Leśniewski's Foundations of Mathematics

Sobociński's first position in the United States was in Saint Paul, Minnesota, at the College of Saint Thomas. He was only there for the spring semester of 1949–1950. During that semester he gave several talks.[1] This one has no title, but I have entitled it "Lesniewski's Foundations of Mathematics." It is an undated double spaced typescript that appears to be two lectures (pp. 1—11 and 12–26).

This is an introductory presentation on Leśniewski's three systems: Protothetic, Ontology, and Mereology. Sobociński lists three things that Leśniewski wanted for his systems; comments such as this have not been included in Leśniewski's publications or other papers about the system. He gives numerous examples and an explanation of semantic categories and the rules for definitions. There are no deductions except for a sketch of the equivalence of bivalence and extensionality. Even higher epsilons are mentioned.

[1] The other manuscript is entitled "Philosophy Seminar" and is in three parts. It deals with the history of logic.

Leśniewski's Foundations of Mathematics, part I, pp 1–11

The object of this lecture is the system of the foundations of mathematics elaborated by Stanisław Leśniewski, /born in 1886, died in 1939/ who was from 1919 up to his death professor of philosophy of mathematics in Warsaw University. This system which differs on many points from other contemporary systems is yet very little known for several reasons. The main reason ~~for it~~ is the fact that Leśniewski presented the results of ~~his xkmxfamx~~ considerations in such an exact, formal, and, at the same time, laconic form that it is almost impossible to understand it without an extensive introduction. Such an extensive introduction and commentary has been prepared by me, unfortunately, however, it has been destroyed during the Warsaw rising in 1944.

The starting point for Leśniewski's interest in the structure of the deductive systems was the problem of antinomies. The way which he adopted for the solution of this problem led him almost inevitably to those conclusions which found their expression in his theory. Therefore in order to understand well his ideas we must first realize and understand Leśniewski's view of the problem of antinomies. I explained at some length this matter in my long essay, "L'analyse de l'antinomie russelliene par Leśniewski", publishd recently in the Italian logical Review "Methodos". Here I shall be able to give only the most essential points of this problem.

1949

The word "antinomie" was understood by Leśniewski according to the definition, given by L.Nelson: The antinomie is a contradiction, which we deduce starting from premises, in whose truth we believe, and using the methods the correctness of which we recognize. Therefore the antinomy is not a simple contradiction, but a contradiction having a psychological aspect. The mechanical removal of contradiction, which we can obtain, for instance, by some restrictions of our theory, does not solve the problem, because we retain the belief in the truth of these factors, which give a ~~combination~~ contradiction. We shall obtain an annihilation of an antinomy only in such case, when we shall cease to believe in some premises or methodes, from which the antinomy arises. In that case, the tragic situation, in which we found ourselves owing to the discovery of the antinomy, vanishes and only the bare fact remains namely that this particular set of assumptions leads to contradiction. Certainly, this fact is a positive attainment of our knowledge, but now our task will be the construction of such a deductive system, which would not be contradictory and in whose truth we would believe. Having adopted this point of view and disagreeing with the solutions, proposed by others Leśniewski begun to analyse the question of the antinomies independently from anyone. This analysis, which we cannot repeat here, more particularly the analysis of Russell's antinomy, led him in 1915 to the construction of his first deductive theory, called later mereology. In years 1919-21 he constructed his theory of terms or ontology, and about 1925 he conceived his protothetic, which is an enlargement of the calculus of propositions. In this way, Leśniewski's system of the foundations of mathematics came into being, as a sum of three deductive theories; i.e. 1/ protothetic, 2/ ontology, 3/ mereology. The whole system of contemporary mathematics is based by Leśniewski on these theories, all conceived and constructed by him. Every theory, belonging to this system, has its proper axiomatic with corresponding rules of procedure, and all these theories are closely connected with each other. Protothetic, as a general theory of connections existing between propositions is not based on any earlier theory. In all the other deductive theories we must use, in the proofs theses of prothotetic itself /for instance in ontology/, or theses of protothetic and ontology /for instance in mereology, in arithmetic/, or theses of all three theories /for instance in certain systems of geometry/ a.s.o.. In this way there arises a certain hierarchy of deductive theories, in which protothetic takes the first place, ontology the second, mereology, arithmetic and, for instance, theories of groups the third, some systems of geometry the fourth and so on.

Concerning these above mentioned three theories, which constitute a base for Leśniewski's system, we can state, that:

1/ They have an accuracy and conciseness of constructions, symbolism and rules of procedure unknown till now in any deductive system.

2/ They differ from some fundamental conceptions and principles generally accepted after the discovery of B.Russell's antinomy. Nevertheless they allow us to formulate and to prove on their ground all theorems of contemporary logic. The resources of its semantical forms surpass other above mentioned theories.

3/ It can be shown by clear and easy proofs that these theories neither contain any internal contradiction nor are in any mutual contradiction, that consequently no known logical antinomy can arise out of them.

