Ancestral Relations

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These are Rickey's notes of a seminar given by Sobociński in 1967-1968. The notes are not dated, but the file that they are in is dated.

The full title for these notes is "Completely formalized systems of ancestral relations." 20 pages.

This material has not been published.

Prof Sobociński

Completely Formalized Systems of Ancestral Relations.

Every thing which is known is proveable from his three axioms.

This theory based on the calculus of relations, He doesnt mean the theory of relns. (CR ⊆ TR).

Reln: functors of two name arguments.

CR is formulated w/o use of s.l.l's & int quants. So we only have MGQ in CR. In CR we have things like. $S \subset T . R \subset S . \supset . R \subset$

But $[xy] . xRy . \supset . xSy : [xy] : xSy . \supset . xTy : \supset : [xy] : xRy . \supset . xTy$

is a thing of TR not CR.

DeMorgan, Peirce, & Schroder all studied TR. Tarski gave an Ax syst for CR.

Accept some Axiom syst for Boolean Algebra & to this we add (prim terms: $=$, $\subset$, $\overline{R}$ (compliment), 0, 1).

Prim to calc of relns we take converse: $[xy] : x\breve{R}y . \equiv . yRx$.

also take relative multiplication as prim: $[xy] . xR|Sy . \equiv . [\exists z] . xRz . zSy$

Rel product is akin to composition of fncts. Accept the constant relns I: $[xy] : xIy . \equiv . x = y$

J: $[xy] : xJy . \equiv . \sim(x = y)$.

Besides the Axioms for B.A we accept the following axioms:
relns are the elts of the B.A.

1. $\breve{\breve{R}} = R$

2. $\breve{(R|S)} = \breve{S}|\breve{R}$

3. $(R|S)|T = R|(S|T)$

4. $R|I = R$

5. $R|1 = 1 . \vee . 1|\overline{R} = 1$

6. $R|S \cap \breve{T} = 0 . \supset . (S|T) \cap \breve{R} = 0$

5. plays a very important role.

Suppose the set A is our universe. Each reln R can be presented as a (possibly infinite) matrix of side $|A| \times |A|$.

$R|1$ takes all rows wherein there are elts of R; $1|R$ takes every col. in which there is an elt of R. So Ax 5 says R has elts in every row or $\bar{R}$ in every column. Schroder knew this thm, but its not mentioned in P.M.

Bernes Conod in Math J. several years ago. Shortest Ax Syst for B.A.

Note We have no rule of extentionality.

Thus far we have the elementary calculus of relations.

Now we generalize this by adding ancestral relations as a primitive term.

Ancestral Relns are introduced independently at about the same time by Dedekind (Was sind und was sollen die Zahlen). In his 167 props he discusses kettentheory (theory of chains) & as prop 59 he established full induction using this chain notion. He didn't discuss this fully but in a very mathl way. Schroder elaborated the

At the same time Frege in Begriffsschrift gives a definition. This is the basis of the chapters concerning ancestral relations in P.M.

$\Phi_\alpha(R) \equiv: x = y \,.\, v \,.\, xRy \,.\, v \,.\, xR|Ry \,.\, v \,.\, xR|R|Ry \ldots$

i.e.

$\equiv, \; I \vee R \vee R^2 \vee R^3 \vee \ldots$

Dedekind & Schroder denoted such relns by R_0. R & W by R_* & Sobocinski shall use $\Phi_\alpha(R)$.

When we drop the $x = y$ disjunct we get another reln, called R_{00} by D & S. R_{po} by R & W. & We use $\Phi_\beta(R)$.

P.M. doesn't even mention the dual of those which were discussed by D & S. The dual of R_0 is $\overline{(\overline{R_0})} = R_1$:

$\overline{I} \wedge R \wedge R \dagger R \wedge R \dagger R \dagger R \wedge \ldots$

where $[xyRS]: xR \dagger Sy \,.\, \equiv [v]: xRv \,.\, v \,.\, vSy$.

