Leśniewski’s Foundations of Mathematics

Sobociński’s first position in the United States was in Saint Paul, Minnesota, at the College of Saint Thomas. He was only there for the spring semester of 1949–1950. During that semester he gave several talks.¹ This one has no title, but I have entitled it “Lesniewski’s Foundations of Mathematics.” It is an undated double spaced typescript that appears to be two lectures (pp. 1—11 and 12–26).

This is an introductory presentation on Leśniewski’s three systems: Prothetetic, Ontology, and Mereology. Sobociński lists three things that Leśniewski wanted for his systems; comments such as this have not been included in Leśniewski’s publications or other papers about the system. He gives numerous examples and an explanation of semantic categories and the rules for definitions. There are no deductions except for a sketch of the equivalence of bivalence and extensionality. Even higher epsilons are mentioned.

¹ The other manuscript is entitled “Philosophy Seminar” and is in three parts. It deals with the history of logic.
The object of this lecture is the system of the foundations of mathematics elaborated by Stanislaw Leśniewski, born in 1886, died in 1939, was from 1919 up to his death professor of philosophy of mathematics in Warsaw University. This system which differs on many points from other contemporary systems is yet very little known for several reasons. The main reason for it is the fact that Leśniewski presented the results of his inexact considerations in such an exact, formal, and at the same time, laconic form that it is almost impossible to understand it without an extensive introduction. Such an extensive introduction and commentary has been prepared by me. Unfortunately, however, it has been destroyed during the Warsaw rising in 1944.

The starting point for Leśniewski's interest in the structure of the deductive systems was the problem of antinomies. The way which he adopted for the solution of this problem led him almost inevitably to those conclusions which found their expression in his theory. Therefore in order to understand well his ideas we must first realize and understand Leśniewski's view of the problem of antinomies. I explained at some length this matter in my long essay, "L'analyse de l'antinomie russellienne par Leśniewski", published recently in the Italian logical Review "Methodos". Here I shall be able to give only the most essential points of this problem.

The word "antinomie" was understood by Leśniewski according to the definition, given by L. Nelson: The antinomie is a contradiction, which we deduce starting from premises, in whose truth we believe, and
using the methods the correctness of which we recognise. Therefore
the antinomy is not a simple contradiction, but a contradiction hav-
ing a psychological aspect. The mechanical removal of contradiction,
which we can obtain, for instance by some restrictions of our theory,
retain
does not solve the problem, because we retain the belief in the truth
of these factors, which give a combination. We shall obtain an amni-
ilation of an antinomy only in such case, when we shall cease to believe
in some premises or methods, from which the antinomy arises.
In that case, the tragic situation, in which we found ourselves owing
to the discovery of the antinomy, vanishes and only the bare fact
remains namely that this particular set of assumptions leads to contra-
diction. Certainly, this fact is a positive attainment of our knowledge,
but now our task will be the construction of such a deductive system,
which would not be contradictory and in whose truth we would believe.
Having adopted this point of view and disagreeing with the solutions,
proposed by others, Leśniewski began to analyse the question of the
antinomy, independently from anyone. This analysis, which we cannot
repeat here, more particularly the analysis of Russell's antinomy,
led him in 1915 to the construction of his first deductive theory, cal-
led later mereology. In years 1919-21 he constructed his theory of
terms or ontology, and about 1925 he conceived his protothetic, which
is an enlargement of the calculi of propositions. In this way, Leś-
niewski's system of the foundations of mathematics came into being, as
a sum of three deductive theories; i.e., 1/ protothetic, ontology,
2/ mereology. The whole system of contemporary mathematics is based by
Leśniewski on these theories, all conceived and constructed by him.
Every theory, belonging to this system, has its proper axiomatic with
corresponding rules of procedure, and all these theories are closely connected with each other. Protothetic, as a general theory of connections existing between propositions is not based on any earlier theory. In all the other deductive theories we must use, in the proofs theses of protothetic itself (for instance in ontology), or theses of protothetic and ontology (for instance in mereology, in arithmetic), or theses of all three theories (for instance in certain systems of geometry and so on). In this way there arises a certain hierarchy of deductive theories, in which protothetic takes the first place, ontology the second, mereology, arithmetic and, for instance, theories of groups the third, some systems of geometry the fourth and so on.

Concerning these above mentioned three theories, which constitute a base for Leśniewski's system, we can state, that:

1/ They have an accuracy and conciseness of constructions, symbolism and rules of procedure unknown till now in any deductive system.

2/ They differ from some fundamental conceptions and principles generally accepted after the discovery of B. Russell's antinomy. Nevertheless, they allow to formulate and to prove on their ground all theorems of contemporary logic. The resources of its semantical forms surpass other above mentioned theories.

3/ It can be shown by clear and easy proofs that these theories neither contain any internal contradiction nor are in any mutual contradiction, that consequently no known logical antinomy can arise out of them.

Before giving a short exposition of these three theories, I must explain some terms and outline in the most general way some of their common properties. Unfortunately, because of the lack of time, I shall be unable to give an exposition of Leśniewski's own symbolism and of
reasons, which induced him to adopt this or that doctrinal position. Instead of Leśniewski's symbolism I shall use the commonly known symbolism of Peano-Russell.

