverse offerings into the City of Melbourne. You and your partner can realize your food dream and begin your entrepreneurial business, but first we need to learn about the area.

Step 1: Become familiar with Queen Victoria Market by writing down the directions on how to get from:

- General Merchandise E to the Food Court
- Shops & Cafes on Elizabeth Street to Eat at F Shed Shops
- Therry Street shops to Dairy Hall to Victoria Street Shops
- The corner of Elizabeth Street and Therry Street to Peel Street by car.
- Test your business partner by choosing two locations to visit.

Figure 1. Stage one of the design brief.

THE DESIGN BRIEF

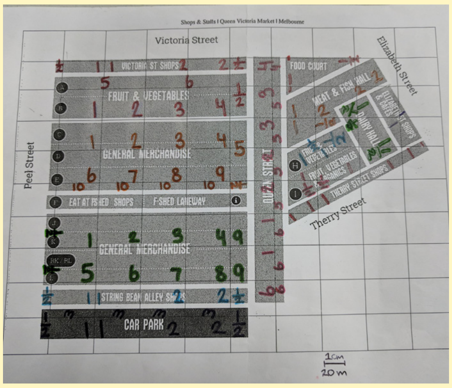
A Design Brief is an outline of a project. It includes ordered steps detailing the tasks they need to complete. In this project, the Design Brief included steps to be a successful food truck vendor. This helped students keep track of their own progress and to know what was expected of them. The design brief also serves as a term overview for teachers. It kept us on track, ensuring we focussed on the skills and strategies we wanted students to learn. Having a pre-planned unit helped teachers feel more confident and prepared.

Our 5/6 team created a Design Brief by looking at the Victorian Curriculum learning areas and brainstorming an overarching investigation that could include them. We created steps based on what the sequential process might be for setting up a food truck at a market. Each step was its own challenging task or problem-solving activity that was open-ended and creative.

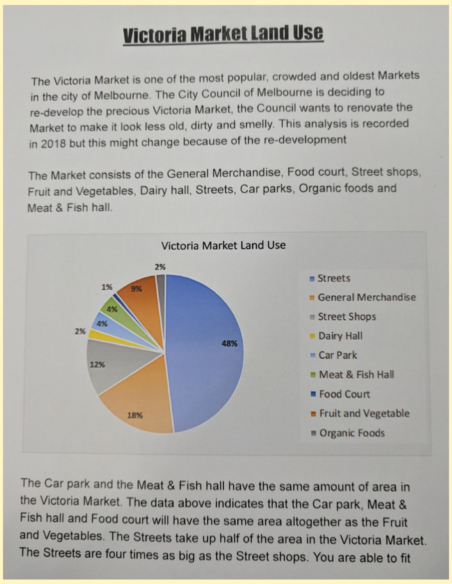
THE FOOD TRUCK PROJECT

The Design Brief for our food truck project consisted of three stages. Stage 1 was focused on becoming familiar with Victoria Market. Using a map of the market, students were asked to find different locations and write down instructions on how to navigate between them. Students recreated the map to include the following elements: border, orientation, legend, title, scale and source. Using the map's scale, we explored the perimeter and area of different sections of the market (see Figure 1).

By laying a 100 square grid over the market map, we were able to calculate the percentage of land used for different purposes in the market. Students were able to access this question at different levels. If one grid square was mainly the merchandise area, some students chose to label that whole square as merchandise. While other students could justify the square being about half merchandise and half Queen Street. Some students were more specific and viewed it as 0.6 merchandise and 0.4 Queen Street. How far they took this, depended on their level of understanding of part/whole and decimal fractions.



The final step was to use Microsoft Excel to create a pie graph showing the percentages of Victorian Market land use. Students then used the pie graph to write an analysis; comparing their data and describing their findings. Throughout Stage 1, we used the same map to look at multiplication, location, directions, mapping, scale, fractions, percentage, perimeter, area, pie graphs, data representation and interpretation.



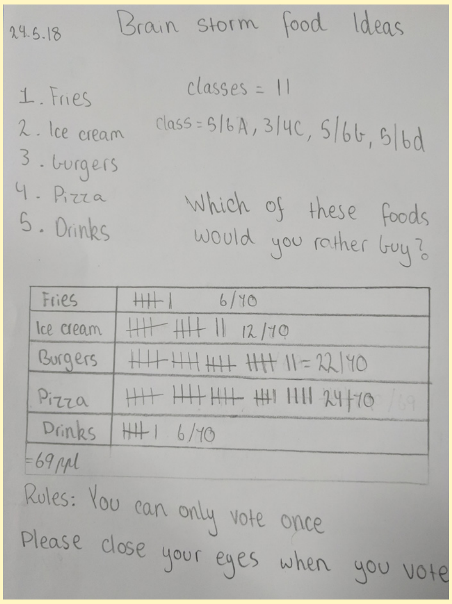
Victoria Market Land Use

The Victoria Market is one of the most popular, crowded and oldest Markets in the city of Melbourne. The City Council of Melbourne is deciding to re-develop the precious Victoria Market, the Council wants to renovate the Market to make it look less old, dirty and smelly. This analysis is recorded in 2018 but this might change because of the re-development