Sobo uses $\Phi_\gamma(R) = R_1$ & $\Phi_\delta(R) = R_{11} = \overline{(\overline{R_{po}})}$ i.e.

$R \wedge R \dagger R \wedge R \dagger R \wedge \ldots$

Now, how to ~~def~~ define such notions in general logic. We go to P.M. They like to give defs in a very condensed way & then prove elucidating thms. S prefers to start with intuitive defs. PM V1. $*90.1$:

$*90.1 \; [xyR]:: xR_*y \,.\, \equiv :. \; x \in C'R : [\mu]: \breve{R}''\mu \subset \mu \,.\, x \in \mu \,.\, \supset . \, y \in \mu.$

$*90.11 \; [xyR]:: xR_*y \,.\,.\, \equiv :. \; x \in \underline{C'R} : [\mu]:. [zw]. \, z \in \mu \,.\, zRw \,.\, \supset . \, w \in \mu : x \in \mu \,.\, \supset . \, y \in \mu.$

field

*91.16. $[xyR] :: x \, R_{po} \, y . \equiv :. [\exists P] . x P y :. [\varphi] :. [S] . \varphi(S) . \supset . \varphi(S|R) : \varphi(R) . \supset . \varphi(P)$

*91.15 $[PR] :: P \in \mathrm{Potid}'R . \equiv :. [\varphi] :. [S] : \varphi(S) . \supset . \varphi(S|R)$
$\varphi(I \upharpoonright C'R) . \supset . \varphi(P)$.

*91.13. $[PR] :: P \in \mathrm{Pot}'R . \equiv . [\varphi] :. [S] : \varphi(S) . \supset . \varphi(S|R) :$
$\varphi(R) . \supset . \varphi(P)$.

Class of all powers of R, beginning with R itself.

These are extremely powerful inductive definitions.

Now for the axioms. Stokes $dp_\alpha(R)$ as primitive (the other three are definable).

A1 $[R] . \overline{dp_\alpha(R)} = dp_\alpha(\breve{R})$ [This is in Schröder & P.M. its also self evident]

A2 $[RS] : R|S \subset S . \supset . dp_\alpha(R)|S \subset S$ [Not in P.M, but to Dedekind & Schröder

This is an extremely powerful Axiom.

A3 $[R] : dp_\alpha(R) = I \cup R | dp_\alpha(R)$ [In P.M]

at first he had to accept
A0: $P = R . \supset . dp_\alpha(R) = dp_\alpha(S)$, but
this follows from A1-3 algebraically

We justify now A2.

D1 $[xy\varphi] : x \, U\varphi \, y . \equiv . [\exists R] . \varphi(R) . x R y$.

$U\varphi$ can't be taken as a prim term in CR. Neither can we take generalized B.A. as the basis.

D2. Potid, as above.

T1. $\phi_\alpha(R) . = \cup \text{ Potid}'R$ (This is in P.M.)

D3. $[\cancel{S}R]. \rho_\alpha\{S\}(R). \equiv . R|S \subset S$

T2. $R|S \subset S . \rho_\alpha\{S\}(\overset{T}{\cancel{R}}) . \supset . \rho_\alpha\{S\}(T|R).$

Hyp(2).⊃.

3) $T|S \subset S.$ [D3, 2]

4) $T|R|S \subset T|S.$ [1, B]

5) $T|R|S \subset S.$ [3,4]

$\rho_\alpha\{S\}(T|R).$ [5. D3]

V. asked why the links in D3. S replied: I am accurate man.

~~T3. R|S ⊂ S.~~

Write $\text{Potid}'\{R\}\{p\}$ for $p \in \text{Potid}'(R)$.

T3 $R|S \subset S . \text{Potid}'\{R\}(P) . \supset . P|S \subset S.$

Hyp(2).⊃.