Before giving a short exposition of these three theories, I must explain some terms and give outline in the most general way some of their common properties. Unfortunatly, because of the lack of time, I shall be unable to give an exposition of Leśniewski's own symbolism and of

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reasons, which induced him to adopt this or that doctrinal position. Instead of Leśniewski's symbolism I shall use the commonly known symbolism of Peano-Russell.

According to the terminology, introduced by Leśniewski and adopted by the whole Warsaw logical school, I use the word "functor" as a name for every operator, belonging to any deductive theory. A functor can be a constant or a variable. Every functor has generally one or several arguments. The following examples will explain it:

1/ In the thesis from the calculus of propositions:

$$\sim p \supset \sim q . \supset . q \supset p$$

the letters p and q are variables, while implication and negation are constants, and the first negation "$\sim p$" is a functor, which has one argument "p", the first implication is a functor, which has two arguments "$\sim p$" and "$\sim q$", the second implication acting here as the principal functor, is a functor, which has two arguments "$\sim p \supset \sim q$" and "$q \supset p$", a.s.o.

2/ In the arithmetical thesis:

$$a + b = b + a$$

"+" is a functor, which has two arguments "a" and "b", "=" is a functor which has two arguments "a + b" and "b + a".

3/ In the logical thesis:

$$\mathrm{sym}\{R\} . = : [ab] : aRb . \supset . bRa$$

"R" is a functor, which has two arguments and at the same time "R" is a variable, "sym" is a functor, which has one argument a.s.o.

A functor, which together with its arguments produces a proposition or a propositional function, I will call a propositional functor, for instance "$\sim$", "$\supset$", "sym" a.s.o. A functor, which together with its arguments produces a term or term function, I will call a term or name functor, for instance "+". Evidently, the propotional functor can have term arguments, for instance a⊂b /a is included by b/.

The notions of functor and of arguments are indispensable for a good understanding of Leśniewski's theory, of semantical categories. He regarded the theory of logical types as conceived by Russell or other autors, as artificial and unnatural constructions because they were adopted only in order to avoid antinomies. This he considered to be the original sin of all these theories, because this reason was a decisive factor in fixing the baundaries of the field of such theories.

If we accept Nelson's definition of antinomy as mentioned above, then according to Leśniewski we must come to the conclusion that no one of the known theories of logical types is able to solve the problem of antinomies, as none of them affects or removes our belief in any of these factors which lead to contradiction. Therefore instead of some theory of logical types, Leśniewski introduced as the fundemental base for his set of deductive theories the conception of semantical categories, which, as he stated most emphatically, he would have to accept even if no antinomies existed. As in any ordinary ~~sentence~~ sentence we distinguish its syntactic parts, for instance, the subject or the predicate, so in every deductive thesis all its parts, which are not quantifiers or paranthesis, belong to some semantical category. That is every variable, constant or argument as well as every functor fall under some determined semantical category. Consequently we can and we must always know the semantical category of every expression, which we use. For instance, in the thesis:

$$\sim p \supset \sim q . \supset . q \supset p$$

we have the following semantical categories:

1/ The variables p and q belong to the category of propositions.

2/ Negation "$\sim$" belongs to the category of propositional functors for one propositional argument.

3/ Implication "$\supset$" belongs to the category of propositional functors for two propositional arguments.

4/ The expresions "$\sim p$", "$\sim q$", "$\sim p \supset \sim q$", "$q \supset p$" and the whole thesis belong to the category of propositions.

In the thesis:

$$a + b = b + a$$

we have following semantical categories:

5/ The variables a and b belong to the category of names.

6/ The functor "+" belongs to the category of name functors for two name arguments.

7/ The expresions "$a + b$" or "$b + a$" belong to the category of names.

8/ The functor "=" belongs to the category of propositional functors for two name arguments.

9/ Whole thesis belongs to the category of propositions.

If we have an expresion in which some part does not belong to some determin~~a~~ted semantical category, then we have ~~a~~ nonsense.

Therefore when building a deductive theory and adopting its axioms we adopt not only their primitive terms, but also determined semantical categories with which these terms appear in our axioms. Evidently in every axiomatic we have a finite number of semantical categories which we adopt as fundamental ones. When it appears that some new semantical category is necessary for the development of our theory we can introduce it only if we are able to construct /on the basis of our theory/ a correct definition of some new constant term belonging to this new semantical category. If we are able to do it and we do it in fact then we have introduced a new category into our theory. From this moment we can use also variables belonging to this category. The following example will illustrate this question.

Let us take the well known axiomatic of the calculus of propositions of Prof. Łukasiewicz, it is:

$$1.\ p \supset q . \supset : q \supset r . \supset . p \supset r$$

$$2.\ \sim p \supset p . \supset p$$

$$3.\ p \supset . \sim p \supset q$$

We have here two primitive terms: negation and implication and three primitive semantical categories: propositions, propositional functors for one propositional argument and propositional functors for two propositional arguments.