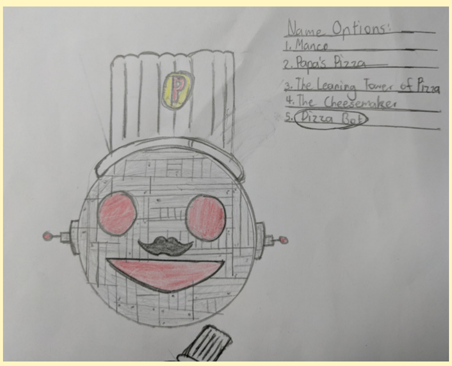
The Market consists of the General Merchandise, Food court, Street shops, Fruit and Vegetables, Dairy hall, Streets, Car parks, Organic foods and Meat & Fish hall.

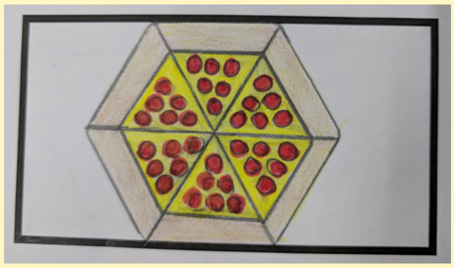
The Car park and the Meat & Fish hall have the same amount of area in the Victoria Market. The data above indicates that the Car park, Meat & Fish hall and Food court will have the same area altogether as the Fruit and Vegetables. The Streets take up half of the area in the Victoria Market. The Streets are four times as big as the Street shops. You are able to fit

Stage 2 of the Design Brief was focused on what food students were going to sell. Students brainstormed possible food options and created a tally sheet to collect data for a survey from multiple classes. A short speech was prepared to detail the purpose and parameters of the survey and presented to these classes.



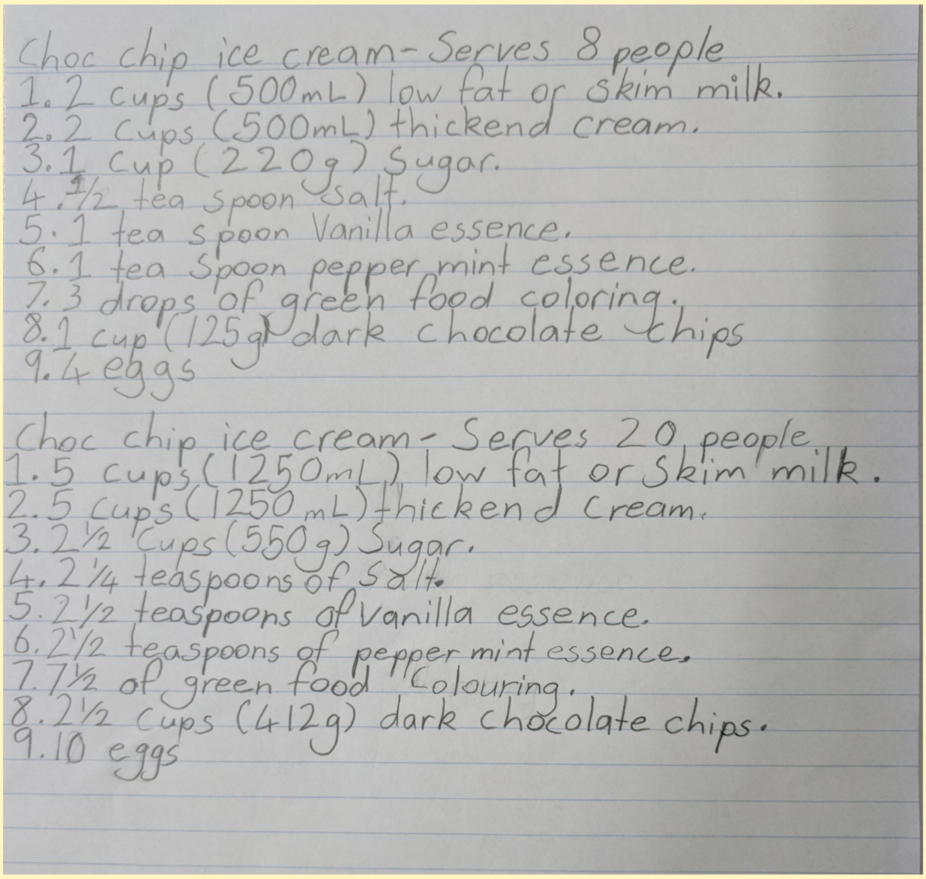
They then used Microsoft Excel to create column and pie graphs to describe their findings. After deciding on the type of food, they began creating a brand name and a simple, yet unique logo.