3) $[\varphi] :. [S] : \varphi(S) . \supset . \varphi(S|R) : \varphi(I) . \supset . \varphi(P).$ [D2, 2]

4) $[T] : \rho_\alpha\{S\}(T) . \supset . \rho_\alpha\{S\}(T|R).$ [T2, 1]

5) $\rho_\alpha\{S\}(I).$ [D3, B]

6) $\rho_\alpha\{S\}(P).$ [3, $\varphi/\rho_\alpha\{S\}$, 4,5]

$P|S \subset S.$ [D3, 6]

T4 $R|S \subset S.\ x(\cup\, \mathrm{Potid}'R)|S\, y.\ \supset.\ x S y.$

Hyp(2). $\supset$:

$[\exists z]$: $x \cup \mathrm{Potid}'R z$ } [2, def of |]

3) $z S y$

4) $[\exists P]$.

5) $\mathrm{Potid} \vdash R \vdash (P)$ } [D1, 3]

6) $x P z$ [T3, 1, 5]

7) $P|S \subset S.$ [6, 4, Def 1]

8) $x P|S\, y.$ [7, 8]

$x S y$

T5. $R|S \subset S.\ \supset.\ \mathrm{dp}_{\alpha}(R)|S \subset S.$ *i.e. A2.*

2nd Seminar

The Axioms have been changed a little:

A0. $P = R . \supset . \phi_\alpha(P) = \phi_\alpha(R)$ [The converse is not true. See, e.g. K61]

A1 $\overline{\phi_\alpha(R)} = \phi_\alpha(\breve{R})$

K1 = A2 $\phi_\alpha(R) = \mathcal{J} \cup R | \phi_\alpha(R)$

K2 = A3 $R|S \subset S . \supset . \phi_\alpha(R)|S \subset S$.

He remarked last week that he could prove A0. Now he has a proof of A1. So we have everything on CR & K1. K2.

Rules: Sub, Det.

New Thms:

K3 $\overset{\text{I}}{\mathcal{J}} \subset \phi_\alpha(R)$. [K1]

K~~3~~ $R | \phi_\alpha(R) \subset \phi_\alpha(R)$. [K1]

[K3, CR]

K8 $B \subset \phi_\alpha(R)|B$.

[K3].

K9 $V \subset V | \phi_\alpha(R)$.

[K9, K4]

K12 $R \subset \phi_\alpha(R)$

K13 $V \subset \phi_\alpha(R)|S . \supset . R|V \subset \phi_\alpha(R)|S$.

Hyp. $\supset$. (1)

2) $R|V \subset R|\phi_\alpha(R)|S$.

$R|V \subset \phi_\alpha(R)|S$ [K4,2]

★ K15′ $R|S \subset S . V \subset S . \supset . \phi_\alpha(R)||V \subset S$. This is very useful.

Hyp $\supset$.

3) $\phi_\alpha(R)|V \subset \phi_\alpha(R)|S$. [2, R] general calc of relns.

4) $\phi_\alpha(R)|S \subset S$. [K2,1]

$\phi_\alpha(R)|| V \subset S$. [3,4].

Now we ~~[illegible]~~ prove A0.

★ K16 $\quad R \subset S . \supset . \phi_\alpha(R) \subset \phi_\alpha(S)$

Hyp(1). $\supset$.

2) $\quad R | \phi_\alpha(S) \subset S | \phi_\alpha(S)$. $\quad$ [1, R]

3) $\quad R | \phi_\alpha(S) . \subset \phi_\alpha(S)$. $\quad$ [2, K4]

4) $\quad \phi_\alpha(R) | I \subset \phi_\alpha(S)$. $\quad$ [K13, 3, K3]

$\quad \phi_\alpha(R) \subset \phi_\alpha(S)$. $\quad$ [4, R].

