CHAPTER 6
PLUMBING ESSENTIAL STATES IN KHOVANOV HOMOLOGY

We prove that every homogeneously adequate Kauffman state has enhancements $X^{\pm}$ in distinct j-gradings whose traces (which we define) represent nonzero Khovanov homology classes over $\mathbb{F}_2$, and that this is also true over $\mathbb{Z}$ when all A-blocks' state surfaces are two-sided. A direct proof constructs $X^{\pm}$ explicitly. An alternate proof, reflecting the theorem's geometric motivation, applies a plumbing (Murasugi sum) operation that has been adapted to the context of Khovanov homology.

6.1 Introduction

Given a link diagram $D \subset S^2$, smooth each crossing in one of two ways, $)(\xleftarrow{A} \times \xrightarrow{B} \asymp$. The resulting diagram x is called a Kauffman **state** of D and consists of *state circles* joined by A- and B-labeled arcs, one from each crossing. **Enhance** x by assigning each state circle a binary label: $\textcircled{1} \xleftarrow{1} \bigcirc \xrightarrow{0} \bigcirc$. Letting R be a ring with unity , the enhanced states from D form an R-basis for a bi-graded chain complex $\mathcal{C}_R(D) = \bigoplus_{i,j\in\mathbb{Z}} \mathcal{C}_R^{i,j}(D)$, which has a differential d of degree $(1,0)$. The resulting (co-)homology groups are link-invariant and are commonly called the **Khovanov homology** of the link [59]. Khovanov homology *categorifies* the Jones polynomial in the sense that the latter is the graded euler characteristic of the former [55, 59, 100]. Section 6.2 reviews Khovanov homology in more detail.

This chapter considers the question: what do nonzero Khovanov homology classes look like? The simplest examples come from all-A states and all-B states which are **ade-**

quate in the sense that each arc joins distinct circles: the all-1 enhancement of the all-A state and the all-0 enhancement of the all-B state represent nonzero homology classes with any coefficients [59]. (This implies that, if L is an *H-H-thin* link with a diagram whose all-A and all-B states both are adequate, then L is adequate [60].) Further, any enhancement of an adequate all-B state with exactly one 1-label represents a nonzero homology class over any ring in which 2 is not a unit, and the sum of all enhancements of an adequate all-A state with exactly one 0-label represents a nonzero homology class over the ring $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

Intriguingly, such states are **essential** in the sense that (all of) their state surfaces are incompressible and ∂-incompressible [86, 27]. Indeed, each of these state surfaces is a plumbing of checkerboard surfaces from reduced alternating link diagrams; such checkerboards are essential, and plumbing respects essentiality. With this motivation (rather than Khovanov homology in the abstract), and letting $\mathcal{C}_R(x)$ denote the submodule of $\mathcal{C}_R(D)$ generated by the enhancements of a state x of D, we ask whether Khovanov homology detects other essential states, in the same sense that it detects adequate all-A and all-B states:

Question 6.1.1. *For which essential states x does $\mathcal{C}_R(x)$ contain a nonzero homology class?*

In order for a state x to be essential, x must necessarily be adequate. Indeed, suppose some state circle x_1 of x contains both endpoints of some crossing arc. Construct a state surface F_x by capping x_1 with a disk on one side of the projection sphere S^2, capping all remaining state circles with disks on the other side of S^2, and joining these disks with half-twist bands, one at each crossing. The surface F_x is ∂-compressible.

For a sufficient condition, let G_x denote the *state graph* obtained from x by collapsing each state circle to a point, while maintaining the A- and B-labels on the edges of G_x, which come from the crossing arcs in x. (Thus, x is adequate iff G_x has no loops.) Cut G_x all at once along its cut vertices (ones whose deletion disconnects G_x); the subsets of x corresponding to the resulting connected components are called the **blocks** of x [21]. The state x decomposes under **plumbing** (of states) into these blocks, and the state surface from x decomposes under plumbing (of surfaces) into the blocks' state surfaces, each of which is a checkerboard surface for its block's underlying link diagram. If each of these checkerboard surfaces comes from an alternating link diagram—i.e., if no block of x contains both A- and B-type crossing arcs—x is called **homogeneous**. (This term first appears in [21], where an oriented link is called *homogeneous* if it has a diagram whose *Seifert state*—the Kauffman state determined by the orientation on the link—is homogeneous in the sense just described.)

If a state x is both adequate and homogeneous, it is called *homogeneously adequate* [89, 27, 28, 12]. In this case, its state surface F_x, a plumbing of checkerboard surfaces from *reduced* alternating link diagrams, is essential. Again, the point is that plumbing respects essentiality [86, 30]. (An alternate proof that F_x is essential, using normal surface theory rather than plumbing, appears in [27, 28]). Our main result states that Khovanov homology over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ detects all homogeneously adequate states:

Theorem 6.1.1. *If x is a homogeneously adequate state, then $C_{\mathbb{F}_2}^{i_x,j_x\pm1}(x)$ both contain (representatives of) nonzero homology classes. If also G_{x_A} is bipartite, then $C_{\mathbb{Z}}^{i_x,j_x\pm1}(x)$ contain such classes as well.*

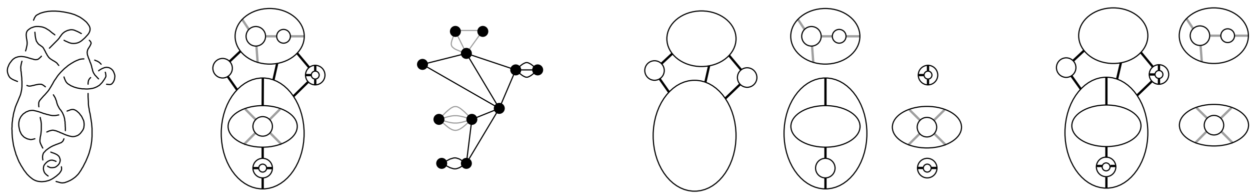


Figure 6.1: From left to right: a link diagram D with a homogeneously adequate state x, and its state graph G_x, blocks, and zones.

Here, i_x, j_x are integers that depend only on the state x, and x_A is the union of the A-type crossing arcs in x and their incident state circles. (When x is homogeneous, the components of x_A are called the A-**zones** of x; x_B and B-zones are defined analogously.) Thus, the theorem's bipartite condition on G_{x_A} is equivalent to the condition that the state surfaces from the A-blocks of x are all two-sided. In case x is adequate and all-A with $G_x = G_{x_A}$ bipartite, the link L can be oriented so that the diagram D is *positive*; if this D is a closed braid diagram, then the main theorem's class from $\mathcal{C}_{\mathbb{Z}}^{i_x, j_{x-1}}(x)$ is Plamenevskaya's *distinguished element* $\psi(L)$ [92]. In general, the condition of homogeneous adequacy is sensitive to changes in the link diagram, as are the homology classes from the main theorem, in the sense that Reidemeister moves generally do not preserve the fact that these classes have representatives in some $\mathcal{C}_R(x)$.

The first half of the paper is fairly straightforward: §6.2 reviews Khovanov homology, following Viro [100]; §6.3 reviews states, state surfaces, and plumbing; and §6.4 offers a direct, constructive proof of the main theorem. The rest of the paper returns to the geometric motivation behind the main theorem: the zones of a homogeneously adequate state x are adequate all-A and all-B states, the state surface from x is a plumbing of these zones' state surfaces, and Khovanov homology detects them all. With this motivation, §6.5 adapts plumbing to the context of Khovanov homology.

Adapting plumbing to Khovanov homology is simple in concept—glue two enhanced states along a state circle where their labels match so as to produce a new enhanced state; then extend linearly—but complicated in execution. The difficulty is that the differential sometimes changes the label on the state circle along which the two plumbing factors are glued together, upsetting the compatibility required for the plumbing. The workaround is to specify, by a *rule of trumps*, whether the labels on the first plumbing factor override those on the second or vice versa. The upshot is a useful identity, which states that plumbing in Khovanov homology behaves roughly like an exterior product followed by interior multiplication:

$$d(X * Y) = dX \; {}_{\diamond}\!* \; Y + (-1)^{|\asymp|_x} X *_{\diamond} dY. \tag{6.1}$$

Section 6.5 develops the notion of plumbing on Khovanov chains far enough to obtain an alternate proof of the main theorem, in which plumbing extends the all-A and all-B cases inductively to the homogeneously adequate case in general, thus fulfilling the theorem's geometric motivation. A discussion at the end of §6.5 describes how the natural property given by (6.1) extends to the entire chain complex $\mathcal{C}_R(D)$ when the plumbing is a connect sum and coefficients are in $\mathbb{F}_2$, but is more localized in general. Section 6.6 then concludes as follows. First, two easy examples show that $\mathcal{C}_R(x)$ may contain nonzero homology classes even when x is inessential. Second, a class of non-homogeneous essential states y indicates that the main question becomes more complicated beyond the homogeneously adequate case. Finally, we pose several questions.

Thank you to the referee from the New York Journal of Mathematics for suggesting many improvements to this chapter's details and overall structure.