Stage 3 of the Design Brief focused on measurement. Students conducted another survey with different pricing options to understand how much people were willing to pay for their product. They created different serving size options and a detailed description of what they were getting for their money. Students found a recipe for their food, noting down the ingredient quantities and serving size. They then calculated the weight and volume of different ingredients required to make 20 of their product using doubling, tripling and halving strategies. For example, if one pizza has 8 slices, then 2.5 pizzas has 20 slices requiring you to double and add half of the ingredients (see Figure 2).

We started projecting revenue in an hour, a week, a month and a year, using the Queen Victoria Market website's estimate of 26,000 visitors at the market every day. The answers varied depending on the extent of their understanding of the question. Some groups worked out



Choc chip ice cream-Serves 8 people
1. 2 cups (500mL) low fat or skim milk.
2. 2 Cups (500mL) thickend cream.
3. 1 Cup (220g) Sugar.
4. 1/2 tea spoon salt.
5. 1 tea spoon Vanilla essence.
6. 1 tea Spoon pepper mint essence.
7. 3 drops of green food coloring.
8. 1 cup (125g) dark chocolate chips
9. 4 eggs

Choc chip ice cream- Serves 20 people
1. 5 cups (1250mL) low fat or Skim milk.
2. 5 cups (1250 mL) thickend cream.
3. 2½ Cups (550g) Sugar.
4. 2¼ teaspoons of Salt.
5. 2½ teaspoons of Vanilla essence.
6. 2½ teaspoons of pepper mint essence.
7. 7½ of green food Colouring.
8. 2½ Cups (412g) dark chocolate chips.
9. 10 eggs

Figure 2. An examples of 'scaling up' ingredient quantities.

26,000 x 7 days, while some realised that the market is closed on Monday and Wednesday. Some took into account the night markets on Wednesday night at certain times during the year. The more advanced students tried to consider how many people out of 26,000 would actually buy their food (see Figure 3).

IMPLEMENTATION

The Design Brief required more pre-planning, but we found the overall term for mathematics was more organised. Our team became more cohesive because the overall direction was clear and there was a common understanding of current and future learning goals and objectives.

While groups were working independently, I conferenced with a few to share their progress and current challenges. This allowed students to listen to other groups' solutions and ways of thinking. It also gave students a chance to present their work, explain their choices and receive feedback. We also ran focus groups to support students who were struggling with certain strategies or concepts.

In planning, we often referred to the Design Brief to know what students needed to learn next to progress in their project. Our weekly mathematics planners were structured to teach skills, strategies and concepts through other challenging tasks, to scaffold student learning so that they may transfer their understandings to their project. Small groups gave us formative data on student progress and helped us to focus our teaching on what students currently need.

THE BENEFITS OF EXTENDED MATHEMATICS PROJECTS

The biggest improvement was in the student discussions. Our Design Brief gave us more opportunities to develop the proficiencies in our classrooms. Working in groups allowed students to improve their reasoning and problem solving by engaging in meaningful maths conversations with their peers. Their discussions encouraged students to distinguish between subjective and objective judgement and challenged them to justify their choices. We heard discussions such as these:

- 'If I charge $4.50 for a slice of pizza, less people would want to buy it.'
- 'The logo needs to be simple, so that more people can recognise it.'
- 'Let's choose hotdogs because less people will buy ice cream in winter.'
- 'Burgers and fries was the most voted food, but it will cost too much to get the deep fryer, freezer and a stove. Let's choose the second favourite, which was milkshakes.'

At our student led conferences, we found many students were able to articulate what they had done in their food truck project and why to their parents or guardian. This demonstrated their reasoning and understanding of what they have learned.

This mathematics project taught students about decision-making. The Design Brief provided a model for students on how to think logically in steps. Instead of a student deciding to sell ice cream impulsively because they like ice cream, the student found out what people want to buy to make an informed decision. The open-endedness gave students control over their choices, allowing for creativity.

Students were able to make connections between what they were learning and its real-life application. When students understood that each lesson's learning goal had a direct impact on their project's success, they became more attentive and motivated to practice their skills and strategies because they knew every lesson was meaningful and purposeful.