We can introduce the following correct definitions:

I. the well known, definition of "and"

$$p . q =_{Df} \sim (p \supset \sim q)$$

and, subsequently, "and" for three propositions:

II. $$\phi(pqr) =_{Df} ((p . q) . r)$$

The first definition gives us a new term "and" which, however, belongs to the semantical category already adopted. As for the second definition it gives us a new term "ϕ" and introduces into our theory a new semantical category, namely a propositional functor for three propositional arguments.

In this way, in every moment of the developement of our theory, we have a finite, known number of semantical categories, but we can introduce into this theory such number of them, ~~which~~ as we need. In regard to ~~contemporary~~ contemporary tendencies which want to limit in this or that way the number or quality of logical types, Leśniewski was of the opinion that this would amount to an unnatural impoverishment of the deductive theories. He considered that they must have at their disposal a language as comprehensive as possible and that they must have a possibility of a continual enrichment of it, as required by a development of the theory. Our current language is being constantly enriched and may be enriched as for semantical forms and there is no reason why we should not allow the same in a deductive theory built in a natural way. However we must have the certitude that the way on which we obtain it does not lead to contradiction. In Leśniewski's system the sole way is by definitions, ~~what guarantees~~ the consistency of this system is guaranteed, of course, if we ~~will~~ construct the definitions correctly.

As for Leśniewski's theory of definitions I can give here only foundamental points. According to him a definition does not belong to the metalanguage of a deductive theory, but to this deductive theory itself. Therefore, it is not noted on the margin of the theory, but it is added to it, as one of its theses. The rules of procedure are formulated in such a way, that they determine the conditions of a correct construction of definitions and simultaneously they allow their addition to the theory, as its theses. In connection with what was said above, the symbol "=Df" has no sense and is useless in Leśniewski's system. Instead of it, the definiendum and definiens of every definition are united by a simple equivalence "≡" or, if we do not have it yet by a corresponding combination of other functors from the calculus of propositions. For these and other reasons, which I cannot explain here, there are also in Leśniewski's system no such well known symbols as "$\hat{x}$", "$\hat{y}$" etc., introduced by Russell. In Leśniewski's system any variable is a bound variable i.e., in any thesis every variable must be ~~curd~~ bound by a suitable quantifier. An expresion with a free variable, is not a thesis. Therefore the above mentioned theses, when correctly formulated, must have the following forms:

I. $[pq] : \sim p \supset \sim q . \supset . q \frown p$

II. $[R] \therefore sym\{R\} . \equiv : [ab] : aRb . \supset . bRa$

III. $[pq] : p . q . \equiv . \sim (p \supset \sim q)$

IV. $[pqr] : \phi\{pqr\} . \equiv . ((pq)r)$

a.s.o.

The precise rules for the construction of definitions as adopted by Leśniewski fix most definitely the number of ~~kinds~~ species of definitions to four, namely:

I. Definitions, which define some propositions, for instance:

$$perm . \equiv : [pq] : p \vee q . \supset . q \vee p$$

and which are called absolute /non-relative/ protothetical definitions.

II. Definitions, which define some protothetical functor, for instance:

$$[ab] \therefore a \subset b . \equiv : [A] : A \varepsilon a . \supset . A \varepsilon b$$

and which are called: relative protothetical definitions.

III. Definitions, which define some name, for instance:

$$[A] : A \varepsilon \bigvee . \equiv . [\exists a] . A \varepsilon a$$

and which are called: absolute ontological definitions.

IV. Definitions, which define some name functor, for instance:

$$[Abc] : A \varepsilon b \frown c . \equiv . A \varepsilon b . A \varepsilon c$$

and which are called: relative ontological definitions.

With these few remarks we must leave the problem of definitions. ~~Unfortunately for~~ the lack of time I cannot enter either into the explanation of this division or into a more profound exposition of the problem of definitions in Leśniewski's system.

Because in this system ~~there are~~, as it was mentioned above, there are ~~xxxxx~~ only bound variables, there are no specific variables. Any ~~xix~~ sign, which is not equimorphous with the terms, adopted already in a certain system, can be used as a variable. Evidently, for a variable, belonging to some semantical category, only variables or constants, belonging to the same semantical category can be substituted.

In turn I must say few words concerning the problem of quantifiers. Here it should be stressed that operational rules concerning quantifiers, rules so well known and commonly used do not appear in Leśniewski's system. Instead we proceed in the following way: if we have already some semantical category, we can for ~~xxxxxx~~ variables, belonging to this category, prove theses, exactly corresponding to these rules, i.e. for every operational rule concerning quantifiers there is for any semantical category a corresponding logical thesis.

For instance, instead of the rule for the general quantifier for propositional variables, we have the following thesis:

$$[pf] : [q] : p \supset f(q) :\equiv: p \supset . [q] . f(q)$$

a.s.o.

Leśniewski's system is extensional to ~~the~~ an extreme degree. ~~That~~ It means that for variables, belonging to any semantical category, we are able to obtain from axioms or from special rules of procedure a thesis, guaranteeing the extensionality of the variables. For instance, if we have propositional variables, then we have also the following thesis:

$$[pq] :. p \equiv q . \equiv : [f] : f(p) \equiv f(q)$$

a.s.o.