Notation: For a diagram Z of any sort and any feature $\odot$ which may appear in such diagram, $|\odot|_Z$ denotes the number of $\odot$'s in Z. For example, if D is a link diagram, then $|\times|_D$ counts the crossings in D.

6.2 Khovanov homology of a link diagram, after Viro

6.2.1 Enhanced states

Index the crossings of a link diagram D as $c^1, \dots, c^{|\times|_D}$, and make a binary choice at each crossing: $)(\xleftarrow{A} \times \xrightarrow{B} \asymp$. The resulting diagram $x \subset S^2$ is called a Kauffman **state** of D and consists of $|\bigcirc|_x$ *state circles* joined by $A-$ and $B-$ labeled arcs, one from each crossing. Index the state circles of x as $x_1, \dots, x_{|\bigcirc|_x}$, and **enhance** x by making a binary choice at each state circle, x_r: $\textcircled{1} \xleftarrow{a_r=1} \bigcirc \xrightarrow{a_r=0} \bigcirc$. Let R be a ring with unity —we consider $R = \mathbb{F}_2$ and $R = \mathbb{Z}$; all results over $\mathbb{Z}$ hold over any ring in which 2 is not a unit—and define $\mathcal{C}_R(x)$ to be the R-module generated by the enhancements of x. Let $V = R[q]/(q^2)$, and associate $\mathcal{C}_R(x)$ with $V^{\otimes|\bigcirc|_x}$ by identifying each enhancement of x with the simple tensor $q^{a_1} \otimes \cdots \otimes q^{a_{|\bigcirc|_x}}$. Define:

$$\mathcal{C}_R(D) = \bigoplus_{\text{states } x \text{ of } D} \mathcal{C}_R(x) = \bigoplus_{\text{states } x \text{ of } D} V^{\otimes|\bigcirc|_x}.$$

6.2.2 Grading

The *writhe* of an oriented diagram D is $w_D = |\times|_D - |\times|_D$. For each state x of D, let $\sigma_x = |)(|_x - |\asymp|_x$ and $i_x = \frac{1}{2}(w_D - \sigma_x)$. For any enhancement X of x, define $\tau_X = |\textcircled{1}|_X - |\bigcirc|_X$

and $j_X = w_D + i_x - \tau_X$. The R-module $\mathcal{C}_R(D)$ carries a bi-grading $\mathcal{C}_R(D) = \bigoplus_{i,j} \mathcal{C}_R^{i,j}(D)$, where each $\mathcal{C}_R^{i,j}(D)$ is generated by the enhancements Y of states y of D with $i = i_y$ and $j = j_Y$. The Jones polynomial $V_L(q)$ of an oriented link L, unnormalized such that $V_{\text{unknot}}(q) = q + q^{-1}$, is given by Kauffman's state sum formula (6.2) [55, 57]. Enhancement expands the terms in this sum in order to express the Jones polynomial as the graded euler characteristic of $\mathcal{C}_R(D)$ (c.f. (6.3), Figure 6.2) [100]:

$$V_L(q) = q^{w_D} \sum_{\text{states } x} (-q)^{i_x} \left(q + q^{-1}\right)^{|\bigcirc|_x} \tag{6.2}$$

$$\begin{aligned} V_L(q) &= \sum_{\text{enhancements } X \text{ of states } x} (-1)^{i_x} q^{j_X} \\ &= \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \operatorname{rk}(\mathcal{C}_R^{i,j}(D)). \end{aligned} \tag{6.3}$$

6.2.3 Homology

With X an enhanced state from a link diagram D, define its differential $d_{c^t} X$ at each crossing c^t of D by the incidence rules in Figure 6.3. (If x has a B-smoothing at c^t, then $d_{c^t} X = 0$.) In general, the differential of an enhanced state $X \in \mathcal{C}_R^{i,j}(D)$ equals the sum $dX = \sum_t (-1)^{|\asymp|_X^t} d_{c^t} Y \in \mathcal{C}_R^{i+1,j}(D)$, where $|\asymp|_X^t$ is the number of crossings c^s with $s < t$ at which X has an A–smoothing. When $R = \mathbb{F}_2$, the differential is simply $dX = \sum_t d_{c^t} X$. Extend R-linearly to obtain the **differential** $d : \mathcal{C}_R(D) \to \mathcal{C}_R(D)$, which has degree $(1,0)$ and obeys $d \circ d \equiv 0$, giving $\mathcal{C}_R(D)$ the structure of a (co-)**chain complex**. The quotients $Kh_R(D) = \ker(d)/\text{image}(d)$ are link-invariant, and are commonly called the **Khovanov homology** of the link, even though this is really set up as a cohomology theory [59].

A subset $\mathcal{B} \subset \mathcal{C}_R(D)$ is called *primitive* if, whenever $r \in R$, $X \in \mathcal{C}_R(D)$, and $rX \in \mathcal{B}$,

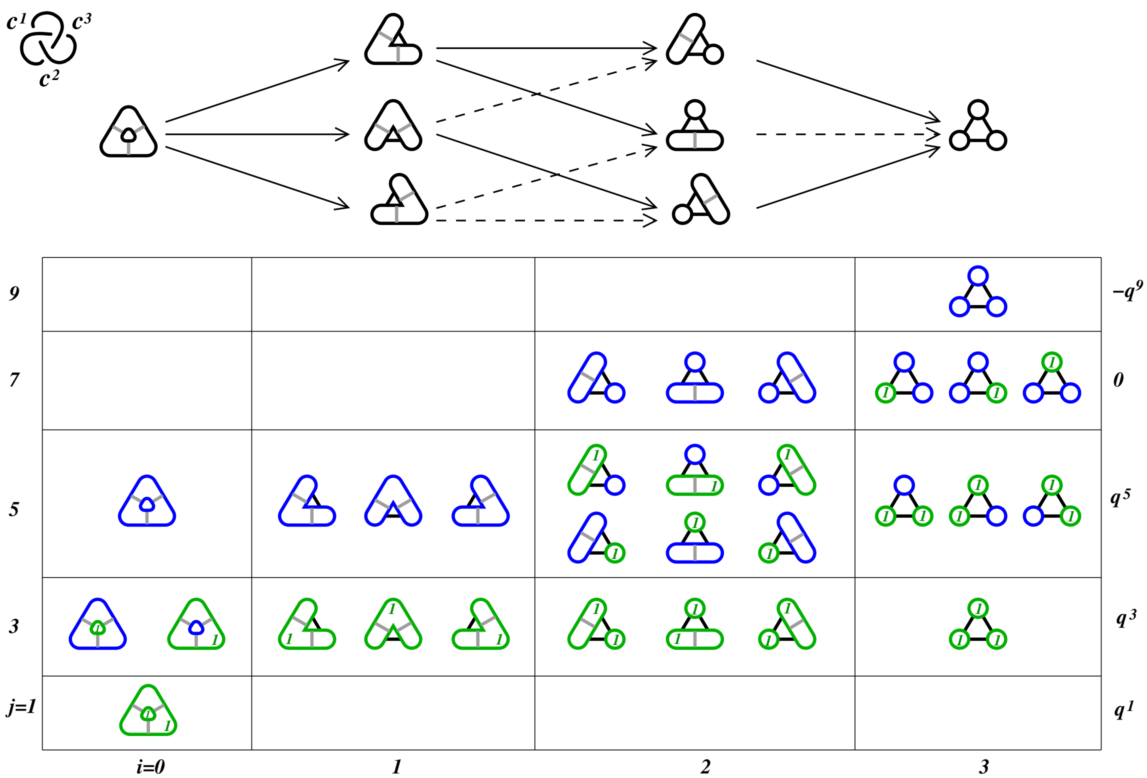


Figure 6.2: The Jones polynomial $V_L(q) = q + q^3 + q^5 - q^9$ of the RH trefoil via Khovanov chains.

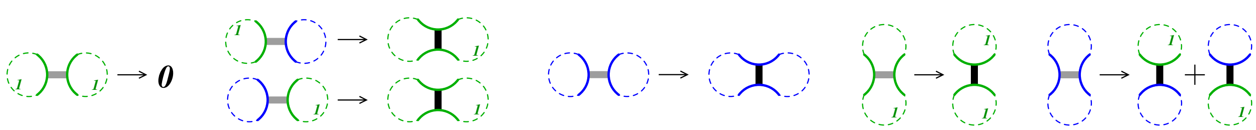


Figure 6.3: Incidence rules for Khovanov's differential map, d: $q \otimes q \mapsto 0$; $1 \otimes q, q \otimes 1 \mapsto q$; $1 \otimes 1 \mapsto 1$; $q \mapsto q \otimes q$; $1 \mapsto q \otimes 1 + 1 \otimes q$.

also $uX \in \mathcal{B}$ for some unit $u \in R$; for example, a collection of enhanced states is primitive. If $\mathcal{B} \subset \mathcal{C}_R(D)$ is primitive, then the **restriction** map $\pi_{\mathcal{B}} : \mathcal{C}_R(D) \to \mathcal{C}_R(D)$ is the R-linear map that sends each chain X to itself when X is in the R-span of $\mathcal{B}$ and to 0 otherwise. The **augmentation** map $\varepsilon : \mathcal{C}_R(D) \to R$ is the R-linear map that sends each enhanced state X to 1. Maps of the form $\varepsilon \circ \pi_{\mathcal{B}} \circ d : \mathcal{C}_R(D) \to R$ will be useful for proving that the main theorem's cycles are not exact.