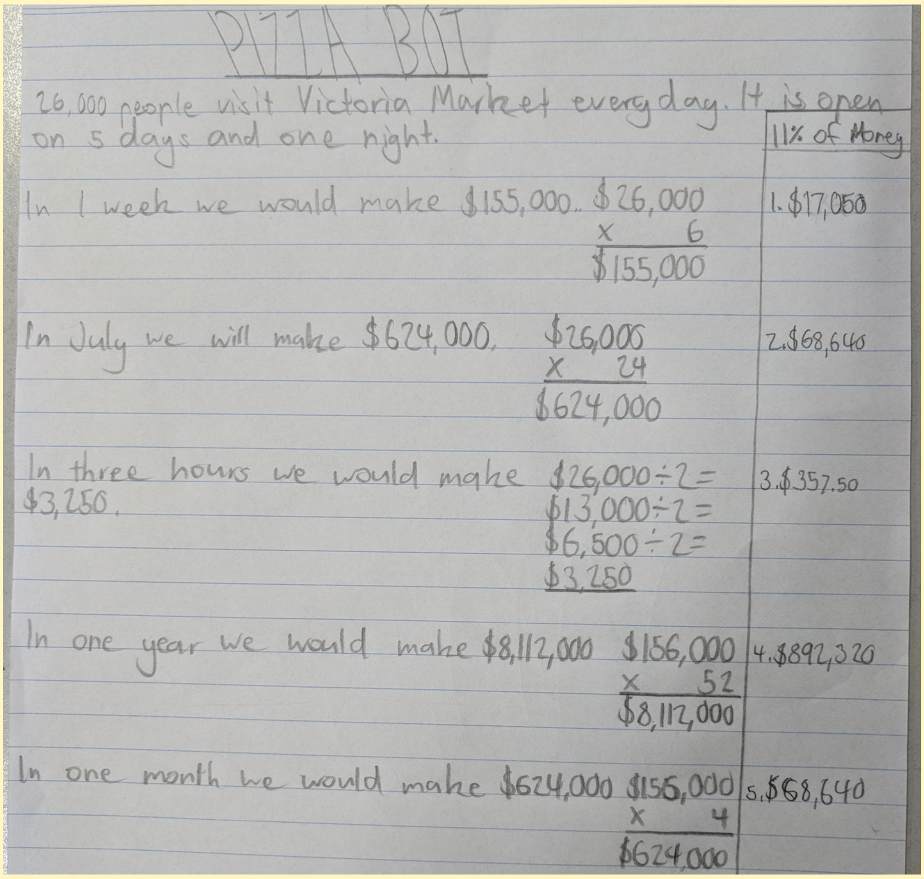


Figure 3. Students calculate the revenue generated by their food truck.

Using the Design Brief's tasks, we were able to explicitly show students the interconnections between different areas of maths. In Stage 1, students used the map's scale to calculate perimeter and area, connecting mapping and measurement. When students investigated Victoria Market's land use, they made connections between percentage, area and mapping. Longer projects allowed us to integrate various areas of number and algebra, measurement and statistics together and make connections across strands using the same resource.

This maths project also extended student learning into other subjects. We used the current issue of Victoria Market's redevelopment as a topic for persuasive writing and debate. Exploring the redevelopment plans in literacy, such as the installation of more toilets and running water, gave students more relevant knowledge and context as to their selling location. In Geography, we discussed heritage buildings in Melbourne, and compared Melbourne City and Victoria Market with other international cities and their markets. We also used our excursion to Melbourne City to link with both our Geography unit and our mathematics project. This transdisciplinary unit gave students greater knowledge and context for their project and made their learning experience more rich and immersive.

Explicitly showing how mathematics connects across strands, subjects and different problems, projects like these give students a 'broader view on numeracy' (Sullivan, 2011). We were able to widen the scope of our investigations and explored multiple challenging tasks under one project. Students were able to talk about their learning because the project was immersive and each lesson was purposeful.

I highlighted at the beginning that students remember the fun and engaging tasks. If that is the case, I believe we should put this one fun and engaging task under the microscope and find different ways we can utilise it to make every mathematics lesson worthwhile.

REFERENCES

Sullivan, P (2011). *Teaching Mathematics: Using research-informed strategies*. Six key principles for effective teaching of mathematics, 24-30.

Queen Victoria Market. Shops and stalls. Retrieved from https://qvm.com.au/shops-and-stalls/.