Here I must interrupt my exposition of the guiding principles, which lie at the foundations of Leśniewski's system, although this exposition ~~is~~ through its shortress is quite ~~un~~sufficient. However before I pass to the discussion of the particular theories conceived by Leśniewski, I must still mention that ~~xxx~~ he elaborated, reaching a very high degree of perfection, the doctrine of the axiomatic of deductive theories. He stresses that there are several conditons which every axiomatic no matter what is its theory, must fulfill if it claims to be perfect.

There is one more and ultimate conclusion flowing from Leśniewski's doctrine of axiomatic. His logical theories do not allow us to draw any conclusion determining the number or the quality of existing objects nor even whether such objects exist at all. For Leśniewski this question belongs to physics and metaphysics. The logic has no right to take any position in this respect.

In his logical theories Leśniewski is concerned only with connections between propositions and between names. His system is purposefully built in such a way that it does not presuppose any philosophical attitude and can be used by any philosophical system. If we want to introduce any assumptions of an existential character we can do it only by explicitly adding them to the axiomatic as separate axioms.

Leśniewski's Foundations of Mathematics, Part II, pp 12 – 26
+12a & 12a-5

II.

As we said, protothetic is a general deductive theory of the connections existing between propositions. We can obtain such a theory by the addition of general quantifiers and variable functors for propositional variables to the whole calculus of propositions. Let us recall the axiomatic of this calculus as formulated by Łukasiewicz. According to Leśniewski's postulates we have no free variables but only bound variables. In this way our axiomatic has the following form:

P_1 $[pqr] :. p \supset q . \supset : q \supset r . \supset . p \supset r$

P_2 $[p] : \sim p \supset p . \supset p$

P_3 $[pq] : p \supset . \sim p \supset q$

Now let us add to these theses a following one, which establishes extensionality for the propositions:

P_4 $[pqf] : p \equiv q . f(p) . \supset f(q)$

From P_1 - P_4 we can easily obtain a following consequence, which I shall state only without proving it.

Theses P_1 - P_3, establish, as we know, the complete axiomatic of the calculus of propositions, and therefore we obtain from them in the ordinary way:

P_5 $[pqrs] :: \sim p . q \supset s : p . r \supset s :. \supset . q . r \supset s$

We can add three following definitions:

D_1 $0 \equiv . [u] . u$

D_2 $1 \equiv : [u] . u . \equiv . [u] . u$

D_3 $[p] : \sim p . \equiv . 0 \equiv p$

Now we can obtain from P_4, D_1 and D_3,

P_6 $[pf] : \sim p . f(0) . \supset . f(p)$

and from P_4, D_2 and D_3

P_7 $[pf] : p . f(1) . \supset . f(p)$

From P_5, P_6 and P_7 we can obtain at once:

$$P_8 \quad [r f] : f(0) . f(1) . \supset . f(r)$$

Conversely ~~On the contrary~~, as can be easily demonstrated P_4, P_5 and P_8 imply P_6.

Therefore, there is an inferential equivalence between P_1, P_2, P_3, P_8 and $S1, S2, S3, S4$

The thesis P_8 is called the bi-value principle for propositions. We can adopt similar principles for every semantical category. Thus, for the propositional functor for one propositional argument we have the following thesis:

$$P_9 \quad [f \varphi] : \varphi\langle \sim \rangle . \varphi\langle \mathrm{as} \rangle . \varphi\langle \mathrm{Fl} \rangle . \varphi\langle \mathrm{As} \rangle . \supset . \varphi\langle f \rangle$$

A very easy proof demonstrates, that if we have the complete calculus of propositions, then P_9 is inferentially equivalent to

$$P_{10} \quad [f g] \therefore [p] : f(p) . \equiv . g(p) : \equiv : [\varphi] : \varphi\langle f \rangle . \equiv . \varphi\langle g \rangle$$

that ~~it~~ is to the thesis, which establishes the extensionality for the variables, belonging to this category.

We can give a general proof, that in the field of the calculus of propositions, for any semantical category, the thesis guaranteeing the extensionality of this exactly category ~~xxx~~ is equivalent to the corresponding thesis, which determines the bi-value principle for this category.

The set of all such theses together with the whole calculus of propositions and with all possible deductive consequences is what we call protothetic. In other words protothetic is the calculus of propositions, enlarged by the variable functors for any semantical category and by the theses, guaranteeing the extensionality for any semantical category, which could appear in its field.

Evidently the deductive consequences of this theory go considerably farther beyond that what we can obtain from the simple calculus of propositions. This last is but a small fragment of the whole protothetic, which contains a great number of theses which do not belong to the calculus of propositions. They are, for instance, theses:

P_{11} $[r f] : f(f(f(r))) . \equiv . f(r)$,

P_{12} $[f] : f(f(0)) . \equiv . [r] . f(r)$,

and all theses which stand for the operational rules concerning quantifiers and others.