K17. $\quad P = R . \supset . \phi_\alpha(R) = \phi_\alpha(S)$. $\quad$ [K16] (try K1 to get the converse

K18. $\quad \phi_\alpha(R) | \phi_\alpha(R) \subset \phi_\alpha(R)$. $\quad$ [K2, K4]

K19. $\quad \phi_\alpha(R) \subset \phi_\alpha(R) | \phi_\alpha(R)$. $\quad$ [K8]

K20 $\quad \phi_\alpha(R) = \phi_\alpha(R) | \phi_\alpha(R)$. $\quad$ [K18, K19]

This is a very strong relation.
So if R is transitive, $\phi_\alpha(R) = \phi_\rho(R)$.
Dedekind, Schroder, Frege all knew this

K21 $\quad I \subset \phi_\alpha(R)$. $\quad$ [K3, R]

K22 $\quad \breve{R} \subset \overline{\phi_\alpha(\breve{R})}$. $\quad$ [K12] 6 $R \subset S . \supset . \breve{R} \subset \breve{S}$

K24 $\quad \breve{R} | \overline{\phi_\alpha(R)} \subset \overline{\phi_\alpha(R)}$

1) $\quad \breve{R} | \overline{\phi_\alpha(R)} \subset \overline{\phi_\alpha(R)} | \overline{\phi_\alpha(R)}$ $\quad$ [K22, R]

2) $\subset \overline{\phi_\alpha(R) | \phi_\alpha(R)}$ $\quad$ [1, R]

3) concl. $\quad$ [2, K18]

<u>K25</u> $\phi_\alpha(\breve{R}) \subset \overline{\phi_\alpha(R)}$ [K15, $R/\breve{R}$, $S/\phi_\alpha(R)$, V/J, K23, K21]

<u>K26</u> $\phi_\alpha(R) \subset \overline{\phi_\alpha(\breve{R})}$ [K25, $R/\breve{R}$, <u>K17</u>]
(Because Ext is not a rule.

<u>K27</u> $\overline{\phi_\alpha(R)} \subset \phi_\alpha(\breve{R})$ [K26]

★ <u>K28</u> $\overline{\phi_\alpha(R)} = \phi_\alpha(\breve{R})$ [K25, K27]

This proves our former axiom A1.

<u>K29</u> $\phi_\alpha(R) = \overline{\phi_\alpha(\breve{R})}$ [K28]

<u>K30</u> $\phi_\alpha(R) = J \cup \phi_\alpha(R)|R$. [This is a turn around on K1]

1) $\phi_\alpha(R) = \overline{\phi_\alpha(\breve{R})}$ [K29]

2) $= \overline{J \cup \breve{R}|\phi_\alpha(\breve{R})}$ [K1, 1]

3) $= J \cup \overline{\phi_\alpha(\breve{R})}|\breve{\breve{R}}$ [2, R]

$= J \cup \phi_\alpha(R)|R.$ [3, K29, R]

<u>K31</u>. $S|R \subset S . \supset . S|\phi_\alpha(R) \subset S.$ [K2, K28, R]

Anal. to K2.

<u>K36</u>. $S|R \subset S . V \subset S . \supset . V|\phi_\alpha(R) \subset S.$ [K31; Sim K15

~~$\phi_\alpha(R)|R$~~

All of these thms except K2 are essentially known in P.M. Dedekind mentioned K2 in an intuitive way. Schroder gave a three dot proof of it

K38. $\Phi_\alpha(R) \mid R \subset R \mid \Phi_\alpha(R)$ [K15, S/R | $\Phi_\alpha(R)$, K11, K19, V/R]

K39 $R \mid \Phi_\alpha(R) \subset \Phi_\alpha(R) \mid R$. [K36 S/$\Phi_\alpha(R) \mid R$, K37, K27]

★ K40 $R \mid \Phi_\alpha(R) = \Phi_\alpha(R) \mid R$. [K38, K39].

This gives us a great deal of freedom in our calculus.

K41 $\Phi_\alpha(R) \subset \Phi_\alpha(\Phi_\alpha(R))$ [K16, K12]

K42. $\Phi_\alpha(\Phi_\alpha(R)) \subset \Phi_\alpha(R)$. [K15, R/$\Phi_\alpha(R)$, S/$\Phi_\alpha(R)$, V/J K18, K3, R]

★ K43 $\Phi_\alpha(\Phi_\alpha(R)) = \Phi_\alpha(R)$ [K41, K42]

Now something previously unknown. (after a few banalities

K44 $1 = 1 \mid \Phi_\alpha(R)$ [K9, K8]

K45 $1 = \Phi_\alpha(R) \mid 1$ "

Schroder was very near to the following, but there is no mention of it in P.M.