According to the terminology, introduced by Leśniewski and adopted by the whole Warsaw logical school, I use the word "functor" as a name for every operator, belonging to any deductive theory. A functor can be a constant or a variable. Every functor has generally one or several arguments. The following examples will explain it:

1/ In the thesis from the calculus of propositions:
\[ \sim p \circ \sim q \circ p \]
the letters \( p \) and \( q \) are variables, while implication and negation are constants, and the first negation \( \sim p \) is a functor, which has one argument \( "p" \), the first implication \( \sim p \circ \sim q \) is a functor, which has two arguments \( "\sim p" \) and \( "\sim q" \), the second implication acting have as the principal functor, is a functor, which has two arguments \( "\sim p \circ \sim q" \) and \( "q \circ p" \), a.s.o.

2/ In the arithmetical thesis:
\[ a + b = b + a \]
"+" is a functor, which has two arguments \( "a" \) and \( "b" \), "=" is a functor which has two arguments \( "a + b" \) and \( "b + a" \).

3/ In the logical thesis:
\[ \text{sym} \left\{ R, E: [ab]: aRb, \supset bRa \right\} \]
"R" is a functor, which has two arguments and at the same time \( "R" \) is a variable, "sym" is a functor, which has one argument a.s.o.

A functor, which together with its arguments produces a proposition or a propositional function, I will call a propositional functor, for instance \( "\sim" \), \( "\supset" \), "sym" a.s.o. A functor, which together with its arguments produces a term or term function, I will call a term or name/
functor, for instance "*". Evidently, the propositional functor can have
term arguments, for instance a cb /a is included by b/.

The notions of functor and of arguments are indispensable for a
good understanding of Leśniewski's theory of semantical categories.
He regarded the theory of logical types as conceived by Russell or
other authors as artificial and unnatural constructions because they
were adopted only in order to avoid antinomies. This he considered to
be the original sin of all these theories, because this reason was a
decisive factor in fixing the boundaries of the field of such theories.

If we accept Nelson's definition of antinomy as mentioned above,
then according to Leśniewski we must come to the conclusion that no
one of the known theories of logical types is able to solve the problem
of antinomies, as none of them affects or removes our belief in any of
these factors which lead to contradiction. Therefore instead of some
theory of logical types, Leśniewski introduced as the fundamental base
for his set of deductive theories the conception of semantical cate-
gories, which, as he stated most emphatically, he would have to ac-
cept even if no antinomies existed. As in any ordinary sentence
we distinguish its syntactic parts, for instance, the subject or the
predicate, so in every deductive thesis all its parts, which are not
quantifiers or parentheses, belong to some semantical category. That
is, every variable, constant or argument as well as every functor fall
under some determined semantical category. Consequently we can and
we must always know the semantical category of every expression, which
we use. For instance, in the thesis:

\[
\neg p \land (\neg q \land \neg r) \land p
\]

we have the following semantical categories;
1/ The variables \( p \) and \( q \) belong to the category of propositions.

2/ Negation \( \sim \) belongs to the category of propositional functors for one propositional argument.

3/ Implication \( \supset \) belongs to the category of propositional functors for two propositional arguments.

4/ The expressions \( \sim p \), \( \sim q \), \( \sim p \supset q \), \( p \supset q \), and the whole thesis belong to the category of propositions.

In the thesis:
\[
a + b = b + a
\]

we have following semantical categories:

5/ The variables \( a \) and \( b \) belong to the category of names.

6/ The functor \( \text{+} \) belongs to the category of name functors for two name arguments.

7/ The expressions \( \text{a} \text{+} \text{b} \) or \( \text{b} \text{+} \text{a} \) belong to the category of names.

8/ The functor \( \text{=} \) belongs to the category of propositional functors for two name arguments.

If we have an expression in which some part does not belong to some determined semantical category, then we have a nonsense.

Therefore when building a deductive theory and adopting its axioms we adopt not only their primitive terms, but also determined semantical categories with which these terms appear in our axioms. Evidently in every axiomatic we have a finite number of semantical categories which we adopt as fundamental ones. When it appears that some new semantical category is necessary for the development of our theory we can introduce it only if we are able to construct /on the basis of our theory/ a.
correct definition of some new constant term belonging to this new semantical category. If we are able to do it and we do it in fact then we have introduced a new category into our theory. From this moment we can use also variables belonging to this category. The following example will illustrate this question.

Let us take the well known axiomatic of calculus of propositions of prof. Łukasiewicz, it is:

\[ \begin{align*}
1. & \quad p \land q \lor \neg p \lor \neg q \\
2. & \quad \neg p \land q \\
3. & \quad p \land \neg q
\end{align*} \]

We have here two primitive terms: negation and implication and three primitive semantical categories: propositions, propositional functors for one propositional argument and propositional functors for two propositional arguments.