SUCCESSFUL LEARNING AND TEACHING OF MATHEMATICS

**Paul Staniscia, Deputy Principal,
St. Joseph's Primary School, Brunswick West**

MUCH MORE THAN THE CONTENT STRANDS

The definition of mathematics and how it is taught in schools has gone through many changes over the years, from the study of patterns to the language of science. It was Prof. R. Adler in the mid-1960's who suggested that 'mathematics is what mathematicians do' (Milgram, 2007, p. 2) and later Prof. N. Gottleib stated that 'mathematics is the study of precisely defined objects' (Milgram, 2007, p. 2). Determining the best teaching practice of mathematics is another complex area and what is known as successful mathematical learning has also undergone several shifts over the years. In the 1950s and 1960s there was a focus on understanding the structure of mathematics and not just the computational skill side. However, this was quickly followed by 'back to basics', where particular attention was placed on being able to 'compute accurately and quickly'.

The 1980s and 1990s involved a focus on 'reasoning, solving problems, connecting mathematical ideas, and communicating mathematics to others' (National Research Council, 2001, p. 115). It was at this time that educators stressed the importance of memorisation, facility and proving mathematical assertions. The implications of the changing nature of mathematical learning meant teachers relied on their textbooks, the ever changing curriculum documents or their own experiences of mathematics when they were in school when teaching their students.

Throughout the definitions of mathematics, as well as the ever changing learning and teaching of mathematics, there is no mention of content strands that today are defined as Number and Algebra, Measurement and Geometry, and Statistics and Probability. Relating the learning and teaching of mathematics to these strands in isolation encourages our practices to have very little connection with actual mathematics. So what cognitive changes do we want to promote in children so that they can be successful in learning mathematics? Recognising that no term captures completely all aspects of expertise, competence, knowledge, and facility in mathematics, The National Research Council (2001) has chosen the term 'mathematical proficiency', to capture what is necessary for anyone to learn mathematics successfully. So what does it mean to be proficient in mathematics?

MATHEMATICAL PROFICIENCY

Mathematical proficiency cannot be categorised as present or absent, one needs to keep in mind that every mathematical idea can be understood in many levels and many ways. It is something that is acquired over time and as students move through their schooling they should become increasingly proficient. Therefore, to become proficient, students need to spend time doing mathematics (solving problems, justifying their thinking, developing understanding, practicing skills) and building connections between their previous knowledge and new knowledge.

The National Research Council (2001) views mathematical proficiency as something that is developed through five interdependent and interwoven strands - Conceptual Understanding, Procedural Fluency, Strategic Competence, Adaptive Reasoning and Productive Disposition. Van de Walle, Karp & Bay-Williams (2013) reinforce this definition by highlighting how these five proficiency strands 'are the foundation for the Standards of Mathematical Practice described in the Common Core State Standards' (p. 26). Ally & Christiansen (2013) further agree with The National Research Council (2001) in that 'a successful mathematics learner is proficient in mathematics if s/he 'possesses' the five component strands in such a way that they can be brought to bear on different situations' (p 107).

They propose the notion of 'opportunity to develop mathematical proficiency' (OTDMP), which is composed of five categories matching the Mathematical Proficiency Strands of The National Research Council (2001). Schoenfeld (2007) describes mathematically proficient people as 'good problem solvers' (p. 60) who are flexible and resourceful, efficient with what they know and having a certain kind of mathematical disposition. Similarly, he defines mathematical proficiency as having four parts - Part A: Knowledge Base, Part B: Strategies, Part C: Metacognition and Part D: Beliefs and Dispositions. The Victorian Curriculum also includes the Mathematical Proficiency Strands in their curriculum - Understanding, Fluency, Problem Solving, and Reasoning. These 'proficiency strands describe the actions in which students can engage when learning the content' which 'are represented across and within the Level Descriptions, Content Descriptions and Achievement Standards'.

It is therefore evident that learning mathematics is much more than content strands, it is about becoming mathematically proficient, making sense of mathematics, and figuring out and solving problems by working hard on them. Throughout the various inquiry into mathematical proficiency, there are five strands that seem to come across as a common theme:

- Understanding
- Fluency
- Problem solving
- Reasoning and
- Disposition.

SO WHAT IS MATHEMATICAL DISPOSITION?

Disposition is a component of mathematical proficiency that does not have its own emphasis in the Victorian Curriculum, however it is a component that needs to be carefully considered, due to its importance on successful mathematics learning. Productive disposition is the ability to 'see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics' (National Research Council, 2001, p. 131).