★ K63. $\phi_\alpha(0) = \mathfrak{I}$

1) $\phi_\alpha(0) = \mathfrak{I} \cup 0 \mid \phi_\alpha(0)$ [K1]

$= \mathfrak{I}$ [R]

K64 $\phi_\alpha(1) = 1$ [K1]

K65 $\phi_\alpha(\mathfrak{I}) \subset \mathfrak{I}$. [K16, K3]

1) $\phi_\alpha(\mathfrak{I}) \subset \phi_\alpha(\phi_\alpha(0))$. [K43, 1]

2) $\phi_\alpha(\mathfrak{I}) \subset \phi_\alpha(0)$. [K63, 2]

$\phi_\alpha(\mathfrak{I}) \subset \mathfrak{I}$.

K66 $\phi_\alpha(\mathfrak{I}) = \mathfrak{I}$ [K65, K3]

K67 $\phi_\alpha(\mathfrak{f}) = 1$

1) $\phi_\alpha(\mathfrak{f}) = \mathfrak{I} \cup \mathfrak{f} \mid \phi_\alpha(\mathfrak{f})$. [K1]

$= \mathfrak{I} \cup \mathfrak{f} \mid (\mathfrak{I} \cup \phi_\alpha(\mathfrak{f}))$ [1, K1]

$= \mathfrak{I} \cup \mathfrak{f} \cup \mathfrak{f} \mid \phi_\alpha(\mathfrak{f})$

$= 1$

K61 $\phi_\alpha(0) = \phi_\alpha(\mathfrak{I})$. [K63, K66]

K62 $\phi_\alpha(1) = \phi_\alpha(\mathfrak{f})$ [K64, K67].

Now for the formula of general induction. (The proof is Schroder's. S states it in an elegant way: Schroder was Mathematician,

K70 $R \mid [(\phi_\alpha(R) \mid V) \cap S] \subset \phi_\alpha(R) \mid V$

1) $R \mid [(\phi_\alpha(R) \mid V) \cap S] \subset R \mid (\phi_\alpha(R) \mid V)$ [R]

2) $\subset \phi_\alpha(R) \mid V$ [1, K10]

~~K71 [illegible]~~

K72 $R \mid [(\phi_\alpha(R) \mid V) \cap S] \subset S \,.\, V \subset S \,.\supset.$

$\phi_\alpha(R) \mid V \subset S.$

Hyp(2). $\supset$.

3) $V \subset (\phi_\alpha(R) \mid V \cap S).$ [K8, 2]

4) $\cancel{R} \mid [(\phi_\alpha(R) \mid V) \cap S] \subset [(\phi_\alpha(R) \mid V) \cap S]$ [K70; 1]

5) $\phi_\alpha(R) \mid V \subset [(\phi_\alpha(R) \mid V) \cap S]$ [K15, 4, 3]

$\phi_\alpha(R) \mid V \subset S.$ [5]

11-27-67

We give the motivation for K72 (in full logic):

1) $[xyzt]: xRz \,.\, z\,\phi_\alpha(R|t) \,.\, tVy \,.\, zSy \,.\supset:\, xSy:$

2) $[hk]: hVk \,.\supset.\, hSk :\supset.$

3) $[xvz]: x\,\phi_\alpha(R)\,z \,.\, zVv \,.\supset.\, xSy.$

K76. $\phi_\alpha(\mathcal{I} \cup R) = \phi_\alpha(R)$

K77 $\phi_\alpha(\mathcal{I} \cap R) = \mathcal{I}$ Schröder

K80 $\phi_\alpha(\mathcal{J} \cup R) = \phi_\alpha(\mathcal{J}).$ B.S.

K79 $\phi_\alpha(\mathcal{J} \cap R) = \phi_\alpha(R)$ Sch.