We can introduce the following correct definitions:

I. the well known definition of "and"

\[ p \land q \equiv \neg (p \lor \neg q) \]

and, subsequently, "and" for three propositions:

II. \[ \phi (pqr) \equiv \neg ((p \land q) \lor \neg r) \]

The first definition gives us a new term "and" which belongs to the semantical category already adopted. As for the second definition it gives us a new term "ϕ" and introduces into our theory a new semantical category, namely a propositional functor for three propositional arguments.

In this way, in every moment of the development of our theory, we have a finite, known number of semantical categories, but we can introduce into this theory such number of them, which we need. In regard
to contemporary tendencies which want to limit in this or that way the
number or quality of logical types, Leśniewski was of the opinion that
the impoverishment of
this would amount to an unnatural...

He considered that they must have at their disposal a language as com-
prehensive as possible and that they must have a possibility of continu-
al enrichment of it, as required by a development of the theory. Our
current language is being constantly enriched and may be enriched as
for semantical forms and there is no reason why we should not allow
the same in a deductive theory built in a natural way. However we
must have the certitude that the way on which we obtain it does not
lead to contradiction. In Leśniewski’s system the sole way is by defi-
nitions, which guarantee the consistency of this system, of course, if
we will construct the definitions correctly.

As for Leśniewski’s theory of definitions I can give here only
fundamental points. According to him a definition does not belong to
the metalanguage of a deductive theory, but to this deductive theory
itself. Therefore, it is not noted on the margin of the theory, but
it is added to it as its thesis. The rules of procedure are formul-
ated in such a way, that they determine the conditions of a correct
construction of definitions and simultaneously they allow their addi-
tion to the theory, as its theses. In connection with what was said
above, the symbol E-chair “*g*” has no sense and is useless in Leśniewski’s
system. Instead of it, the definiendum and definiens of every defini-
tion are united by a simple equivalence “*g*” or, if we do not have it
yet by a corresponding combination of other functors from the calculus
of propositions. For these and other reasons, which I cannot explain
here, there are also in Leśniewski’s system no such well known symbols
as “*x*”, “*y*” etc., introduced by Russell. In Leśniewski’s system any
variable is a bound variable i.e., in any thesis every variable must
be bound by a suitable quantifier. An expression with a free variable, is not a thesis. Therefore above mentioned theses, when correctly formulated, must have the following forms:

I. \[ \text{[pq]} : \sim p \lor \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim 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the problem of definitions in Łeśniewski's system.

Because in this system there are only bound variables, there are no specific variables. Any sign, which is not equinomorphic with the terms adopted already in a certain system, can be used as a variable. Evidently, for a variable belonging to some semantical category, only variables or constants belonging to the same semantical category can be substituted.

In turn I must say few words concerning the problem of quantifiers. Here it should be stressed that operational rules concerning quantifiers, rules so well known and commonly used do not appear in Łeśniewski's system. Instead we proceed in the following way: if we have already some semantical category, we can for variables belonging to this category, prove theses, exactly corresponding to these rules.

i.e. for every operational rule concerning quantifiers there is for any semantical category a corresponding logical thesis.

For instance, instead of the rule for the general quantifier for propositional variables, we have the following thesis:

\[ [p, f] \supset [q] : p \supset f(q) : \exists ! p : [q] \supset f(q) \]

Łeśniewski's system is exten
dional to the extreme degree. It means that for variables, belonging to any semantical category, we are able to obtain from axioms or from special rules of procedure a thesis, guaranteeing the extensibility of the variables. For instance, if we have propositional variables, then we have also the following thesis:

\[ [p, q] : p \equiv \equiv f [f] : f(p) \equiv f(q) \]
Here I must interrupt my exposition of the guiding principles, which lie at the foundations of Łeśniewski's system, although this exposition, through its shortness is quite insufficent. However before I pass to the discussion of the particular theories conceived by Łeśniewski, I must still mention that he elaborated, reaching a very high degree of perfection, the doctrine of the axiomatic of deductive theories. He stresses that there are several conditions which every axiomatic no matter what is its theory, must fulfill if it claims to be perfect.

There is one more and ultimate conclusion flowing from Łeśniewski's doctrine of axiomatic. His logical theories do not allow us to draw any conclusion determining the number or the quality of existing objects nor even whether such objects exist at all. For Łeśniewski this question belongs to physics and metaphysics. The logic has no right to take any position in this respect.

In his logical theories Łeśniewski is concerned only with connections between propositions and between names. His system is purposely built in such a way that it does not presuppose any philosophical attitude and can be used by any philosophical system. If we want to introduce any assumptions of an existential character we can do it only by explicitly adding them to the axiomatic as separate axioms.
As we said, protothetic is a general deductive theory of the connexions existing between propositions. We can obtain such a theory by the addition of general quantifiers and variable functors for propositional variables to the whole calculus of propositions. Let us, recall the axiomatic of this calculus as formulated by Łukasiewicz. According to Leśniewski's postulates we have no free variables but only bound variables. In this way our axiomatic has the following form:

\[ P_1 \: [pqr] \cdot p \implies q \implies q \implies r \implies p \implies r \]
\[ P_2 \: [p] \cdot \neg p \implies p \implies p \]
\[ P_3 \: [pq] \cdot r p \implies \neg p \implies q \]

Now let us add to this theses a following one, which establishes extensionality for the propositions:

\[ P_4 \: [pqf] \cdot r p q f(p) \implies f(q) \]

From \( P_1 \rightarrow P_4 \) we can easily obtain a following consequence, which I shall state only without proving it.