The theory described this way can ~~by~~ be presented in the precise form of a deductive theory and can possess ~~have~~ its own axiomatic. It goes without saying that we can base this theory on various primitive terms. We can, for instance, choose the implication, as a sole primitive term of the protothetic. In this case, the following theses can serve as its axiomatic:

P_{13} $[rq] : r \supset . q \supset r$

and P_8.

But for the reasons, which I cannot explain here, the most convienient primitive term, on which we can found the protothetic, is equivalence. Thanks to the result of Dr.A.Tarski, who proved, that in the protothetic we can define a conjunction by equivalence only, in the following way

D_4 $[rq] \therefore r . q . \equiv : [f] : f(r) \equiv f(q) . \equiv q$

we can use this term, as a sole primitive term of protothetic. Researches conducted many years by Leśniewski, dr.M.Wajsberg and myself, led eventually qfter succesive simplifications to the following shortest sole axiom of protothetic, discovered by me in 1945:

$$814\ [p\,q] :: p \equiv q . \equiv :. [f] :. f(p\, f(p\, [u] . u)) . \equiv : [r] : f(q\, r) . \equiv . q \equiv p$$

From this thesis, through rather complicated and difficult deductions, we can derive, applying only the rule of procedure of protothetic, all theses, belonging to this theory.

I cannot explain here, how is formulated the rule of procedure of protothetic. I shall only mention, that there are proofs of consistency and completness of this theory. Protothetic is complete in the meaning, that any expression, which is senseful in its field, for instance:

$$[p] . \varphi(p)$$

either itself or its negation, it is:

$$\sim ([p] . \varphi(p))$$

is a thesis of protothetic.

Protothetic as conceived by Leśniewski is nothing else than a consistent, immense enlargement of the calculus of propositions whose fundamental assumptions, however, do not differ from other contemporary conceptions of this calculus. On the contrary, Leśniewski's calculus of terms or ontology, as he called it, contains ideas which differ fundamentally from the generally accepted opinions. Here, as in other instances of various deductive theories constructed by him, a starting point for Leśniewski was his analysis of the current language. As a result of such analysis he selected some linguistic forms corresponding most adequately to his ideas and later used these forms often, having them strictly defined. So, for instance, the analysis of the individual /atomic/ proposition:

A is b

Socrates is white

led him to the following interpretation of this expression:

If anybody says

A is b

then in fact he says three following things:

I. A is unempty,

II. If anything is A, then it is b,

III. A is unique.

It is, if I say "Socrates is white", I say:

1/ There is in the world a object, which is Socrates,

2/ If something is Socrates, then it is white,

3/ Socrates is an unique such object in the world.

According to this definition of the individual proposition, the proposition:

Jove is a god

is false, because it does not satisfy the first condition; ~~xxxxxxxxx~~ the proposition:

Socrates is an American

is false, because it does not satisfy the second condition; the proposition:

man is mortal

is false, because it does not satisfy the third condition.

The meaning of individual propositions understood and established in this way was in complete harmony with the opinions, which were held by Leśniewski concerning the logic of terms. The conception of modern logic, according to which the subject and the predicate of the individual proposition belong to the different logical types was considered by Leśniewski as unnatural. This conception introduced into the logic for the first time by Peano in order to avoid some difficulties and

~~and~~ addopted later by Russell, was, for him, an intolerable impoverishment of our logical language. This unnatural construction, introduced only to avoid some antinomies, is, as he demonstrated, by no means indispensable for this aim. We can construct a noncontradictory system of logic, in which ~~the~~ both these parts of individual propositions will belong to the same semantical category. I will show later that in such a system it is impossible to construct Russell's antinomy. This point is closely connected with Leśniewski's opinion concerning the unit unique-class and the null-class. According to him, if we accept the existence of the unique-class, as an object, which is different from its own unique element, then we accept the existence of an unnatural being. The unique-class is the same object as its own unique element. Evidently, this position solves in advance the question of the null-class. In Leśniewski's logical language and system there are no null-classes, because in the world there are no contradictory or empty objects. We can into our language introduce the notion of contradictory object, but we cannot build a theory based on such notion. Leśniewski used to say, that he cannot agree, with, nor understand those theorists of set theory, who build this theory, as a theory of nothing. For him the well known constructions

$$\{0\},\ \{\{0\}\},\ \{\{0\},\{\{0\}\}\} \qquad \text{a.s.o.}$$

are un impermissible hypostases.

There is a complete harmony between these considerations and Leśniewski's interpretation of the individual proposition. According to this interpretation and following the practice of the current language he wanted to have in his logical system as senseful propositions such propositions as for instance:

Aristotle is the author of Organon

and

The author of Organon is Aristotle.

He wanted also to have included in his system the following form of reasoning:

If A is B and B is C, then ~~B~~ A is C.

Having established the sense of the individual proposition, Leśniewski faced the problem how ~~could he~~ build a system of logic which would harmonize with these considerations.