In summary, disposition is understanding how mathematics works, knowing that it works, and believing that if it does not make sense, then persistence will pay off in seeing how it makes sense. Therefore, in order to develop disposition, one needs frequent opportunities to make sense of mathematics and participate in an environment where the teacher shows sensitivity towards learners' previous difficulties, encourages persistence, and accepts mistakes as part of learning (Ally & Christiansen, 2013). Once a learner feels safe to ask questions, seek understanding and make mistakes, they can then build beliefs and dispositions towards mathematics and make sense of it. Hence why Schoenfeld's (2007) components of mathematical proficiency includes 'Part D: Beliefs and Dispositions' (p. 68), as students need the time to form beliefs about mathematics in order to develop their disposition.

Obviously, the more mathematical concepts students understand, the more sensible mathematics becomes. However, this is not an excuse to continually teach the same skills over and over again nor an excuse to teach something until it is memorised as 'when students are seldom given challenging mathematical problems to solve, they come to expect that memorising rather than sense making paves the road to learning mathematics' (National Research Council, 2001, p. 131).

SO WHAT DOES THIS ALL MEAN?

After an analysis of the mathematical proficiency strands, it is easy to develop a misconception that some are more important than others, or that there is some sort of hierarchy. One needs to remember that the components or strands are 'not independent: they represent different aspects of a complex whole' (National Research Council, 2001, 116) and as such, they heavily rely on each other in order to develop mathematical proficiency.

One needs to also keep in mind that we do not simply teach mathematical proficiency, not because it develops over time, but because students need to be given opportunities to develop their mathematical proficiency. They need opportunities to make sense of mathematics, explore it for themselves and be taught how to develop these proficiency strands through explicit teaching.

This does not mean that we no longer need to teach the content strands of mathematics (Number & Algebra, Measurement & Geometry, and Statistics & Probability); in fact the same approach to teaching the mathematical proficiency strands needs to be taken with teaching the content strands, as students should be given opportunities to make sense of the content, explore it for themselves, and come to understand these ideas through explicit teaching and challenging tasks.

Therefore, it can be said that the mathematical proficiency strands are not only interwoven and interdependent, but they are also interwoven across the content strands of mathematics. Hence, successful mathematics learning should be viewed as a web rather than a continuum. The learner can move from one content strand to another content strand, from a content strand to a proficiency, and from one proficiency to another proficiency, all within one problem.

If this is how successful mathematics learning can be viewed, then how can it be taught? How do we move from the traditional classroom, where mathematics is taught through memorisation of concepts, within content strands of a curriculum and placed on a continuum, to a classroom where successful mathematics is taught through a web, where links are made between concepts, ideas, content strands and proficiency strands?

LET'S PROBLEM SOLVE!

It is at this point that we introduce the idea of teaching successful mathematics learning through problem solving. Students should be exposed to non-routine problems rather than just routine problems, as through non-routine problems the learner does not immediately know the solution and it is at that point where we begin to assist students in developing mathematical proficiency. Therefore, we want students to be engaged in productive struggle and selecting tasks or problems that will do this is paramount to successful mathematics learning.

So, how do we know if we have a problem to solve? According to Van de Walle, Karp & Bay-Williams (2013) a problem is defined as 'any task or activity for which the students have no prescribed or memorised rules or methods, nor is there a perception by students that there is a specific 'correct' solution method' (p. 34). We also need to keep in mind that a problem is where the engaging aspect is due to the mathematics, and that it must require justifications and explanations. If a task or problem contains all of these elements, then it can be considered a problem and the learner can be immersed in 'problem solving'.

When students are engaged in problem solving they undertake specific steps. There have been multiple studies into the 'Problem Solving Steps', although most relate to the work of George Polya, who put forth his summary in the core steps of problem solving:

1 - Understand the problem

2- Devise a plan

3 - Carry out the plan

4 - Look back

SUCCESSFUL LEARNING AND TEACHING OF MATHEMATICS (CONT.)

So, how do these problem solving steps link in with successful mathematics learning and our concept of a 'web' style approach?

1 - UNDERSTAND THE PROBLEM

'To represent a problem accurately, students must first understand the situation, including its key features. They then need to generate mathematical representation of the problem that captures the core mathematical elements and ignores the irrelevant features' (National Research Council, 2001, p. 124).

2 - DEVISE A PLAN

When we look at Part B: Strategies of Schoenfeld's (2007) four parts of mathematical proficiency, we see how he presented the idea that students need to explore and develop the various strategies used when solving problems, such as making a list, drawing a picture, working backwards, etc. But how does this relate to 'devise a plan'?

When students devise a plan they must select a strategy, this can be as simple as a strategy used to solve one of the four operations or as complex as one of the strategies used when problem solving. When students are selecting a strategy they are developing their fluency, another mathematical proficiency, as they need to rely on what they know and use it in relation to the mathematics they are engaged in.