Theses \( P_1 \rightarrow P_3 \), establish, as we know, the complete axiomatic of the calculus of propositions, and therefore we obtain from them in the ordinary way:

\[ P_5 \: [pqrs] \cdot \neg p \implies q \implies r \implies s \implies q \implies r \implies s \]

We can add three following definitions:

\[ D_1 \: \Omega \: [u] \cdot u \]
\[ D_2 \: \equiv \: [u] \cdot u \equiv [u] \cdot u \]
\[ D_3 \: [p] \cdot \neg p \equiv 0 \equiv p \]

Now we can obtain from \( P_4, D_1 \) and \( D_3 \):

\[ P_6 \: [p] f : \neg p \cdot f(0) \implies f(p) \]

and from \( P_4, D_2 \) and \( D_3 \):

\[ P_7 \: [p] f : p \cdot f(0) \implies f(p) \]
From $P_5$, $P_6$ and $P_7$ we can obtain at once:

$$P_8 \ [\vdash \phi] : \{0\} \cdot \{1\} \cdot \{\phi\}$$

On the contrary, as can be easily demonstrated $P_4$, $P_5$ and $P_6$ imply $P_8$.

Therefore, there is an inferential equivalence between $P_1$, $P_2$, $P_3$, $P_6$ and $S_1, S_2, S_3, S_4$.

The thesis $P_8$ is called the bi-value principle for propositions. We can adopt a similar principles for every semantical category. This, for the propositional functor for one propositional argument we have the following thesis:

$$P_9 \ [\vdash \phi] : \phi \langle \sim \rangle, \phi \langle \circ \rangle, \phi \langle \psi \rangle, \phi \langle \lambda \rangle, \phi \langle \omega \rangle, \phi \langle \chi \rangle$$

A very easy proof demonstrates, that if we have the complete calculus of propositions, then $P_9$ is inferentially equivalent to

$$P_{10} \ [\vdash \phi] : \{\gamma\} \cdot \{\phi\} \cdot \{\gamma\} \cdot \{\phi\} : \phi \langle \sigma \rangle$$

It is to the thesis, which establishes the extensionality for the variables, belonging to this category.

We can give a general proof, that in the field of the calculus of propositions, for any semantical category, the thesis guaranteeing the extensionality of this exactly category is equivalent to the corresponding thesis, which determines the bi-value principle for this category.

The set of all such theses together with the whole calculus of propositions and with all possible deductive consequences is what we call protothetic. In other words protothetic is the calculus of propositions, enlarged by the variable functors for any semantical category and by the theses, guaranteeing the extensionality for any semantical category, which could appear in its field.
Evidently the deductive consequences of this theory go considerably farther beyond that what we can obtain from the simple calculus of propositions. This last is but a small fragment of the whole protothetic, which contains a great number of theses which do not belong to the calculus of propositions. There are, for instance, theses:

\[ P_{11} \quad \{p \}: f\{(f(r))\} = \neq f(r) \]

\[ P_{12} \quad \{f\}: f\{f(c)\} = \cdot \{c\} \cdot \{f(c)\} \]

all theses which stand for the operational rules concerning quantifiers and others.

The theory described this way can be presented in the precise form of a deductive theory and can possess its own axiomatic. It goes without saying that we can base this theory on various primitive terms. We can, for instance, choose the implication as a sole primitive term of the protothetic. In this case, the following theses can serve as its axiomatic:

\[ P_{15} \quad \{r\} : r \rightarrow q \rightarrow r \]

and \( P_{6} \).

But for the reasons, which I cannot explain here, the most convenient primitive term, on which we can found the protothetic, is equivalence. Thanks to the result of Dr. A. Tarski, who proved, that in the protothetic we can define a conjunction by equivalence only, in the following way

\[ Q_{4} \quad \{r\} : r \cdot q \equiv f \{f\} : f(r) = f(q) = q \]

we can use this term, as a sole primitive term of protothetic. Researches conducted many years by Leśniewski, Dr. M. Wajsberg and myself, led eventually after successive simplifications to the following shortest sole axiom of protothetic, discovered by me in 1945:
From this thesis, through rather complicated and difficult deductions, we can derive, applying only the rule of procedure of protothetic, all theses, belonging to this theory.

I cannot explain here, how is formulated the rule of procedure of protothetic, I shall only mention, that there are proofs of consistency and completeness of this theory. Protothetic is complete in the meaning, that any expression, which is senseful in its field, for instance:

\[ [t] \cdot \psi(t) \]

either itself or in negation, it is:

\[ \sim ([t] \cdot \psi(t)) \]

is a thesis of protothetic.