1/ He would not adopt this method of moderm logic, by which we build the calculus of terms having established only the sens of name functions, without any axiomatic explicity formulated. According to Leśniewski's conception of the deductive theory no semantic category /or logical type/ can be accepted, unless based on some axiomatic.

2/ He stressed that the propositional function might appear in two different forms namely

$\varphi/x/$ and $x \varepsilon \alpha$

These two logical forms are entirely different, ~~what~~ which is also acknowledged by the current usage of our language in which we say for instance

"Mary loves" and "Aristotle is a Greek".

It is obvious that these two sentences have a completely different gramatical structure, and ~~besides~~ so there are no reasons why we should not have in a logical theory two coresponding forms.

Unfortunately these two forms are not distinguisted by the modern logic ~~what~~ which Leśniewski regarded as the peak of ~~xxxxx~~ confusion and misunderstanding.

Therefore, he wanted to build ~~such~~ a calculus of terms, which would satisfy the following conditions:

1/ It should have its own axiomatic, which would establish its primitive terms, and determine its initial semantical categories.

2/ It should distinguish and define exactly two forms of individual proposition, that it is $\varphi(x)$ and $x \varepsilon \alpha$

3/ The subject and the predicate of the individual proposition should belong to the same semantical category, namely to the category of names.

4/ It should not contain neither unique class nor null-class.

The above mentioned analysis of individual propositions adopted by Leśniewski, was a starting point for the construction of the logical theory, possessing such properties. He tried to express the conditions, which are equivalent to the individual proposition:

A is b

in such a way, as to express them by the copula "is". In other words, having established the meaning of the word "is", he tried to employ this word as the sole primitive term of his ontology. In this way, he addopted the following meanings of those formerly mentioned conditions:

I. A is unempty .≡. [∃ B] .B is A.

II. If anything is A, then it is b.≡:

[C] :.If C is A, then C is b

III. A is unique .≡: [DE] :If D is A and E is A, then D is E

Using the symbol "ε" for the word "is", we can now write the following thesis:

01 $[Aa] \therefore A\varepsilon a . \equiv : [\exists B] . B\varepsilon A : [B] : B\varepsilon A . \supset . B\varepsilon a : [BC] : B\varepsilon A . C\varepsilon A . \supset . B\varepsilon C$

As the subsequent investigations have shown this thesis can serve as the sole axiom of Leśniewski's ontology. My later investigations demonstrated in 1929, that the following thesis can also serve as the sole axiom of this theory:

02 $[Aa] : A\varepsilon a . \equiv . [\exists B] . A\varepsilon B . B\varepsilon a$

From this one thesis applying only the rule of procedure of ontology we can derive theses, corresponding to all theses of the traditional logic, of algebra of classes and of other parts of calculus of terms and relations. When compared, for instance, with "Principia Mathematica", ontology has a considerably greater wealth of logical forms.

Unfortunatly, I can only mention few points concerning this theory:

I. Ontology is a theory based on protothetic. Its rule of procedure is constructed in such manner that there is a close connection between both theories.

II. Its sole primitive term is "ε", but we can adopt also other terms, as primitive. Initial semantical categories are two: the semantical category of names and the semantical category of propositional functors for two name arguments.

III. Ontology contains several peculiar theses, as, for instance,:

03 $[ABa] : A\varepsilon B . B\varepsilon a . \supset . A\varepsilon a$

04 $[Aa] : A\varepsilon a . \supset . A\varepsilon A$

05 $[ABa] : A\varepsilon B . B\varepsilon a . \supset . B\varepsilon A$

As ontology has two different terms:

/1/ "identic" used in respect to objects, and

/2/ "equal to" used in respect to general names there are two following definitions:

D5 $[AB] : A = B . \equiv . A \varepsilon B . B \varepsilon A$

D6 $[ab] :. a \circ b . \equiv : [A] : A \varepsilon a . \equiv . A \varepsilon b$

IV. For variables beloging to any semantical category we can establish notions and theses, coresponding to the properties of, so called descriptions. Moreover, in the field of ontology we must distinguish the functors, which are connexions between general names, for instance:

$$a \circ b$$

and functors, which are relations between objects, for instance:

$$A = B$$

In other words, the relation is such a functor, which has only objects as its arguments. We can define the relation functor for two arguments in the following way:

D7 $[f] :. \Psi\{f\} . \equiv : [AB] : f\{AB\} . \supset . A \varepsilon A . B \varepsilon B$

V. In the field of ontology for any semantical category we can define a term, possessing properties ~~analogical~~ analogous to the properties of the primitive term "ε". For every such term we can prove theses, which are analogical to the axioms of ontology and to the definitions previously adopted. It means that, for instance, for the propositional functors for two name arguments:

We can introduce the following definition:

D8 $[\varphi\psi] :. \varphi \overset{\times}{\varepsilon} \psi . \equiv : [\exists ab] . \varphi\{ab\} . \psi\{ab\} : [abcd] : \varphi\{ab\} . \varphi\{cd\} . \supset . a \circ c . b \circ d$

Subsequently, we can prove the following thesis:

O6 $[\varphi\psi] : \varphi \overset{\times}{\varepsilon} \psi . \equiv . [\exists\Psi] . \varphi \overset{\times}{\varepsilon} \Psi . \Psi \overset{\times}{\varepsilon} \psi$

which is analogical to the axiom and theses analogical to ~~the~~ all definitions already adopted for names. In this way, we have the certitude, that, for instance, in the field of relations we can obtain all theses, analogical to those which we have for the names.