6.2.4 Reduced homology

Let D be a link diagram, R a ring with unity , and p a point on D away from crossings. For each state x of D, define $\mathcal{C}_{R,1}(x)$ and $\mathcal{C}_{R,0}(x)$ to be the submodules of $\mathcal{C}_R(x)$ generated by those enhancements of x in which the state circle containing the point p has the indicated label. Also define $\mathcal{C}_{R,1}(D) = \bigoplus_x \mathcal{C}_{R,1}(x)$ and $\mathcal{C}_{R,0}(D) = \bigoplus_x \mathcal{C}_{R,0}(x)$. (After natural modifications of the differential map and shifts in the j-grading, these chain complexes yield a (co-)homology theory, which is commonly called the *reduced Khovanov homology* of the link; reduced Khovanov homology with integer coefficients is known to detect the unknot, and its graded euler characteristic equals the normalized Jones polynomial, $V_L(D)/(q+q^{-1})$ [59, 61].) Section 6.5 will use the decomposition of chain groups $\mathcal{C}_R(D) = \mathcal{C}_{R,1}(D) \oplus \mathcal{C}_{R,0}(D)$ to adapt the operation of plumbing to the context of Khovanov homology.

6.3 Further background

Recall the notions of blocks, zones, homogeneity, and adequacy from §6.1: given a state x, construct its state graph G_x by collapsing state circles to points, while retaining

A- and B-labels on arcs. Cut components of G_x correspond to *blocks* of x. The state x is *homogeneous* if no block contains both A- and B-type arcs, and x is *adequate* if G_x lacks loops. In general, the subset $x_B \subset x$ is the union of all B-type crossing arcs and their incident state circles; x_A is defined analogously. In case x is homogeneous, x_A and x_B are the respective unions of the A- and B-blocks of x, and the connected components of x_A and x_B are called the A- and B-*zones* of x, respectively (c.f. Figure 6.1).

6.3.1 Equivalences and traces

Define the equivalence relations $\sim_A$, $\sim_B$ on enhanced states to be generated by  $\sim_A$  and  $\sim_B$ , respectively, so that $X \sim_A Y$ (resp. $X \sim_B Y$) iff X can be changed to Y by a sequence of moves, each of which switches the labels on two state circles joined by an A-type (resp. B-type) crossing arc. Let $[X]_A, [X]_B$ denote the associated equivalence classes. Note that $[X]_A, [X]_B \subset \mathcal{C}_R^{i_X, j_X}(x)$.

Proposition 6.3.1. *If X enhances a homogeneous state x, then* $[X]_A \cap [X]_B = \{X\}$.

Proof. Given a homogeneous state x, each circle of $x_A \cap x_B$ abuts at least one A-zone and at least one B-zone. Assign *heights* to the circles of $x_A \cap x_B$ as follows. A circle of $x_A \cap x_B$ has height 0 if it is the only circle of $x_A \cap x_B$ in any of the zones it abuts; and recursively a circle of $x_A \cap x_B$ has height n if its height is not less than n and if, in any of the incident zones of x, all other circles of $x_A \cap x_B$ have height less than n.

Suppose X and Y both enhance x. If $X \sim_A Y$, then X and Y must be identical in $x \setminus x_A$, and each A-zone of x must have equal numbers of 0-labeled circles in X and in Y. Likewise, if $X \sim_B Y$, then X and Y are identical in $x \setminus x_B$, and each B-zone of x has equal numbers of 0-labeled state circles in X and in Y. Hence, if $Y \in [X]_A \cap [X]_B$, then X and Y are identical in all of $(x \setminus x_A) \cup (x \setminus x_B) = x \setminus (x_A \cap x_B)$ and have the same number of 0-labeled state circles in each zone of x. This implies that X and Y assign the same label to each circle of $x_A \cap x_B$ with height 0. That, in turn, implies that X and Y also assign the same label to each circle of $x_A \cap x_B$ with height 1. Continuing inductively completes the proof. □

Let X be an enhancement of an arbitrary state, x. With coefficients in $\mathbb{F}_2$, define the **trace** of X to be $\mathrm{tr}_{\mathbb{F}_2} X = \sum_{Y \sim_A X} Y$. (The term is chosen in rough analogy with the field trace.) To extend this notion to $R = \mathbb{Z}$, suppose every non-bipartite component of x_A is either all-1 or all-0 in X. Then, for each $Y \sim_A X$, define $\mathrm{sgn}(X \to Y)$ to be 1 or -1 according to whether an even or odd number of $)(\leftrightarrow)($ moves take X to Y. Define the trace over $\mathbb{Z}$ of such an enhanced state X to be $\mathrm{tr}_{\mathbb{Z}} X = \sum_{Y \sim_A X} \mathrm{sgn}(X \to Y) Y$. (The trace $\mathrm{tr}_{\mathbb{Z}} X$ is undefined if there is a non-bipartite component of x_A with both 0- and 1-labels in X.) This notion of trace is generic to the main question in the following sense:

Proposition 6.3.2. *Over $R = \mathbb{F}_2$ (resp. $R = \mathbb{Z}$), every cycle of the form $X \in \mathcal{C}_R(x)$ is a sum of traces, $X = \sum_r \mathrm{tr}_R X_r$, in which every component of x_A in each X_r is adequate and either all-1 or (bipartite) with exactly one 0-label.*

Proof. Recall the incidence rules for the differential (c.f. Figure 6.3). Suppose that X_0 enhances x and c is a crossing at which x has an A-type arc α. Observe that (i) if α joins two state circles with opposite labels, then $d_c X_0 = d_c X_r$ for exactly two enhancements X_r of x, namely X_0 and the enhancement obtained from X_0 by reversing the labels on the two

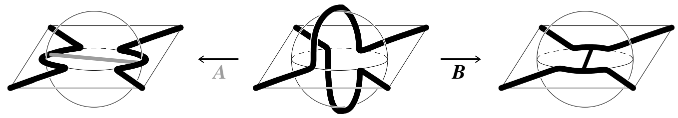


Figure 6.4: Use crossing balls $C = \bigsqcup C^t$ to embed a link and its states' circles in $(S^2 \setminus C) \cup \partial C$.

circles incident to α; and (ii) if α joins two 0-labeled circles of X_0 or joins the same circle of X_0 to itself, then $d_c X_0 \neq d_c X_r$ for any enhancement $X_r \neq X_0$ of x. From (i) it follows that any cycle $X \in \mathcal{C}_R(x)$ is a sum of traces (and in particular that these traces are defined), and (ii) implies that each component of x_A is adequate with at most one 0-label in each summand of X. □

6.3.2 State surfaces

Given a link diagram D on $S^2 \subset S^3$, embed the underlying link L in S^3 by inserting tiny, disjoint balls $\bigsqcup C^t = C$ at the crossing points c^t and pushing the two arcs of $D \cap C^t$ to the hemispheres of $\partial C^t \setminus S^2$ indicated by the over-under information at c^t [69]. In this setup, the states x of D correspond to the closed 1-manifolds $S^2 \cap L \subset x' \subset S^2 \cap (L \cup \partial C)$: deleting the crossing arcs from x gives x'. Observe that $x' \cup L$ is a trivalent spatial graph which intersects each ∂C^t in a circle. Cap each such circle with a disk in C^t, called a *crossing band*, and cap the components of x' with disks whose interiors are disjoint from one another and from $S^2 \cup C$. The resulting unoriented surface F_x *spans* L, meaning that $\partial F_x = L$, and is called a **state surface** from x.

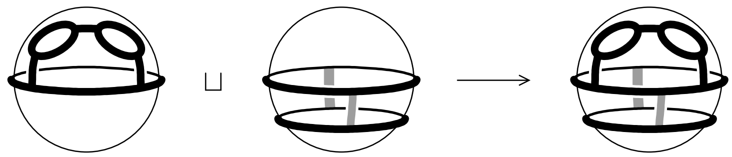


Figure 6.5: A gluing map $g:(S^2,x)\sqcup(S^2,y)\to(S^2,z)$ for a plumbing of states $x*y=z$.