Protothetic as conceived by Leśniewski is nothing else than a consistent, immense enlargement of the calculus of propositions whose fundamental assumptions however do not differ from other contemporary conceptions of this calculus. On the contrary Leśniewski's calculus of terms or ontology, as he called it, contains ideas which differ fundamentally from the generally accepted opinions. Here, as in other instances of various deductive theories constructed by him, a starting point for Leśniewski was his analysis of the current language. As a result of such analysis he selected some linguistic forms corresponding most adequately to his ideas and later used these forms only having them strictly defined. So, for instance, the analysis of the individual /atomic/ proposition:

\[ A \text{ is } b \]

Socrates is white

led him to the following interpretation of this expression:
If anybody says

A is b

then in fact he says three following things:

I. A is unempty,
II. If anything is A, then it is b,
III. A is unique.

It is, if I say "Socrates is white", I say:

1/ There is in the world a\text{\_object\_} which is Socrates,
2/ If something is Socrates, then it is white,
2/ Socrates is unique such object in the world.

According to this definition of the individual proposition, the proposition:

Jove is a god

is false, because it does not satisfy the first condition; the proposition:

Socrates is an American

is false, because it does not satisfy the second condition;
the proposition:

man is mortal

is false, because it does not satisfy the third condition,

The meaning of individual propositions understood and established in this way was in complete harmony with the opinions, which we held by Leśniewski concerning the logic of terms. The conception of modern logic, according to which the subject and the predicate of the individual proposition belong to the different logical types was considered by Leśniewski as unnatural. This conception introduced into the logic for the first time by Peano in order to avoid some difficulties and
and adopted later by Russell, was, for him, an intolerable impoverishment of our logical language. This unnatural construction, introduced only to avoid some antinomies, is, as he demonstrated, by no means indispensable for this aim. We can construct a noncontradictory system of logic, in which the both these parts of individual propositions will belong to the same semantical category. I will show later that in such a system it is impossible to construct Russell's antinomy. This point is closely connected with Leśniewski's opinion concerning the unique-class and the null-class. According to him, if we accept the existence of the unique-class as an object, which is different from its own unique element, then we accept the existence of an unnatural being. The unique-class is the same object as its own unique element. Evidently, this position solves in advance the question of the null-class. In Leśniewski's logical language and system there are no null-classes, because in the world there are no contradictory or empty objects. We can introduce the notion of contradictory object, but we cannot build a theory based on such notion. Leśniewski used to say, that he cannot agree with, nor understand those theorists of set theory, who build this theory, as a theory of nothing. For him the well known constructions

\[
\{ 0 \}, \{ 1 \}, \{ 2 \}, \{ 3 \}, \cdots
\]

are impermissible hyponyses.

There is a complete harmony between these considerations and Leśniewski's interpretation of individual proposition. According to this interpretation and following the practice of the current language he wanted to have in his logical system as senseful propositions
such propositions as for instance:

Aristotle is the author of Organon

and

The author of Organon is Aristotle.

He wanted also to have included in his system the following form of reasoning:

\[ \text{If } A \text{ is } B \text{ and } B \text{ is } C, \text{ then } A \text{ is } C. \]

Having established the sense of individual proposition, Leśniewski faced the problem how could he build a system of logic which would harmonize with these considerations:

1/ He would not adopt this method of modern logic, by which we build the calculus of terms having established only the sense of name functions, without any axiomatic explicit formulation. According to Leśniewski's conception of the deductive theory no semantic category/ or logical type/ can be accepted, unless based on some axiomatic.

2/ He stressed that the propositional function might appear in two different forms namely

\[ f(x) \quad \text{and} \quad x \in A \]

These two logical forms are entirely different, what is also acknowledged by the current usage of our language in which we say for instance "Mary loves" and "Aristotle is a Greek".

It is obvious that these two sentences have a completely different grammatical structure, and besides there are no reasons why we should not have in a logical theory two corresponding forms.

Unfortunately these two forms are not distinguished by the modern logic what Leśniewski regarded as the peak of confusion and misunderstanding.
Therefore, he wanted to build such a calculus of terms, which would satisfy the following conditions:

1/ It should have its own axiomatic, which would establish its primitive terms, and determine its initial semantical categories.

2/ It should distinguish and define exactly two forms of individual proposition, it is \( \phi(x) \) and \( \lambda \).

3/ The subject and the predicate of the individual proposition should belong to the same semantical category, namely to the category of names.

4/ It should not contain neither unique class nor null-class.

The above mentioned analysis of individual propositions adopted by \( \text{Laśniewski} \), was a starting point for the construction of the logical theory, possessing such properties. He tried to express the conditions, which are equivalent to the individual proposition:

\[ A \text{ is } B \]

in such a way, as to express them by the copula "is". In other words, having established the meaning of the word "is", he tried to employ this word as the sole primitive term of his ontology. In this way, he adopted the following meanings of those formerly mentioned conditions:

I. A is unempty \( \exists \mathcal{E} B \). B is A.

II. If anything is A, then it is \( B \).

\[ \exists B \text{ If } C \text{ is } A \text{, then } C \text{ is } B \]

III. A is unique \( \exists \mathcal{E} \exists \mathcal{G} \text{ If } D \text{ is } A \text{ and } E \text{ is } A \text{, then } D \text{ is } E \).