K89 $\phi_\alpha(1|R) = \mathcal{I} \cup 1|R$

K90 $\phi_\alpha(R|1) = \mathcal{I} \cup R|1$

K93 $\phi_\alpha(0 \dagger R) = \mathcal{I} \cup (0 \dagger R)$

K94 $\phi_\alpha(R \dagger 0) = \mathcal{I} \cup (R \dagger 0)$

From P.M. VI § 93 we have

B10 $R|S = S|R \,.\supset.\, \phi_\alpha(R)|\phi_\alpha(S) = \phi_\alpha(S)|\phi_\alpha(R)$

B1 $R|S \subset S|R . \supset . R|S|\phi_\alpha(R) \subset S|\phi_\alpha(R)$

Hyp $\supset$.

2) $R|S|\phi_\alpha(R) \subset S|R|\phi_\alpha(R)$. [1, B62]

$\subset S|\phi_\alpha(R)$. [K1, 2]

B2 $R|S \subset S|R . \supset . \phi_\alpha(R)|S \subset S|\phi_\alpha(R)$. [K17, K9]

B4 $S|R \subset R|S . \supset . S|\phi_\alpha(R) \subset \phi_\alpha(R)|S$ [K36, K10]

B5′ $R|S = S|R . \supset . \phi_\alpha(R)|S = S|\phi_\alpha(R)$. [B2, B4]

B6 $R|S = S|R . \supset . R|S|\phi_\alpha(R)|\phi_\alpha(S) \subset \phi_\alpha(R)|\phi_\alpha(S)$.

Hyp. $\supset$.

2) $\phi_\alpha(R)|S = S|\phi_\alpha(R)$. [B5′, 1]

3) $R|S|\phi_\alpha(R)|\phi_\alpha(S) \subset S|R|\phi_\alpha(R)|\phi_\alpha(S)$ [1]

4) $\subset S|\phi_\alpha(R)|\phi_\alpha(S)$. [K1, B62]

5′) $\subset \phi_\alpha(R)|S|\phi_\alpha(S)$ [2, 4]

$\subset \phi_\alpha(R)|\phi_\alpha(S)$ [K4, 5]

Clay: the B5′ $R|S$ $S|\phi_\alpha(R)$

B14 $\rightleftarrows\langle R\rangle . \supset . \phi_\alpha(R \cup \breve{R}) = \phi_\alpha(R) \cup \phi_\alpha(\breve{R})$

B11 $\rightleftarrows\langle R\rangle . \supset . [\phi_\alpha(R) \cup \phi_\alpha(\breve{R})]|(R \cup \breve{R}) \subset \phi_\alpha(R) \cup \phi_\alpha(\breve{R})$

<u>B12</u> $\rightleftarrows\langle R\rangle . \supset . \phi_a(R\cup\breve{R}) \subset \phi_a(R)\cup\phi_a(\breve{R})$

<u>B13</u> $\phi_a(R)\cup\phi_a(\breve{R}) \subset \phi_a(R\cup\breve{R})$ [K16]

<u>B21</u> $\rightleftarrows\langle R\rangle . \supset . \phi_a(\breve{R})|\phi_a(R) = \phi_a(R)|\phi_a(\breve{R})$

<u>lemmas</u> $\leftleftarrows\langle R\rangle . \supset . \phi_a(R)|\phi_a(\breve{R}) . = \phi_a(R)\cup\phi_a(\breve{R})$

$\rightrightarrows\langle R\rangle . \supset . \phi_a(\breve{R})|\phi_a(R) = \phi_a(R)\cup\phi_a(\breve{R})$

This ends the material motivated by P.M.

Dr Vučković suggests

$J \cup R \cup R f R \cup \dots$

This can do in gen logic & can do a fin no of steps.
But <u>not</u> on inf seq. (Since we have no distribution of f by $\cup$).

$P_2(PQR) = R, \quad P|R|Q, \quad P^2|R|Q^2, \dots$

set of cousins of the same generation.