VI. Like protothetic, ontology is an extensional system. For variables, belonging to any semantical category we are able to obtain by special rules of procedure a thesis, guaranteeing the extensionality of these variables. For instance, if we have name variables, then we have also the following thesis:

07 $[ab]:[A]:A\varepsilon a.\equiv.A\varepsilon b:\equiv:[\varphi]:\varphi(a)\equiv\varphi(b)$

For propositional functions ~~variables~~ for one name argument we have the following thesis:

08 $[\varphi\chi]\therefore[a]:\varphi(a)\equiv\chi(a):\equiv:[\psi]:\psi\{\varphi\}\equiv\psi\{\chi\}$

a.s.o.

VII. We have an easy and short proof of consistency of ontology. More particularly we cannot construct in its field any reasoning similar to Russell's antinomy. This is due to the rule of procedure, which allows the construction of ontological definitions only in this way, ~~that~~ that:

If any variables is a subject of the definiendum of some ontological definitions, the definiens of this definition~~s~~ must imply, that this variable is an object.

Using this condition or an equivalent one we can for instance define the following terms:

D9 $[Aa]:A\varepsilon\sim(a).\equiv.\sim(A\varepsilon a).A\varepsilon A$

D10 $[A]:A\varepsilon V.\equiv.A\varepsilon A$

D11 $[A]:A\varepsilon\Lambda.\equiv.\sim(A\varepsilon A).A\varepsilon A$

a.s.o.

Without the above described condition, we would immediately get a vulgar contradiction.

According to Leśniewski's opinion, the essence of Russell's antinomy does not consist in the irregular construction of definitions, but it is the result of equivocality of the notions "class" and "element". He usually formulated Russell's antinomy in the following way:

The assumptions:

R1. $[a]:[\exists A].\, A \varepsilon Kl(a)$

R2. $[AaB]: A \varepsilon Kl(a).\, B \varepsilon a.\, \supset.\, B \varepsilon el(A)$

R3 $[AaB]: A \varepsilon Kl(a).\, B \varepsilon el(A).\, \supset.\, B \varepsilon a$

. it is:

I. For any a, there is the class of a;

II. For any A, α and B, if A is the class of α and B is α, then B is element of A;

III. For any A, α and B, if A is the class of α and B is an element of A, then B is α;

seem evident. Nevertheless from these assumptions using correct logical methods, we can obtain a contradiction in the following way:

R4 $[ABab]: A \varepsilon Kl(a).\, A \varepsilon Kl(b).\, B \varepsilon b.\, \supset.\, B \varepsilon a$ (R2, R3)

We introduce the following correct definition:

DI $[A] \therefore A \varepsilon *.\, \equiv: A \varepsilon A: [a]: A \varepsilon Kl(a).\, \supset.\, \sim(A \varepsilon a)$

and we obtain:

R5 $[A]: A \varepsilon Kl(*).\, \supset.\, \sim(A \varepsilon *)$ /DI/

R6 $[Aa]: A \varepsilon Kl(*).\, A \varepsilon Kl(a).\, \supset.\, \sim(A \varepsilon a)$ /R4; R5/

R7 $[A]: A \varepsilon Kl(*).\, \supset.\, A \varepsilon *$ /R6; DI/

R8 $[A].\, \sim(A \varepsilon Kl(*))$ /R5, R6/

We obtain a contradiction because R1 is contrary to R8.

Therefore our contradiction is included in the assumptions R1 – R3

I can give here only final conclusions of Leśniewski's analysis of this problem. According to him in the assumptions R1 - R3 which are apparently evident, occurs the confusion of two entirely different meanings of the notion "class", namely "distributive class" and "collective class" In Leśniewski's opinion the notion "distributive class" is an unreal term, used without any need and instead of a general name. If for instance, we use the word "class" in this meaning in the sentence: "A is an element of the class of b", all we want to say is: "A is one of the objects belonging to b", or simply "A is B". Thus the notion "distributive class" can be replaced by the purely logical notion of "general name" and consequently it is superfluous.

On the contrary, the notion "collective class" is a real term, which denotes an existing real object. It is, if we use the word "class" in this meaning and we say "A ε Kl(b)", then we say, that A is an object, which consists of these and only these objects, which are b. Thus, if I say "A is the class of books" then I say, that A is a real object, which consists of all books existing in the world, no matter where these books are. If we use the term "class" in this meaning, the expression " A is an element of class od b" signifies, that A is a part of this class or that A is this whole object, which is the class of b. However not necessarily A must be b. Thus, if A is an element of the class of men, then A can be any section of the class of b. For instance, A can be the object consisting of all Americans or A can be my head etc.