Using the symbol "\( \text{is} \)" for the word "is", we can now write the following thesis:
As the subsequent investigations have shown this thesis can serve as the sole axiom of Leśniewski's ontology. My later investigations demonstrated in 1939, that the following thesis can also serve as the sole axiom of this theory:

02 \([A\vec{a}]: A_{\vec{a}} \equiv [\satisfies \vec{b}]. h_{\vec{a}} A : [b]: h_{\vec{a}} A \cdot h_{\vec{a}}. A_{\vec{a}}. \not h_{\vec{a}} A \cdot h_{\vec{a}} C\]

From this one thesis applying only the rule of procedure of ontology we can derive theses, corresponding to all theses of the traditional logic of algebra of classes and of other parts of calculus of terms and relations. When compared, for instance, with "Principia Mathematica", ontology has a considerably greater wealth of logical forms.

Unfortunately, I can only mention few points concerning this theory:

I. Ontology is a theory based on protothetic. Its rule of procedure is constructed in such manner that there is a close connexion between both theories.

II. Its sole primitive term is "\(\not\)", but we can adopt also other terms, as primitive. Initial semantical categories are two: the semantical category of names and the semantical category of proposition- nal functors for two name arguments,

III. Ontology contains several peculiar theses, as, for instance:

03 \([A B \vec{a}]: A \lor B. \not h_{\vec{a}} A. A_\vec{a} \]
04 \([A \vec{a}]: A_\vec{a} \cdot A_{\vec{a}} \]
05 \([A B \vec{a}]: A \lor B. \not h_{\vec{a}} A. \not h_{\vec{a}} B \]

As ontology has two different terms:

/1/ "Identical" used in respect to objects, and

/2/ "equal to" used in respect to general names there are two
following definitions:

\[ A \circ b : A = b \Rightarrow A \circ (a \circ b) \]

\[ a \circ b : a = b \Rightarrow (a \circ b) \]

IV. For variables belonging to any semantical category we can establish notions and theses corresponding to the properties of, so called descriptions. Moreover, in the field of ontology we must distinguish the functors, which are connexions between general names, for instance:

\[ a \circ b \]

and functors, which are relations between objects, for instance:

\[ A = B \]

In other words, the relation is such a functor, which has only objects as its arguments. We can define the relation functor for two arguments in the following way:

\[ \{ A \} : Y \times Y \Rightarrow (A \circ b) \Rightarrow (A \circ b) \]

V. In the field of ontology for any semantical category we can define a term, possessing properties analogical to the properties of the primitive term \( \Sigma \# \). For every such term we can prove theses, which are analogical to the axioms of ontology and to the definitions previously adopted. It means that, for instance, for the propositional functors for two name arguments:

We can introduce the following definitions:

\[ [A \circ b] : \phi \circ \# \psi \Rightarrow \{ \{ A \} \circ \phi \circ \# \psi \} \Rightarrow \{ \{ A \} \circ \phi \circ \# \psi \} \]

Subsequently, we can prove the following thesis:

\[ \phi \circ \# \psi \Rightarrow \phi \circ \# \psi \Rightarrow \phi \circ \# \psi \]

which is analogical to the axiom and theses analogical to the definitions already adopted for names. In this way, we have the certitude, that, for instance, in the field of relations we can obtain all theses,
analogical to those which we have for the names.

VI. Like protothetic, ontology is an extentional system. For
variables, belonging to any semantical category we are able to obtain
by special rules of procedure a thesis, guarantying the extentionality
of these variables. For instance, if we have name variables, then
we have also the following thesis:

07 \[ a : b \] : \[ a \] : A \rightarrow a \equiv A \sim b : \equiv : \[ b \] : \[ a \] (a) \equiv \[ b \] (b)

For propositional variables for one name argument we have the following
thesis:

08 \[ x \] : \[ a \] : \[ x \] (a) \equiv \[ x \] (b) : \equiv : \[ b \] : \[ x \] (b) \equiv \[ a \] (x)

VII. We have an easy and short proof of consistency of ontology.
More particularly we cannot construct in its field any reasoning simi-
lar to Russell's antinomy. This is due to the rule of procedure, which
allows the construction of ontological definitions only in this way, that
that:

If any variable is a subject of the definiendum of some onto-
logical definitions, the definiens of this definition must imply, that
this variable is an object.

Using this condition or an equivalent one we can for instance
define the following terms:

09 \[ a : b \] : A \rightarrow \sim (A \sim a) \sim A

100 \[ a : b \] : A \rightarrow \sim (A \sim a) \sim A

11 \[ a : b \] : A \rightarrow \sim (A \sim a) \sim A

Without the above described condition, we would immediately get a
typical contradiction.
According to Leśniewski's opinion, the essence of Russell's antinomy does not consist in the irregular construction of definitions, but it is the result of equivocality of the notions "class" and "element". He usually formulated Russell's antinomy in the following way:

The assumptions:

1. For any \( a \), there is the class of \( a \):

2. For any \( A \), \( a \) and \( B \), if \( A \) is the class of \( a \) and \( B \) is \( a \), then \( B \) is element of \( A \).