R to be cousin

P to be child of

Q to be parent of

Dec 3

$\phi_\beta = R_{oo} = R_{po}$

Def $\phi_\beta = R \cup R^2 \cup R^3 \cup \dots$

So $I \cup \phi_\beta = \phi_\alpha$.

D1 $[R]: \phi_\beta(R) . \equiv . R | \phi_\alpha(R)$.

C1 $\phi_\beta(R) = \phi_\alpha | R$.

C2 $\phi_\beta(R) \subset \phi_\alpha(R)$.

C3 $\phi_\alpha(R) = I \cup \phi_\beta(R)$. Hence ϕ_β could also be taken as a prim term.

And of Ax1.

C4 $\phi_\beta(R) = R \cup R | \phi_\beta(R)$.

C5 $R \subset \phi_\beta(R)$.

This shows that ϕ_β is not as convenient as a prim term as ϕ_α, where we have $I \subset \phi_\alpha(\dots$

C6. $\phi_\beta(1) = 1$

C7. $\phi_\beta(0) = 0$ remember $\phi_\alpha(0) = I$

C11 $\phi_\beta(I) = I$

C1 $\sim(I = 1) . \supset . \phi_\beta(I) = 1$

Can't prove $\phi_\beta(I) = 1$ w/o the assumption that there are at least two objects

Hyp(1). $\supset$.

2) $\phi_\beta(I) = I | \phi_\alpha(I)$

3) $= I | 1$

$= 1$ use hyp here.

$\underline{C8}$ $\quad R \subset S . \supset . \ \phi_\beta(R) \subset \phi_\beta(S).$

$\underline{C9}$ $\quad R = S . \supset . \ \phi_\beta(R) = \phi_\beta(S).$

Remember Extentionality is not an Axiom

don't get = . as with ϕ_α

$\underline{C\chi}$ $\quad \phi_\beta(R) | \phi_\beta(R) \subset \phi_\beta(R)$

$\underline{C24}$ $\quad \phi_\beta(R) = \phi_\beta(\phi_\beta(R)).$

$\underline{C26}$ $\quad \phi_\alpha(R) = \phi_\alpha(\phi_\beta(R)).$

$\underline{C28}$ $\quad \phi_\alpha(R) = \phi_\beta(\phi_\alpha(R)).$

$\underline{C29}$ $\quad \overline{\phi_\beta(R)} = \phi_\beta(\breve{R})$

$\underline{C35'}$ $\quad R|S = S|R . \supset . \ \phi_\beta(R|S) \subset \phi_\alpha(R) | \phi_\alpha(S),$

but not = as with ϕ_α.

Now there are two functions which are not discussed in P.M. They are the duals of ϕ_α + ϕ_β.

$\underline{D2}$ $. [R] . \phi_\gamma(R) \equiv \overline{\phi_\alpha(\overline{R})}$

$\underline{F1}$. $\quad \overline{\phi_\gamma(\overline{R})} = \phi_\alpha(R)$

$\underline{F3}$. $\quad \phi_\gamma(R) = J \cap R \cap \overline{\overline{R} | \phi_\alpha(\overline{R})}$

This generates.

$$J \cap R \cap \underbrace{R \dagger R}_{= R_2} \cap R \dagger R \dagger R \cap \ldots$$

So $J \cap R \cap R_2 \cap R_3 \cap \ldots \quad = \phi_\gamma(R).$

where $xR \dagger Sy \equiv : [u] . xRu . v . uSy \equiv \overline{x\overline{R}|\overline{S}y}$

~~The~~ Thus $R|(S\cup T) = R|S \cup R|T$

$R\dagger(S\cap T) = R\dagger S \cap R\dagger T$

no equality when we distrib wrt $\cap$, $\cup$ resp in these two

allow the above statements for any finite number of conjuncts.