When L is a knot, its linking number with a co-oriented pushoff $\hat{L}$ in F_x is called the *boundary slope* of F_x and equals $2i_x$. When $L=\bigsqcup_r L_r$ is an oriented link with components L_r, $2i_x$ is the sum of the component-wise boundary slopes of F_x, which do not depend on the orientation on L, and twice the link components' pairwise linking numbers, which do:

$$2i_x=\sum_r \mathrm{lk}(L_r,\hat{L}_r)+2\sum_{r<s}\mathrm{lk}(L_r,L_s).$$

6.3.3 Plumbing

In the context of 3-manifolds, *plumbing*, or *Murasugi sum*, is an operation on states, links, and spanning surfaces [79]. Plumbing two states x, y simply involves gluing these states along a single state circle in such a way that the resulting diagram is also a state, z. A plumbing of states can be described *externally* by taking x and y to be states on separate projection spheres and gluing them by a map $g:(S^2,x)\sqcup(S^2,y)\to(S^2,z)$ (c.f. Figure 6.5). Such a plumbing can also be described *internally* by identifying x with $g(x)$ and y with $g(y)$, so that x and y are seen as subsets of z, and writing $x*y=z$. We will use the internal notion of plumbing rather than the external notion, in order to simplify notion. Be careful, though: unlike the free product for groups, plumbing is not a well-

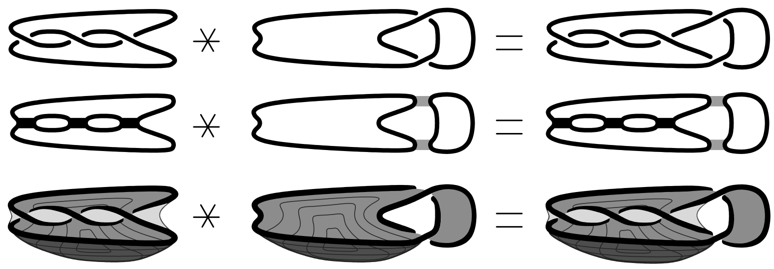


Figure 6.6: Plumbings of link diagrams, states, and surfaces.

defined binary operation on states. Rather, plumbing depends on a gluing map, which the notation $x * y = z$ encodes implicitly.

If $x * y = z$ is a plumbing of states, then there are associated plumbings of link diagrams, $D_x * D_y = D_z$, and of the underlying links (c.f. Figure 6.6). There is also an associated plumbing of state surfaces, $F_x * F_y = F_z$; here is how this works. Let U be the disk that the state circle $x \cap y$ bounds in the state surface F_z. There is an embedded sphere $Q \subset S^3$ transverse to the projection sphere S^2 with $Q \cap F_z = U$ and $Q \cap S^2 = x \cap y$; let B_x, B_y denote the (closed) balls into which Q cuts S^3, such that $x \subset B_x$, $y \subset B_y$. The surfaces $F_x := F_z \cap B_x$, $F_y := F_z \cap B_y$ are the state surfaces for x, y, respectively, and plumbing these surfaces along Q produces $F_x * F_y = F_z$.

For general interest, we briefly describe two notions of (de-)plumbing for spanning surfaces. These notions are more general than the notion of plumbing of states. The simplest reason is that some spanning surfaces (e.g. any surface whose complement is not a handlebody) cannot be realized as state surfaces. In general, characterizing all the ways to de-plumb a given spanning surface is an interesting and difficult problem.

First, suppose F spans a link $L \subset S^3$ and $Q \subset S^3$ is a sphere which intersects F (non-transversally) in a disk $U = Q \cap F$. If B_0, B_1 are the (closed) balls into which Q cuts S^3 and $F_0 = B_0 \cap F$, $F_1 = B_1 \cap F$, so that $F_0 \cap F_1 = F \cap B_0 \cap B_1 = F \cap Q = U$, then the sphere Q is said to *de-plumb* F as $F = F_0 * F_1$.

A second notion of plumbing, which is better suited for iteration, views a regular neighborhood of $\text{int}(F)$ in the link complement $S^3 \setminus L$ in terms of a line bundle $\rho : N \to \text{int}(F)$ and allows de-plumbing along a sphere Q which (i) is transverse in S^3 to L and in $S^3 \setminus K$ to the fibers of ρ; and which (ii) intersects N in a disk U which is (the image of) a local section of ρ. Letting B_0, B_1 denote the balls into which Q cuts S^3, the resulting plumbing factors are $\rho(N \cap B_0)$, $\rho(N \cap B_1)$.

6.4 Direct proof of the main theorem

Throughout this section, fix a homogeneously adequate state x of a link diagram D. Here is the plan. Several propositions will establish two conditions on enhancements X of x which together guarantee that $\text{tr}_{\mathbb{F}_2} X$ represents a nonzero homology class: each A-zone must contain at most one 0-labeled circle, and each B-zone must contain at most one 1-labeled circle. These conditions also suffice over $R = \mathbb{Z}$ when G_{x_A} is bipartite. An explicit construction will then fashion enhancements $X^{\pm}$ of x which satisfy these conditions, with $j(X^+) = j(X^-) + 2$.

Proposition 6.4.1. *If X enhances x with at most one 0-labeled circle in each A-zone, then $d(tr_{\mathbb{F}_2} X) = 0$. Further, $d(tr_{\mathbb{Z}} X) = 0$ if $tr_{\mathbb{Z}} X$ is defined (i.e. if every non-bipartite A-zone of x is all-1 in X).*

Proof. Let c be an arbitrary crossing of the link diagram D; it will suffice to show that $d_c(\mathrm{tr}_R X) = 0$ for $R = \mathbb{F}_2$, and for $R = \mathbb{Z}$ if $\mathrm{tr}_{\mathbb{Z}} X$ is defined. Assume that x has an A-type crossing arc at c, or else we are done. Partition the enhanced states in $[X]_A$ as follows. Let one equivalence class consist of all enhancements for which both state circles incident to c are labeled 1; $d_c(X') = 0$ for each X' in this class. Partition any remaining enhanced states in $[X]_A$ into pairs $\{X_s, X_{s'}\}$ which are identical except with opposite labels on the two state circles incident to c. For each such pair, $d_c(X_s) = d_c(X_{s'})$ over both $R = \mathbb{F}_2$ and $R = \mathbb{Z}$; also, $\mathrm{sgn}(X \to X_s) = -\mathrm{sgn}(X \to X_{s'})$ in case $R = \mathbb{Z}$. Conclude in both cases $R = \mathbb{F}_2$ and $R = \mathbb{Z}$:

$$d\,(\mathrm{tr}_R X) = \sum_{X' \in [X]_A} \mathrm{sgn}(X \to X') d_c X' = \sum_{\text{pairs } \{X_s,\, X_{s'}\}} \mathrm{sgn}(X \to X_s)\,(d_c X_s - d_c X_{s'}) = 0.$$

□

Proposition 6.4.2. *If X enhances x so that no B-zone contains more than one 1-labeled circle, then*

$$\varepsilon \circ \pi_{[X]_B} \circ d : \mathcal{C}_R(D) \to 2R.$$

Proof. Let Y be any enhanced state from D. If $\pi_{[X]_B} \circ d(Y) \neq 0$, then the underlying state y of Y must differ from x at precisely one crossing, c, at which y must have an A-smoothing with one incident state circle. This circle must be labeled 0 in Y because each $X' \in [X]_B$ has at most one 1-labeled circle in each component of x_B. Thus, $\pi_{[X]_B} \circ d_c(Y) = \pm(X_s + X_{s'})$, where $X_s, X_{s'}$ are identical except with opposite labels on the two state circles of x incident to c. In particular, $\varepsilon \circ \pi_{[X]_B} \circ d(Y) = \pm(1+1) \in 2R$. □

Proposition 6.4.3. *If X enhances x with at most one 0-labeled state circle in each A-zone and at most one 1-labeled state circle in each B-zone, then $\mathrm{tr}_{\mathbb{F}_2} X$ represents a nonzero homology class. Further, if every A-zone containing a 0-labeled circle in X is bipartite, then $\mathrm{tr}_{\mathbb{Z}} X$ also represents a nonzero homology class.*

Proof. Such $\mathrm{tr}_{\mathbb{F}_2} X$, $\mathrm{tr}_{\mathbb{Z}} X$ are cycles by Proposition 6.4.1. If $\mathrm{tr}_R X$ were exact over $R = \mathbb{F}_2$ or $R = \mathbb{Z}$, say $\mathrm{tr}_R X = dY$, then Propositions 6.3.1 and 6.4.2 would imply that 2 is a unit in R:

$$1 = \varepsilon(X) = \varepsilon\left(\sum_{X' \in [X]_A \cap [X]_B} X'\right) = \varepsilon \circ \pi_{[X]_B}(\mathrm{tr}_R X) = \varepsilon \circ \pi_{[X]_B} \circ dY \in 2R.$$

□

In particular, the chains described in the introduction behave as advertised:

Corollary 6.4.1. *Let x be an adequate, all-A state, and let y be an adequate, all-B state. The all-1 enhancement of x represents a nonzero homology class over both $R = \mathbb{F}_2$, $\mathbb{Z}$, as does any enhancement of y with no more than one 1-labeled circle. Moreover, if X enhances x and has exactly one 0-labeled circle, then $\mathrm{tr}_{\mathbb{F}_2} X$ always represents a nonzero homology class; the trace $\mathrm{tr}_{\mathbb{Z}} X$ does too, if it is defined.*

Putting together Propositions 6.4.1–6.4.3 proves the main theorem, which states in part that Khovanov homology over $\mathbb{F}_2$ detects every homogeneously adequate state x in two distinct gradings, $(i_x, j_x \pm 1)$, where:

$$j_x = w_D + i_x + |\bigcirc|_{x_B} - |\bigcirc|_{x_A} + \#(B\text{-zones of } x) - \#(A\text{-zones of } x).$$

This proof and the one in §6.5 will establish the following, which is stronger than the version from §6.1.