3. For any \( A \), \( a \) and \( B \), if \( A \) is the class of \( a \) and \( B \) is an element of \( A \), then \( B \) is \( a \).

It is evident. Nevertheless from these assumptions using correct logical methods, we can obtain a contradiction in the following way:

We introduce the following correct definitions:

and we obtain:

We obtain a contradiction because \( R1 \) is contrary to \( R3 \).

Therefore our contradiction is included in the assumptions \( R1 - R3 \)

I can give here only final conclusions of Leśniewski's analysis of this
problem. According to him in the assumptions R1 - R5 which are apparently evident, occurs the confusion of two entirely different meanings of the notion "class", namely "distributive class" and "collective class". In Łeśniewski's opinion the notion "distributive class" is an unreal term, used without any need and instead of a general name. If, for instance, we use the word "class" in this meaning in the sentence: "A is an element of the class of b", all we want to say is: "A is one of the objects belonging to b", or simply "A is B". Thus the notion "distributive class" can be replaced by the purely logical notion of "general name" and consequently it is superfluous.

On the contrary, the notion "collective class" is a real term, which denotes an existing real object. It is, if we use the word "class" in this meaning and we say "A ∈ Kl(b)", then we say, that A is an object which consist of these and only these objects, which are b. Thus, if I say "A is the class of books" then I say, that A is a real object, which consists of all books existing in the world, no matter where these books are. If we use the term "class" in this meaning, the expression "A is an element of class od b" signifies, that A is a part of this class or that A is this whole object, which is the class of b. However not necessarily A must be b. Thus, if A is an element of the class of men, then A can be any section of the class of b. For instance, A can be the object consisting of all American or A can be my head etc.

Now we adopt Łeśniewski's distinction between the "distributive class" and the "collective class", then a short analysis will convince us, that some of our assumptions /R1 - R5/ are false. In this way we get rid of Russell's antinomy, which was the result of the confusion of two different notions expressed one word. Some of these assumptions, which lead to the contradiction are true for "distributive class" and
these belong to logic, some are true for "collective class" and these belong to mereology.

Mereology is a deductive theory built by Łeśniewski in order to establish the properties of the collective class.

We can formulate a number of intuitive theses, which determine the properties of this notion, for instance:

M1. [\exists \bar{A} | \exists \bar{A} | \exists \bar{A} | \exists \bar{A}] \quad A \in \bar{A}

M2. [\exists \bar{A} | \exists \bar{A} | \exists \bar{A} | \exists \bar{A}] \quad A \in \bar{A}, B \in \bar{A}, C \in \bar{A}, D \in \bar{A}

M3. [\exists \bar{A} | \exists \bar{A} | \exists \bar{A} | \exists \bar{A}] \quad A \in \bar{A}, B \in \bar{A}, C \in \bar{A}, D \in \bar{A}, E \in \bar{A}, F \in \bar{A}, G \in \bar{A}, H \in \bar{A}, I \in \bar{A}, J \in \bar{A}, K \in \bar{A}, L \in \bar{A}, M \in \bar{A}, N \in \bar{A}, O \in \bar{A}, P \in \bar{A}, Q \in \bar{A}, R \in \bar{A}, S \in \bar{A}, T \in \bar{A}, U \in \bar{A}, V \in \bar{A}, W \in \bar{A}, X \in \bar{A}, Y \in \bar{A}, Z \in \bar{A},

M4. [\exists \bar{A} | \exists \bar{A} | \exists \bar{A} | \exists \bar{A}] \quad A \in \bar{A}, B \in \bar{A}, C \in \bar{A}, D \in \bar{A}, E \in \bar{A}, F \in \bar{A}, G \in \bar{A}, H \in \bar{A}, I \in \bar{A}, J \in \bar{A}, K \in \bar{A}, L \in \bar{A}, M \in \bar{A}, N \in \bar{A}, O \in \bar{A}, P \in \bar{A}, Q \in \bar{A}, R \in \bar{A}, S \in \bar{A}, T \in \bar{A}, U \in \bar{A}, V \in \bar{A}, W \in \bar{A}, X \in \bar{A}, Y \in \bar{A}, Z \in \bar{A},

We can also introduce several definitions, for instance:

D12. [\exists \bar{A} | \exists \bar{A} | \exists \bar{A} | \exists \bar{A}] \quad A \in \bar{A}, B \in \bar{A}, C \in \bar{A}, D \in \bar{A}, E \in \bar{A}, F \in \bar{A}, G \in \bar{A}, H \in \bar{A}, I \in \bar{A}, J \in \bar{A}, K \in \bar{A}, L \in \bar{A}, M \in \bar{A}, N \in \bar{A}, O \in \bar{A}, P \in \bar{A}, Q \in \bar{A}, R \in \bar{A}, S \in \bar{A}, T \in \bar{A}, U \in \bar{A}, V \in \bar{A}, W \in \bar{A}, X \in \bar{A}, Y \in \bar{A}, Z \in \bar{A},