3 - CARRY OUT THE PLAN

When exploring problem solving or strategic competence as one of the mathematical proficiency strands, we are looking at how students 'carry out a plan'. When learners are engaged in problem solving they have to understand it, identify the mathematics and attempt to solve the problem. It is through this step that students are able to develop their mathematical proficiency through the problem solving strand.

4 - LOOK BACK

This is the final problem solving step and as the name suggests, it is at this point where students look back at what they have done, check to see if the plan they devised solved the problem and if their solution makes sense and is reasonable. It is also at this point where the learner must justify and explain their thinking about how they solved the problem and not just what their solution is. This is very similar to our earlier definition of the reasoning proficiency.

WHAT ABOUT DISPOSITION?

The only mathematical proficiency we have neglected to acknowledge through the problem solving steps is 'disposition'. However, when one looks carefully, 'disposition' is developed through all of the problem solving steps, as when students solve a problem they make sense of the mathematics involved, make links to prior knowledge and use persistence to come to a solution.

Therefore, successful mathematics learning can be taught through problem solving and explained through a mathematical 'web'. Within the 'web' students are exposed to and emerged in the various mathematical content strands and given opportunities to develop their mathematical proficiency through the problem solving steps.

REFERENCES

Ally, N. & Christiansen, I. M. (2013). Opportunities to develop mathematical proficiency in Grade 6 mathematics classrooms in KwaZulu-Natal. *Perspectives in Education 2013*, *31*(3), 106 – 121.

Mathematics - Content structure. (n.d.). Retrieved March from http://victoriancurriculum.vcaa.vic.edu.au/Mathematics/Overview/Content-structure.

Milgram, R. J. (2007). What is mathematical proficiency? *Assessing mathematical proficiency. MSRI Publications*, *53*(1), 1 - 25.

National Research Council. (2001). Adding it up: *Helping children learn mathematics*. J. Kilpatrick, J. Swafford, and B. Findell (Eds.). Mathematics Learning Study Committee, Centre for Education, Division of Behavioral and Social Sciences. Washington, DC: National Academy Press.

Schoenfeld, A. H. (2007). What is mathematical proficiency and how can it be assessed? *Assessing mathematical proficiency. MSRI Publications*, *53*(1), 59 – 73.

Van de Walle, J., Karp, K. & Bay-Williams, J. (2013). *Elementary and middle school mathematics*. *Teaching Developmentally*. New Jersey: Pearson Education, Inc.

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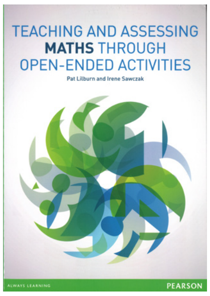

TEACHING AND ASSESSING MATHS THROUGH OPEN-ENDED ACTIVITIES

F–6

This book contains activities and real student work samples as a guide, to help improve student learning. It relates mathematical strategies to real life situations for students and caters for different needs, allowing for differentiated learning. Designed for teachers to help students practise investigation, conduct assessment and create reports.

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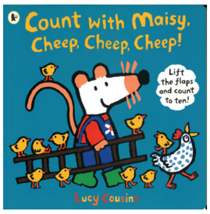

COUNT WITH MAISY, CHEEP, CHEEP, CHEEP!

K–2

Count from one to ten in this farm-themed lift-the-flap book from the multi-award-winning creator of Maisy. It's bedtime but where have all of Mummy Hen's chicks gone? Maisy goes in search for them, and you can help too! Lift the flaps along the way to see who's in the stable, in the tractor, or up in the apple tree. Cluck, cluck, cheep, cheep - find all ten chicks and make sure that they get home safely! It's more fun with Maisy!

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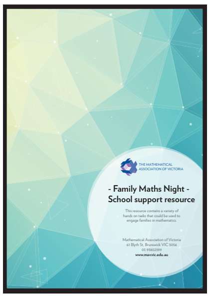

FAMILY MATHS NIGHT RESOURCE (BOOK FORMAT)

F–6

The MAV Family Maths Night Resource (book format) contains a variety of hands on tasks that could be used to engage families in mathematics. The activities within the collection are clearly laid out with simple instructions for families to follow. The activities also identify the materials required for each task (such as dice). There is also a handy resource section to support the organisation of your schools event. This version is in book format. It has been designed for busy teachers to quickly and easily obtain quality mathematics activities.

Optional extra: A PDF version of this resource is available to those who request it.