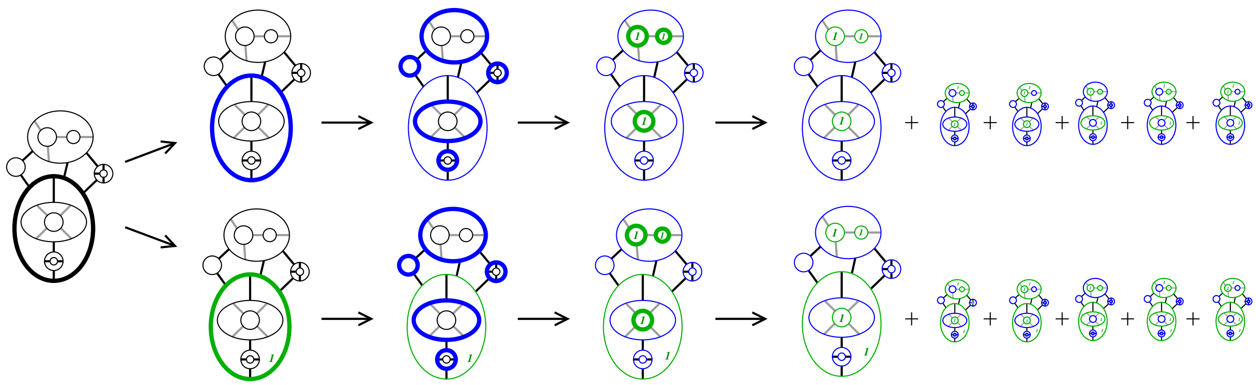


Figure 6.7: Constructing $X^{\pm} \in \mathcal{C}_{\mathbb{F}_2}(x)$, as in the direct proof of the main theorem.

Theorem 6.4.1. *If x is a homogeneously adequate state, then for any $p \in x$ away from crossing arcs, x has enhancements $X^- \in \mathcal{C}_{R,1}(x)$, $X^+ \in \mathcal{C}_{R,0}(x)$, identical away from p, such that both $tr_{\mathbb{F}_2}X^{\pm}$ represent nonzero classes in $Kh^{i_x,j_x\pm1}_{\mathbb{F}_2}(x)$. If also G_{x_A} is bipartite, then both $tr_{\mathbb{Z}}X^{\pm}$ represent nonzero classes in $Kh^{i_x,j_x\pm1}_{\mathbb{Z}}(x)$.*

Proof. Construct $X^{\pm}$ as follows (c.f. Figure 6.7). First, label the state circle containing p: 1 for X^-, 0 for X^+. Second, for both X^- and X^+, label all remaining state circles in the zone(s) containing p: 1 for any state circle in an A-zone containing p, 0 for B-. Repeat in this manner, progressing by adjacency of zones until every circle of x is labeled: in each zone which abuts a labeled one, label all remaining circles 1 or 0, according the type (A- or B-, respectively) of the zone.

The resulting enhancements $X^{\pm}$ are identical away from p and satisfy the hypotheses of Proposition 6.4.3. Therefore, both of $\mathrm{tr}_{\mathbb{F}_2}X^{\pm}$ represent nonzero homology classes, as do $\mathrm{tr}_{\mathbb{Z}}X^{\pm}$ if they are defined. □

6.5 Plumbing Khovanov chains

Following the main theorem's geometric motivation, this section adapts plumbing to Khovanov homology and develops this notion far enough to obtain an inductive proof of the main theorem, extending the adequate all-A and all-B cases (c.f. Corollary 6.4.1) to homogeneously adequate states in general. The geometric motivation is this: a homogeneously adequate state x is a plumbing of adequate all-A and all-B states, which Khovanov homology detects, and whose state surfaces are essential. Plumbing respects the essentiality of these state surfaces; perhaps plumbing operates similarly in Khovanov homology. All results in §6.5, except the main theorem, pertain to plumbings of arbitrary states, not just homogeneously adequate ones.

6.5.1 Plumbing Khovanov chains

Let $x*y = z$ be a plumbing of states, and let $D_x*D_y = D_z$ be the associated plumbing of link diagrams. Index the crossings c_z^t of D_z so that the crossings from D_x precede those from D_y: let $c_z^t = c_x^t$ for $1 \leq t \leq |\times|_x$, and let $c_z^t = c_y^{t-|\times|_x}$ for $1+|\times|_x \leq t \leq |\times|_x+|\times|_y$. Likewise, index the state circles z_r of z so that the state circles from x precede the state circles from y: let $z_r = x_r$ for $1 \leq r \leq |\bigcirc|_x$, and let $z_r = y_{r+1-|\bigcirc|_x}$ for $|\bigcirc|_x \leq r \leq |\bigcirc|_x + |\bigcirc|_y - 1$. Note that $z_{|\bigcirc|_x} = x_{|\bigcirc|_x} = y_1 = x \cap y$.

With this indexing, let X, Y enhance x, y, and write $X = q^{a_1} \otimes \cdots \otimes q^{a_{|\bigcirc|_x}}$, $Y = q^{b_1} \otimes \cdots \otimes q^{b_{|\bigcirc|_y}}$, with each $a_r, b_r \in \{0, 1\}$ according to whether the associated state circle is labeled 0 or 1, as in §6.2. Define the **plumbing** of the chains X and Y (associated to the plumbing of states $x * y = z$) to be the enhancement of $x * y$ which matches X on the state circles from x and which matches Y on those from y, *if such an enhancement exists*:

$$X{*}Y = \begin{cases} q^{a_1}\otimes\cdots\otimes q^{a_{|\bigcirc|_x-1}}\otimes q^{b_1}\otimes q^{b_2}\otimes\cdots\otimes q^{b_{|\bigcirc|_y}} & \text{if } a_{|\bigcirc|_x}=b_1,\\ \text{undefined} & \text{if } a_{|\bigcirc|_x}\neq b_1. \end{cases}$$

Extend this plumbing of chains R-linearly to obtain the following isomorphism of R-modules:

$$\begin{aligned} *:(\mathcal{C}_{R,1}(x)\otimes\mathcal{C}_{R,1}(y))\oplus(\mathcal{C}_{R,0}(x)\otimes\mathcal{C}_{R,0}(y)) &\to \mathcal{C}_R(x*y),\\ X\otimes Y &\mapsto X*Y. \end{aligned} \tag{6.4}$$

6.5.2 Differentials of plumbings

More generally, let $x*y=z$ be a plumbing of states with the indexing of crossings and state circles from §6.5.1, and let x', y' be arbitrary states of the underlying link diagrams D_x, D_y. Whether or not x',y' contain the state circle $x\cap y$, let $x'{*}y$ denote the state of $D_x*D_y=D_z$ whose smoothings match those of x' and y, and let $x{*}y'$ denote the state of D_z whose smoothings match those of x and y'. Note that $x*y'$ is a plumbing of x and y' if and only if $y'\supset x\cap y$, and $x'{*}y$ is a plumbing of x' and y if and only if $x'\supset x\cap y$. Regardless of whether these states are bona fide plumbings, however, they are well-defined and will prove convenient.

Suppose further that X, X', Y, Y' respectively enhance the states x, x', y, y'. Define the *left-trump plumbing* $X'{}_{\diamond}{*}Y$ and the *right-trump plumbing* $X{*}_{\diamond}Y'$ as follows: $X'{}_{\diamond}{*}Y$ is the enhancement of $x'{*}y$ which assigns each state circle of x' the same label that X' does, and which assigns each state circle of y—*except possibly* $x\cap y$, which need not be a state

Figure 6.8: If $x{*}y$ is a plumbing of states and $X \in \mathcal{C}_R(x)$, $Y \in \mathcal{C}_R(y)$ are cycles with $X * Y \in \mathcal{C}_R(x*y)$, then $X * Y$ is also a cycle (c.f. Proposition 6.5.1).

circle of $x' * y$—the same label that Y does. Likewise, $X{*_\diamond}Y'$ is the enhancement of $x * y'$ that matches x, *except possibly along* $x \cap y$, and which assigns matches y' everywhere. That is, $X'{_\diamond{*}}Y$ and $X{*_\diamond}Y'$ are the respective enhancements of $x' * y$ and $x * y'$ which match X', Y and X, Y' away from $x \cap y$; along $x \cap y$, the labels from X' *trump* the labels from Y in $X'{_\diamond{*}}Y$, and the labels from Y' *trump* those from X in $X{*_\diamond}Y'$. This gives R-module homomorphisms:

$$\begin{gathered} {_\diamond{*}} : \mathcal{C}_R(D_x) \otimes \mathcal{C}_R(y) \to \mathcal{C}_R\left(D_x{*}D_y\right) \\ X' \otimes Y \mapsto X' {_\diamond{*}}\, Y, \end{gathered} \tag{6.5}$$

$$\begin{gathered} {*_\diamond} : \mathcal{C}_R(x) \otimes \mathcal{C}_R(D_y) \to \mathcal{C}_R\left(D_x{*}D_y\right) \\ X \otimes Y' \mapsto X *_\diamond Y'. \end{gathered} \tag{6.6}$$

In general, neither map extends to all of $\mathcal{C}_R(D_x) \otimes \mathcal{C}_R(D_y)$. In the simplest case, however, it does. Section 6.5.6 discusses this case: connect sum, #.