F4 $\phi_\gamma(R) \subset \mathcal{J}$

F5 $\phi_\gamma(R) \subset R$

F7 $\phi_\gamma(R) \subset R \dagger \phi_\gamma(R)$

F9 $\phi_\gamma(\breve{R}) = \overline{\phi_\gamma(R)}$

F10 $R \subset S . \supset . \phi_\gamma(R) \subset \phi_\gamma(S)$

F11 $\phi_\gamma(0) = 0$

F12 $\phi_\gamma(1) = \mathcal{J}$

F13 $\phi_\gamma(\mathfrak{I}) = 0$

F14 $\phi_\gamma(\mathcal{J}) = \mathcal{J}$

F71 $\phi_\gamma(R) \dagger \phi_\gamma(R) = \phi_\gamma(R)$

F72 $\phi_\gamma(R) = \phi_\gamma(\phi_\gamma(R))$.

As ϕ_γ is the dual of ϕ_α the proofs are analogous.

There are also, what Schröder called, pseudo-duals.

F43 $\quad \phi_\alpha(R) = \phi_\gamma(\breve{\bar{R}}) \dagger \phi_\alpha(R)$

F45' $\quad \phi_\alpha(R) = \phi_\gamma(\breve{\bar{\bar{R}}}) \dagger \phi_\alpha(R) \dagger \phi_\alpha(\breve{\bar{\bar{R}}})$

F47 $\quad \phi_\gamma(R) = \phi_\gamma(R) | \phi_\alpha(\breve{\bar{\bar{R}}})$

F63 $\quad \phi_\gamma(R) \dagger V = \phi_\alpha(\breve{\bar{R}}) | [\phi_\gamma(R) \dagger V]$

The second dual is a dual for ϕ_β. Schröder uses R_{11}

D3 $\quad \phi_\delta(R) \; .\equiv. \; \overline{\phi_\beta(\bar{R})}$

Intuitively $\phi_\gamma(R) = R \cap R_2 \cap R_3 \cap \ldots$

We get, naturally, dual thms of those for ϕ_β.

H4 $\quad \phi_\delta(R) \subseteq R$

H8 $\quad \phi_\delta(\breve{R}) = \overline{\phi_\delta(R)}$

H9 $\quad \phi_\delta(1) = 1$

H10 $\quad \phi_\delta(0) = 0$

H11 $\quad \phi_\delta(\mathcal{J}) = \mathcal{J}$ $\quad$ note the change

H12 $\quad \sim(\mathcal{J} = 1) . \supset . \; \phi_\delta(\mathcal{J}) = 0.$

There are difficulties involving

$\phi_\alpha(R) = \mathcal{J} \cup R \cup R_2 \cup R_3 \cup \ldots$

The difficulty is that $\dagger$ doesn't dist thru $\cap$, so he can't give a def which eqs ands into any fin no of terms of this union.

Def $[xyR] :: \rho_\alpha\langle R\rangle\{xy\}. \equiv :. [\exists P]. xPy :. [\varphi] :. \varphi(R). \supset \varphi(P).$
$[S] : \varphi(S). \supset. \varphi(S \dagger R);$

We can define $\rho_\beta, \rho_\gamma, \rho_\delta$ sim to $\phi_\beta . \phi_\gamma . \phi_\delta$

where
$$\rho_\beta = R \cup R_2 \cup R_3 \cup \ldots$$
$$\rho_\gamma = \mathcal{I} \cap R \cap R|R \cap \ldots$$
$$\rho_\delta = \quad R \cap R|R \cap \ldots$$

This notion ρ_α is akin, but not dual to ϕ_α.

I hasn't managed to develop ρ_α into a theory, even in full logic.

In particular how can ρ_α be defined in the calc of relations?

$$T_\beta(R,P,Q) = R \cup P|R|Q \cup P^2|R|Q^2 \cup \ldots$$

Then $\phi_\beta(RPQ)$ can define ϕ_α

by $\phi_\alpha(R) = T_\beta\langle I R I\rangle$

So would be interesting to axiomatize T_β.

$$R \cup R_2 \cup R^3 \cup R_4 \cup R^5 \cup \ldots$$

is akin to Kleene Hierarchy