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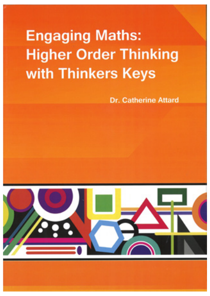

ENGAGING MATHS: HIGHER ORDER THINKING WITH THINKING KEYS

3–6

All students should be encouraged to engage with higher order thinking tasks. This book uses a critical and creative thinking tool called Thinkers Keys, adapted to be specifically mathematical. The keys indicate the kind of thinking needed to do the task: 'reverse' thinking, 'what if' thinking, 'invention' thinking, 'prediction thinking', even 'ridiculous' thinking. Could your students invent a calibrated measuring device to measure liquid? Or list reasons why a mental strategy might be better than a written strategy for a particular problem? A comprehensive table explains the different keys and gives an example activity. There are a number of carefully annotated student work samples. Engage your students with creative mathematical thinking!

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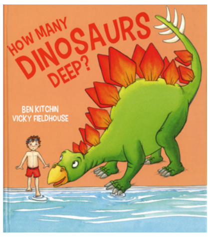

HOW MANY DINOSAURS DEEP?

1–4

Jim is not quite sure that he is ready to move from the baby pool to the middle sized pool.

Can a group of splashing, sploshing dinosaurs help him face his fear?

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THE ART OF CLEAN UP

F–8

Swiss comedian and cabaret artist Ursus Wehrli loves organisation in the extreme. *In The Art of Clean Up*, Wehrli arranges a bowl of alphabet soup, a group of pool-goers, a spruce branch, and other elements of our chaotic world into neat rows sorted by colour, size, shape or type. This eye-catching work of inventive organisation reassembles the everyday world as you know it. An inexpensive gift book, it will appeal not only to designers and artists, but to anyone willing to see the world in a new way.

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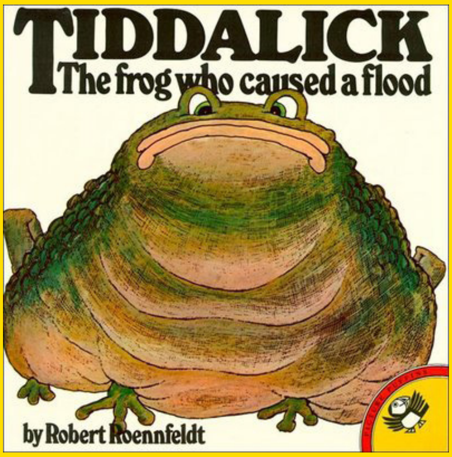
INVESTIGATIONS

Toby Russo, Bell Primary School and James Russo, Monash University

TIDDALICK

Read the Dreamtime story to your class - or watch an online animated version - to launch these mathematical investigations. You may wish to encourage students to do some online research to support their mathematical work. You might also like to connect these investigations to the Uluru image taks in this edition.

http://www.redpixels.com.au/dreamtime-stories-tiddalick-the-frog/ and https://www.youtube.com/watch?v=VVODbc7j6OM



CHALLENGING TASK 1: YEARS 3 AND UP

At the beginning of the story, Tiddalick was already a big frog. In fact, he was much bigger than a regular frog: three times taller, four times longer and five times wider! Thinking about the size of an ordinary frog, estimate how big Tiddalick might be. Can you draw a picture of Tiddalick next to an ordinary frog?

ENABLING PROMPT

An Australian Green Frog grows to around 10cm long, 4cm high and 5cm wide. Using this information, how big is Tiddalick at the start of the story?

EXTENDING PROMPT

Based on his size at the start of the story, can you estimate how much heavier Tiddalick might be than an ordinary frog? How much do think Tiddalick might weigh?

CHALLENGING TASK 2: YEARS 5 AND UP

He drank all the water from a river.

He drank the water from the billabongs.

He drank the water from the lakes.

He drank until the last drops of water from the land had disappeared into his vast mouth.

First, Tiddalick drank all of the Murray River! Next, he drank all the water from the billabongs, which was equal to one quarter of the Murray River. Finally, he drank all the water from the lakes, which was six times the capacity of the billabongs. Estimate how much water he drank?

ENABLING PROMPT

Assume there are 12,000 gigalitres in the Murray River. Now have a go at the question above.

EXTENDING PROMPT

After Tiddalick drank all the water, do you think Tiddalick weighs more or less than Uluru? Explain your reasoning.

SHARE YOUR EXPERIENCE

How did students in your class approach the above investigation? Share your class's experience with the *Prime Number* editorial team (james.russo@monash.edu), for the opportunity to have it published in *Prime Number* as a resource to share with other teachers and students. If possible, try and include photographs of work samples, as well as of students engaging in the task.