Proposition 6.5.1. *If $X{*}Y$ enhances a plumbing of states $x * y$, then $d(X * Y) = dX {_\diamond{*}}\, Y + (-1)^{|\rangle\langle|_x} X *_\diamond dY$. In particular, if X and Y are cycles and $X * Y$ is defined, then $X * Y$ is also a cycle.*

Proof. Let $D_x * D_y = D_z$ be the plumbing of link diagrams associated to the plumbing of states $x*y = z$. Index the crossings as in §6.5.1, and let $|)(|_z^t$ denote the number of crossings c_z^r in D_z with $r < t$ at which z has an A-smoothing. Defining $|)(|_x^t$ and $|)(|_y^t$ analogously, note that $|)(|_z^t = |)(|_x^t$ when $t \leq |\times|_{D_x}$, and that $|)(|_z^t = |)(|_x + |)(|_y^{t-|\times|_{D_x}}$ when $t > |\times|_{D_x}$. Thus:

$$\begin{aligned} d\left(X*Y\right) &= \sum_{t=1}^{|\times|_{D_x}} (-1)^{|)(|_z^t} d_{c_z^t}(X*Y) + \sum_{t=1+|\times|_{D_x}}^{|\times|_{D_x}+|\times|_{D_y}} (-1)^{|)(|_z^t} d_{c_z^t}\left(X*Y\right) \\ &= \sum_{t=1}^{|\times|_{D_x}} (-1)^{|)(|_x^t} d_{c_x^t} X \,{}_{\diamond}{*}\, Y + (-1)^{|)(|_x} \sum_{s=1}^{|\times|_{D_y}} (-1)^{|)(|_y^s} X *_{\diamond} d_{c_y^s} Y \\ &= dX \,{}_{\diamond}{*}\, Y + (-1)^{|)(|_x} X *_{\diamond} dY. \end{aligned}$$

□

In particular, if $X * Y$ enhances a plumbing of states $x * y$, then with coefficients in $\mathbb{F}_2$:

$$d(X * Y) = dX \,{}_{\diamond}{*}\, Y + X *_{\diamond} dY. \tag{6.7}$$

6.5.3 Cycles

In the context of a plumbing $x = x * \bigcirc$ of a state x with the state $\bigcirc$ of the trivial diagram, let p be a point on the state circle $\bigcirc \subset x$ and away from crossing arcs. Observe that two chains $X \in \mathcal{C}_{R,1}(x)$, $X' \in \mathcal{C}_{R,0}(x)$ are identical away from p if and only if $X *_{\diamond} \bigcirc = X'$, or equivalently $X' *_{\diamond} \textcircled{1} = X$. Further:

Observation 6.5.1. *If $X, X' \in \mathcal{C}_R(x)$ are identical away from p, and if $x * y = z$ is a plumbing of states with $p \in x \cap y$, then $X *_{\diamond} Y = X' *_{\diamond} Y$ for any $Y \in \mathcal{C}_R(D_y)$.*

Figure 6.9: If $x * y$ is a plumbing of states and X, X', $Y + Y'$ are cycles, with $X, X' \in \mathcal{C}_R(x)$ identical away from $x \cap y$ and $X * Y + X' * Y' \in \mathcal{C}_R(x * y)$, then $X * Y + X' * Y'$ is also a cycle (c.f. Proposition 6.5.2).

Proposition 6.5.2. *If $x * y$ is a plumbing of states and X, X', $Y + Y'$ are cycles, with $X \in \mathcal{C}_{R,1}(x)$, $X' \in \mathcal{C}_{R,0}(x)$ identical away from $p \in x \cap y$ and $Y \in \mathcal{C}_{R,1}(y)$, $Y' \in \mathcal{C}_{R,0}(y)$, then $X * Y + X' * Y'$ is also a cycle.*

Proof. Observation 6.5.1 implies that $X *_{\diamond} dY' = X' *_{\diamond} dY'$. This and Proposition 6.5.1 now yield:

$$\begin{aligned} d\Big(X * Y + X' * Y'\Big) &= dX \,{}_{\diamond}{*}\, Y + dX' \,{}_{\diamond}{*}\, Y' + (-1)^{|\rangle\langle|_x} (X *_{\diamond} dY + X' *_{\diamond} dY') \\ &= (-1)^{|\rangle\langle|_x} (X *_{\diamond} dY + X *_{\diamond} dY') \\ &= (-1)^{|\rangle\langle|_x} (X *_{\diamond} d(Y + Y')) = 0. \end{aligned}$$

□

Observation 6.5.2. *Let $x = x * \bigcirc$ be a plumbing of a state x with the trivial state $\bigcirc$. If $X \in \mathcal{C}_R(x)$ and $\bigcirc \cap x_A = \varnothing$, so that the state circle $\bigcirc$ abuts no A-type arcs in x, then for each $X \in \mathcal{C}_R(x)$:*

$$d(X *_{\diamond} \textcircled{1}) = dX *_{\diamond} \textcircled{1} \qquad \text{and} \qquad d(X *_{\diamond} \bigcirc) = dX *_{\diamond} \bigcirc.$$

In particular, if X is a cycle with $\bigcirc \cap x_A = \varnothing$, then $X *_{\diamond} \textcircled{1}$ and $X *_{\diamond} \bigcirc$ are both cycles.

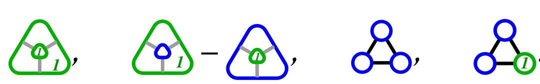
The point is that, because the state circle $\bigcirc$ is incident to no A-type crossing arcs, every enhanced state $Y \in \mathcal{C}_R(D_x)$ with $\pi_{RY} \circ dX \neq 0$ contains $\bigcirc$ and assigns it the same label that X does.

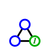
Proposition 6.5.3. *Let $z = x * y$ be a plumbing of states in which $x \cap y \subset y \setminus y_A$. If X, Y enhance x, y such that both $tr_R X$ and $tr_R Y$ are cycles and $X * Y$ is defined, then $tr_R(X * Y)$ is also a cycle.*

Proof. Proposition 6.3.2 implies that each component of x_A, y_A is adequate and has at most one 0-labeled circle in X, Y. Using the indexing from §6.5.1, so that the state circles in z from x precede those from y, with $x \cap y = x_{|\bigcirc|_x} = y_1$, write $X = \bigotimes_{r=1}^{|\bigcirc|_x} q^{a_r}$, $Y = \bigotimes_{r=1}^{|\bigcirc|_y} q^{b_r}$. Define $Y' := 1 \otimes \bigotimes_{r=2}^{|\bigcirc|_y} q^{b_r}$, $Y'' := q \otimes \bigotimes_{r=2}^{|\bigcirc|_y} q^{b_r}$, so that Y', Y'' are identical away from y_1, and one of Y', Y'' equals Y. Thus, one of $\mathrm{tr}_R Y'$, $\mathrm{tr}_R Y''$ equals $\mathrm{tr}_R Y$, which is a cycle. The assumption that $x \cap y \subset y \setminus y_A$ further implies that $\mathrm{tr}_R Y'$ and $\mathrm{tr}_R Y''$ are *both* cycles, by Observation 6.5.2. Taking p to be a point on the circle $x \cap y$ and away from crossing arcs, write $\mathrm{tr}_R X = X' + X''$, where $X' \in \mathcal{C}_{R,1}(x)$ and $X'' \in \mathcal{C}_{R,0}(x)$. Proposition 6.5.2 now implies that the chain $\mathrm{tr}_R(X * Y) = X' * \mathrm{tr}_R Y' + X'' * \mathrm{tr}_R Y''$ is a cycle, as claimed. □

6.5.4 Boundaries

Consider the following chains from Figure 6.2:



All four are closed; are they exact? The first three cannot be exact, because their B-type crossing arcs, if there are any, join distinct 0-labeled circles (c.f. Figure 6.3); this holds over both $R = \mathbb{F}_2, \mathbb{Z}$. To see that $X :=$  is not exact over $R = \mathbb{F}_2, \mathbb{Z}$, consider the map

$\varepsilon \circ \pi_{[X]_B} \circ d : \mathcal{C}_R(D) \to R$. If W is an enhanced state from D with $\varepsilon \circ \pi_{[X]_B} \circ d(W) \neq 0$, then W is one of , , , and $\varepsilon \circ \pi_{[X]_B} \circ d(W) = 1 + 1 \in 2R$. Together with the homogeneity of  and Proposition 6.3.1, this implies that $X = \mathrm{tr}_R X$ is not exact:

Proposition 6.5.4. *If X enhances a state x of a diagram D such that $[X]_A \cap [X]_B = \{X\}$ (e.g. if x is homogeneous), and if $\varepsilon \circ \pi_{[X]_B} \circ d : \mathcal{C}_R(D) \to 2R$ with $2R \subsetneq R$, then $\mathrm{tr}_R X$ is not exact.*

Proof. If $\mathrm{tr}_R X$ were exact, say $\mathrm{tr}_R X = dY$, $Y \in \mathcal{C}_R(D)$, then 2 would be a unit in R, contrary to assumption:

$$1 = \varepsilon(X) = \varepsilon \circ \pi_{[X]_B}(\mathrm{tr}_R X) = \varepsilon \circ \pi_{[X]_B} \circ d(Y) \in 2R.$$

□

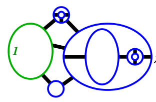
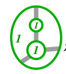
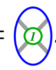
Thus, $\mathrm{tr}_R X = X =$  is not exact because $[X]_A \cap [X]_B = \{X\}$ and $\varepsilon \circ \pi_{[X]_B} \circ d : \mathcal{C}_R(D) \to 2R$. Plumbing preserves homogeneity, which implies the former property, by Proposition 6.3.1. The next proposition states that, with an extra condition, plumbing also preserves the latter property.

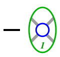
Proposition 6.5.5. *Let $x * y = z$ be a plumbing of states with $x \cap y \subset x \setminus x_B$. If X, Y enhance x, y with $\varepsilon \circ \pi_{[X]_B} \circ d : \mathcal{C}_R(D_x) \to 2R$, $\varepsilon \circ \pi_{[Y]_B} \circ d : \mathcal{C}_R(D_y) \to 2R$ and $X * Y = Z$, then $\varepsilon \circ \pi_{[Z]_B} \circ d : \mathcal{C}_R(D_z) \to 2R$.*

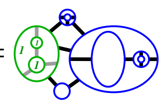
Proof. Observe the following consequence of the incidence rules for the differential. Suppose W enhances a state w of a diagram D. Then $\varepsilon \circ \pi_{[W]_B} \circ d : \mathcal{C}_R(D) \to 2R$ if and only if each component of w_B contains at most one 1-labeled circle. The proposition follows immediately from this observation. □

6.5.5 The main theorem via plumbing

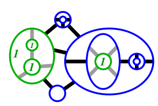
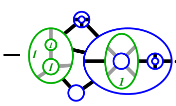
Two examples will show how plumbing is used to build up the main theorem's homology classes. First, with either $R = \mathbb{F}_2$ or $R = \mathbb{Z}$, consider:

$X_1 =$ , $X_2 =$ , $X_3 =$ .

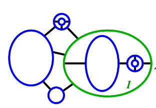
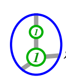
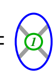
Each of $\mathrm{tr}_R X_1 = X_1$, $\mathrm{tr}_R X_2 = X_2$, $\mathrm{tr}_R X_3 = X_3 -$  is a cycle; also each $[X_r]_A \cap [X_r]_B = \{X_r\}$, and $\varepsilon \circ \pi_{[X_r]_B} \circ d$ maps to $2R$; Proposition 6.5.4 implies that $\mathrm{tr}_R X_1$, $\mathrm{tr}_R X_2$, $\mathrm{tr}_R X_3$ represent nonzero homology classes. Propositions 6.5.3, 6.5.5 further imply that

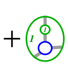
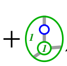
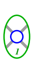
$\mathrm{tr}_R(X_1 * X_2) = X_1 * X_2 =$ 

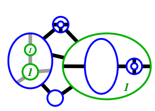
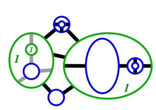
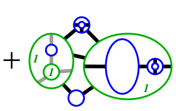
also represents a nonzero homology class, as does

$\mathrm{tr}_R(X_1 * X_2 * X_3) =$  $-$ .

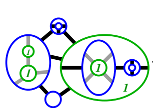
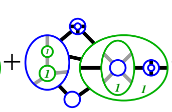
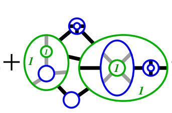
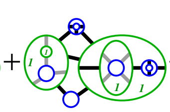
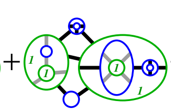
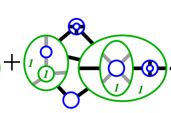
While the previous example holds over both $\mathbb{F}_2$, $\mathbb{Z}$, the next example works over $\mathbb{F}_2$ only. Let

$Y_1 =$ , $Y_2 =$ , $Y_3 =$ .

By the same reasoning as the last example, $\mathrm{tr}_{\mathbb{F}_2} Y_1 = Y_1$, $\mathrm{tr}_{\mathbb{F}_2} Y_2 = Y_2 +$  $+$ , and $\mathrm{tr}_{\mathbb{F}_2} Y_3 = Y_3 +$  represent nonzero homology classes, as do

$\mathrm{tr}_{\mathbb{F}_2}(Y_1 * Y_2) =$  $+$  $+$ 

and

$\mathrm{tr}_{\mathbb{F}_2}(Y_1 * Y_2 * Y_3) =$  $+$  $+$  $+$  $+$  $+$ .

Theorem 6.5.1. *If z is a homogeneously adequate state, then for any $p \in z$ away from crossing arcs, z has enhancements $Z^- \in \mathcal{C}_{R,1}(x)$, $Z^+ \in \mathcal{C}_{R,0}(z)$, identical away from p, such that both $tr_{\mathbb{F}_2} Z^\pm$ represent nonzero classes in $Kh^{i_z, j_z \pm 1}_{\mathbb{F}_2}(z)$. If also G_{z_A} is bipartite, then both $tr_{\mathbb{Z}} Z^\pm$ represent nonzero classes in $Kh^{i_z, j_z \pm 1}_{\mathbb{Z}}(z)$.*

Proof. For both $R = \mathbb{F}_2, \mathbb{Z}$, we argue by induction on the number of zones in z that z has enhancements $Z^+ \in \mathcal{C}_{R,0}(z)$, $Z^- \in \mathcal{C}_{R,1}(z)$, identical away from p, such that both traces $\mathrm{tr}_R Z^\pm$ are cycles, and $2R$ contains the images of $\varepsilon \circ \pi_{[Z^\pm]_B} \circ d$, implying that neither $\mathrm{tr}_R Z^\pm$ is exact, by Proposition 6.5.4.

When z has a single zone, it is adequate and either all-A or all-B, and the theorem holds with $Z^\pm$ as described in the introduction, by Corollary 6.4.1. For the inductive step, de-plumb $z = x * y$ such that $x \cap y \subset x_A \cap y_B$ or $x \cap y \subset x_B \cap y_A$, i.e. such that the state circle $x \cap y$ is incident only to A-type arcs in x and to B-type arcs in y, or vice versa. In either case, x and y have fewer zones than z does.

If $p \in x \cap y$, then apply the inductive hypothesis to x and y to obtain $X^+ \in \mathcal{C}_{R,0}(x)$, $X^- \in \mathcal{C}_{R,1}(x)$, identical away from p, and $Y^+ \in \mathcal{C}_{R,0}(y)$, $Y^- \in \mathcal{C}_{R,1}(y)$, identical away from p, such that all four of $\mathrm{tr}_R X^\pm$, $\mathrm{tr}_R Y^\pm$ are cycles, and such that $2R$ contains the images of all four of $\varepsilon \circ \pi_{[X^\pm]_B} \circ d$, $\varepsilon \circ \pi_{[Y^\pm]_B} \circ d$. Let $Z^- := X^- * Y^-$, $Z^+ := X^+ * Y^+$. Then Z^- and Z^+ are identical away from p; $\mathrm{tr}_R Z^-$ and $\mathrm{tr}_R Z^+$ are cycles by Proposition 6.5.3; and $2R$ contains the images of $\varepsilon \circ \pi_{[Z^-]_B} \circ d$ and $\varepsilon \circ \pi_{[Z^+]_B} \circ d$ by Proposition 6.5.5.