Now if we adopt Leśniewski's distinction between the "distributive class" and the "collective class", then a short analysis will convince us, that some of our assumptions /R1 - R3/ are false. In this way we get rid of Russell's antinomy, which was the result of the confusion of two different notions expressed by one word. Some of these assumptions, which lead to the contradiction are true for "distributive class" and these belong to logic, some are true for "collective class" and these belong to mereology.

Mereology is a deductive theory built by Leśniewski in order to establish the properties of the collective class.

We can formulate a number of intuitive theses, which determine the properties of this notion, for instance:

M1. $[a] : [\exists A] . A \varepsilon a . \equiv . [\exists A] . A \varepsilon Kl(a)$

M2. $[ABa] : A \varepsilon Kl(a) . B \varepsilon Kl(a) . \supset . A = B$

M3. $[Aa] : A \varepsilon Kl(a) . \supset . A = Kl(a)$

M4. $[A] : A \varepsilon V . \supset . A = Kl(A)$

a.s.o.

We can also introduce several definions, for instance:

D 12 $[AB] : A \varepsilon el(B) . \equiv . [\exists a] . B \varepsilon Kl(a) . A \varepsilon a$

D 13 $[AB] : A \varepsilon pr(B) . \equiv . A \varepsilon el(B) . \sim(A = B)$

it is: A is a part of B. a.s.o.

And we can also establish some these, concerning these terms, for inst.:

M5. $[ABC] : A \varepsilon el(B) . B \varepsilon el(C) . \supset . A \varepsilon el(C)$

M6. $[AB] : A \varepsilon el(B) . B \varepsilon el(A) . \supset . A = B$

M7. $[AB] : A \varepsilon pr(B) . \supset . \sim(B \varepsilon pr(A))$

a.s.o.

Like all deductive theories built by Leśniewski, also mereology has its own axiomatic. There are various terms, which can be chosen as its primitive terms. For instance, in 1948 I proved, that if we choose as a sole primitive term the term "el" then the following simple thesis:

M8. $[AB] \therefore A \varepsilon el(B) . \equiv \therefore B \varepsilon B \therefore [fa] \therefore [C] :: C \varepsilon f(a) . \equiv \therefore [D] : D \varepsilon a . \supset . D \varepsilon el(C) : [D] : D \varepsilon el(C) . \supset . [\exists EF] . E \varepsilon a . F \varepsilon el(E) . F \varepsilon el(D) :: B \varepsilon el(B) . B \varepsilon a . \supset . A \varepsilon el(f(a))$

can be used as a sole axiom of mereology.

Mereology is based on protothetic and ontology and is constructed in such a way, that it does not decide, what objects do exist in the world. Therefore, it can be used by any physical or philosophical considerations. In mathematics we can use mereology as non-geometrical base for some systems of geometry.

We are in possession of a proof* of the consistency of this theory.

I gave here hardly a short outline of Leśniewski's system of foundations of mathematics. However this system is so rich that in order to understand it fully and above all in order to understand its leading ideas, an extensive and systematic course would be necessary.

$A \varepsilon \sigma(b) \equiv S(A \varepsilon b)$

$$\frac{\Sigma p}{\Sigma p \supset p}$$

$A \varepsilon \sigma(b) \equiv \Sigma p$

~~System Leśniewskiego~~

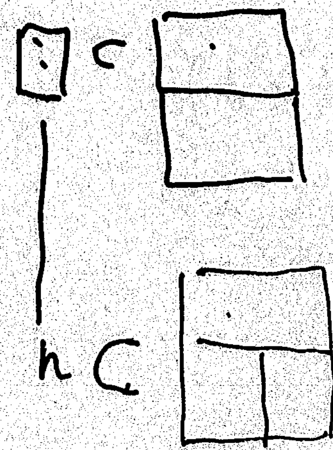
1) Podstawy matematyki
2) System logiki (Leśniewski)
3) Podstawy logiki
3) Zmienne, funktory i argumenty, kwantyfikator ogólny. zmienne wolne i związane.
4) Zdania oznajmujące
5) Operacje nad jednem zdaniem.
6) Operacje nad dwoma zdaniami.
7) Teza
8) Aksjomatyka
9) Dyrektywy.

This page was between pages 12 & 13. I have called it p 12a & moved it to the back.

$(b-c-a) \vee a \; \infty \; (b-c-a) \vee a$

$2a-c \quad C(b-c-a) \quad \vee a$

$n \quad C$



$\Delta \; cca$

$n \, C \sim(ca) \quad m \, C \, a-c$

$[\exists m] . m \, C \, ca . \cancel{m \, c \, n}$ $ca \, n \, c \, n$

$[m] : m \, C \, ca . \supset . \cancel{(m \, c \, n) \supset \ldots \sim (n)}$

$\sim (ca \, c \, n)$

$[\exists v] : v \, C \, ca . \cancel{\sim (v \, c \, n)}$