D13. [\exists \bar{A} | \exists \bar{A} | \exists \bar{A} | \exists \bar{A}] \quad A \in \bar{A}, B \in \bar{A}, C \in \bar{A}, D \in \bar{A}, E \in \bar{A}, F \in \bar{A}, G \in \bar{A}, H \in \bar{A}, I \in \bar{A}, J \in \bar{A}, K \in \bar{A}, L \in \bar{A}, M \in \bar{A}, N \in \bar{A}, O \in \bar{A}, P \in \bar{A}, Q \in \bar{A}, R \in \bar{A}, S \in \bar{A}, T \in \bar{A}, U \in \bar{A}, V \in \bar{A}, W \in \bar{A}, X \in \bar{A}, Y \in \bar{A}, Z \in \bar{A},

It is: A is a part of B.

And we can also establish some theses concerning these terms, for instance:

M5. [\exists \bar{A} | \exists \bar{A} | \exists \bar{A} | \exists \bar{A}] \quad A \in \bar{A}, B \in \bar{A}, C \in \bar{A}, D \in \bar{A}, E \in \bar{A}, F \in \bar{A}, G \in \bar{A}, H \in \bar{A}, I \in \bar{A}, J \in \bar{A}, K \in \bar{A}, L \in \bar{A}, M \in \bar{A}, N \in \bar{A}, O \in \bar{A}, P \in \bar{A}, Q \in \bar{A}, R \in \bar{A}, S \in \bar{A}, T \in \bar{A}, U \in \bar{A}, V \in \bar{A}, W \in \bar{A}, X \in \bar{A}, Y \in \bar{A}, Z \in \bar{A},

M6. [\exists \bar{A} | \exists \bar{A} | \exists \bar{A} | \exists \bar{A}] \quad A \in \bar{A}, B \in \bar{A}, C \in \bar{A}, D \in \bar{A}, E \in \bar{A}, F \in \bar{A}, G \in \bar{A}, H \in \bar{A}, I \in \bar{A}, J \in \bar{A}, K \in \bar{A}, L \in \bar{A}, M \in \bar{A}, N \in \bar{A}, O \in \bar{A}, P \in \bar{A}, Q \in \bar{A}, R \in \bar{A}, S \in \bar{A}, T \in \bar{A}, U \in \bar{A}, V \in \bar{A}, W \in \bar{A}, X \in \bar{A}, Y \in \bar{A}, Z \in \bar{A},

M7. [\exists \bar{A} | \exists \bar{A} | \exists \bar{A} | \exists \bar{A}] \quad A \in \bar{A}, B \in \bar{A}, C \in \bar{A}, D \in \bar{A}, E \in \bar{A}, F \in \bar{A}, G \in \bar{A}, H \in \bar{A}, I \in \bar{A}, J \in \bar{A}, K \in \bar{A}, L \in \bar{A}, M \in \bar{A}, N \in \bar{A}, O \in \bar{A}, P \in \bar{A}, Q \in \bar{A}, R \in \bar{A}, S \in \bar{A}, T \in \bar{A}, U \in \bar{A}, V \in \bar{A}, W \in \bar{A}, X \in \bar{A}, Y \in \bar{A}, Z \in \bar{A},

Like all deductive theories built by Łeśniewski, also mereology has its own axiomatic. There are various terms, which we can be chosen as its primitive terms. For instance, in 1948 I proved, that if we choose as a sole primitive term the term "el" then the following simple thesis:

M8. [\exists \bar{A} | \exists \bar{A} | \exists \bar{A} | \exists \bar{A}] \quad A \in \bar{A}, B \in \bar{A}, C \in \bar{A}, D \in \bar{A}, E \in \bar{A}, F \in \bar{A}, G \in \bar{A}, H \in \bar{A}, I \in \bar{A}, J \in \bar{A}, K \in \bar{A}, L \in \bar{A}, M \in \bar{A}, N \in \bar{A}, O \in \bar{A}, P \in \bar{A}, Q \in \bar{A}, R \in \bar{A}, S \in \bar{A}, T \in \bar{A}, U \in \bar{A}, V \in \bar{A}, W \in \bar{A}, X \in \bar{A}, Y \in \bar{A}, Z \in \bar{A},

\text{can be used as a sole axiom of mereology.}
Mereology is based on protetastic and ontology and is constructed in such a way that it does not decide what objects do exist in the world. Therefore, it can be used by any physical or philosophical considerations. In mathematics we can use mereology as non-geometrical base for some systems of geometry.

We are in possession of a proof of the consistency of this theory.

I gave here hardly a short outline of Leśniewski's system of foundations of mathematics. However this system is so rich that in order to understand it fully and above all in order to understand its leading ideas, an extensive and systematic course would be necessary.
A ∈ Ω(δ) = S(A ∈ δ)

1) Substancja matematyczna
2) System logiki (filozoficzny)
3) Substancja logiki
4) Zdanie oznaczające
5) Operacje nad jednym zdaniem.
6) Operacje nad dwoma zdaniami.
1) Tera
2) Alegjmatyka
3) Dysydzyng.