Assume instead that $p \notin x \cap y$; without loss of generality $p \in x \setminus y$. Apply the inductive hypothesis to obtain $X^+ \in \mathcal{C}_{R,0}(x)$, $X^- \in \mathcal{C}_{R,1}(x)$, identical away from p, such that $\mathrm{tr}_R X^\pm$ are cycles and $2R$ contains the images of $\varepsilon \circ \pi_{[X^\pm]_B} \circ d$. Let b be a point in $x \cap y$. Apply the inductive hypothesis to obtain $Y^+ \in \mathcal{C}_{R,0}(y)$, $Y^- \in \mathcal{C}_{R,1}(y)$ (with the subscript indicating the label at b), identical away from b, such that $\mathrm{tr}_R Y^\pm$ are cycles and $2R$ contains the images of $\varepsilon \circ \pi_{[Y^\pm]_B} \circ d$. Since $X^\pm$ are identical away from p, which is not on $x \cap y$, $X^\pm$ assign the same label to the circle $x \cap y$. If this label is 1, then let $Z^\pm := X^\pm * Y^-$; if this label is 0, then let $Z^\pm := X^\pm * Y^+$. Either way, $Z^\pm$ are identical away from p, and $\mathrm{tr}_R Z^\pm$ are cycles by Proposition 6.5.3. Further, in both cases, Proposition 6.5.5 implies that $2R$ contains the images of both $\varepsilon \circ \pi_{[Z^\pm]_B} \circ d$; Proposition 6.5.4 then implies that neither $\mathrm{tr}_R Z^\pm$ is exact. □

6.5.6 Connect sums of chains

A state z is a connect sum $z = x\#y$ if $x, y \subset z$ are states intersecting in exactly one circle, and there is a simple closed curve $\gamma \subset S^2$ which intersects z in two points, both of them on the circle $x \cap y$ and away from crossing arcs, so that γ separates the crossing arcs in y from those in x. If D_x, D_y, D_z are the underlying link diagrams for x, y, z, then the connect sum $x\#y = z$ also gives connect sums of link diagrams, $D_x \# D_y = D_z$ and of links, as well as a boundary-connect-sum of the associated state surfaces $F_x \natural F_y = F_z$. (The Khovanov homology groups for $D_x \# D_y = D_z$ can be described using a long-exact sequence [59].) Also, these three link diagrams can always be oriented coherently, so that component-wise boundary slope is additive under $\natural$ and in particular $i_x + i_y = i_z$; this is not always possible for plumbings of states.

More importantly (for the purposes of plumbing), each state z' of D_z decomposes (along γ) as a connect sum $z' = x'\#y'$, where x' is a state of D_x and y' is a state of D_y, although the state circle $x' \cap y'$ will not generally equal $x \cap y$. As noted in §6.5.2, this fact does not extend to plumbings of states in general.

If Z' enhances $z' = x'\#y'$, then Z' restricts on x', y' to enhancements X', Y', whose labels match at p: either both are 1 (in which case $j_Z = j_X + j_Y + 1$) or both are 0 (in which case $j_Z = j_X + j_Y - 1$). This extends (6.4), in the case of connect sum $x\#y = z$, to the following R-module isomorphism:

$$\begin{aligned} \# : \left(\mathcal{C}_{R,1}(D_x) \otimes \mathcal{C}_{R,1}(D_y)\right) \oplus \left(\mathcal{C}_{R,0}(D_x) \otimes \mathcal{C}_{R,0}(D_y)\right) &\to \mathcal{C}_R(D_x\#D_y) \\ X' \otimes Y' &\mapsto X'\#Y'. \end{aligned} \tag{6.8}$$

If $X \in \mathcal{C}_{\mathbb{F}_2,1}(D_x)$ and $Y \in \mathcal{C}_{\mathbb{F}_2,1}(D_y)$, then $X\#Y$ is defined and $d(X\#Y) = dX\#Y + X\#dY$: this follows straight from the definition of the differential (c.f. Figure 6.3). Describing $d(X\#Y)$ in case $X \in \mathcal{C}_{\mathbb{F}_2,0}(D_x)$ and $Y \in \mathcal{C}_{\mathbb{F}_2,0}(D_y)$ requires connect sums in left-trumps ${}_\diamond\#$ and right-trumps $\#_\diamond$. These operations are defined the same as ${}_\diamond*$ and $*_\diamond$, except taking the domain to be all of $\mathcal{C}_R(D_x) \otimes \mathcal{C}_R(D_y)$. The maps (6.5) and (6.6) thus extend in the case of connect sum to:

$$\begin{aligned} {}_\diamond\# : \mathcal{C}_R(D_x) \otimes \mathcal{C}_R(D_y) &\to \mathcal{C}_R\left(D_x\#D_y\right) \\ X' \otimes Y &\mapsto X'\,{}_\diamond\# Y \end{aligned} \qquad \begin{aligned} \#_\diamond : \mathcal{C}_R(D_x) \otimes \mathcal{C}_R(D_y) &\to \mathcal{C}_R\left(D_x\#D_y\right) \\ X \otimes Y' &\mapsto X\#_\diamond Y'. \end{aligned}$$

This extension of domain is possible because, with γ decomposing the connect sum of states $z = x\#y$ and link diagrams $D_z = D_x\#D_y$, γ decomposes any other state of D_z as $z' = x'\#y'$ such that $x' \cap y'$ consists of a single state circle. This implies that connect sums of chains are always defined in both trumps. Thus, connect sum and its variants in left-and right-trumps extend to the entire chain complex; Proposition 6.5.1 further implies that they do so in a way that respects the differential:

$$d(X\#Y) = dX\,{}_{\diamond}\#Y + (-1)^{|\rangle\langle|_X} X\#_{\diamond} dY. \tag{6.9}$$

Let $\imath : (\mathcal{C}_{R,1}(D_x)\otimes\mathcal{C}_{R,1}(D_y))\oplus(\mathcal{C}_{R,0}(D_x)\otimes\mathcal{C}_{R,0}(D_y)) \hookrightarrow \mathcal{C}_R(D_x)\otimes\mathcal{C}_R(D_y)$ denote inclusion and $\#^{-1}$ the inverse of the isomorphism from (6.8). With coefficients in $\mathbb{F}_2$, (6.9) states that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{C}_{\mathbb{F}_2}(D_x\#D_y) & \xrightarrow{\quad d \quad} & \mathcal{C}_{\mathbb{F}_2}(D_x\#D_y) \\
& {\scriptstyle \imath\circ\#^{-1}} \searrow \qquad \nearrow {\scriptstyle \#_{\diamond}\circ(\mathbb{1}\otimes d) + {}_{\diamond}\#\circ(d\otimes\mathbb{1})} & \\
& \mathcal{C}_{\mathbb{F}_2}(D_x)\otimes\mathcal{C}_{\mathbb{F}_2}(D_y) &
\end{array}$$

Compare this with the case of plumbing in general, where a similar natural property holds, albeit on a more restricted domain. Let $x*y$ be a plumbing of states, with D_x*D_y the associated plumbing of link diagrams, and let $\mathcal{S}$ denote the collection of states from D_x*D_y that contain the state circle $x\cap y$. Thus, each $z\in\mathcal{S}$ decomposes as a plumbing of states $z = x'*y'$ where x', y' are states of D_x, D_y with $x\cap y = x'\cap y'$. Also let $\imath$: $(\mathcal{C}_{R,1}(D_x)\otimes\mathcal{C}_{R,1}(D_y))\oplus(\mathcal{C}_{R,0}(D_x)\otimes\mathcal{C}_{R,0}(D_y)) \hookrightarrow \mathcal{C}_R(D_x)\otimes\mathcal{C}_R(D_y)$ denote inclusion and $*^{-1}$ the inverse of the isomorphism from (6.4). With coefficients in $\mathbb{F}_2$, (6.7) states that the following diagram commutes:

$$\begin{array}{ccc}
\bigoplus_{z\in\mathcal{S}}\mathcal{C}_{\mathbb{F}_2}(z) & \xrightarrow{\quad d \quad} & \mathcal{C}_{\mathbb{F}_2}(D_x*D_y) \\
& {\scriptstyle \imath\circ *^{-1}} \searrow \qquad \nearrow {\scriptstyle *_{\diamond}\circ(\mathbb{1}\otimes d) + {}_{\diamond}*\circ(d\otimes\mathbb{1})} & \\
& \mathcal{C}_{\mathbb{F}_2}(D_x)\otimes\mathcal{C}_{\mathbb{F}_2}(D_y) &
\end{array}$$

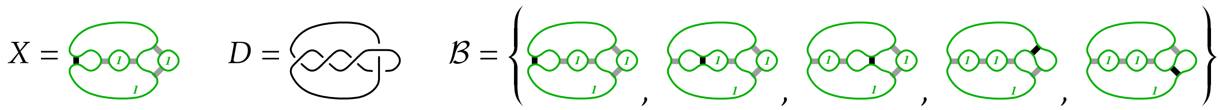
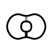
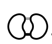


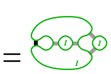
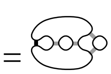
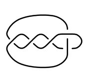
Figure 6.10: Over $\mathbb{F}_2$, this enhancement X of an *inessential* state of D represents a nonzero homology class, since $dX = 0$ and $\varepsilon \circ \pi_{\mathcal{B}} \circ d \equiv 0$, where $\mathcal{B}$ is an $\mathbb{F}_2$-basis for $\mathcal{C}_{\mathbb{F}_2}^{i_X,j_X}(D)$.

6.6 Remarks and questions

A state x need not be essential in order for $\mathcal{C}_{\mathbb{F}_2}(x)$ to contain a (representative of a) nonzero homology class; indeed, x need not even be adequate. Consider two examples. First, for the trivial diagram of two components, ○ ○, the homology groups are

$$Kh_{\mathbb{F}_2}^{0,-2} = \mathbb{F}_2 \cdot ① \ ①, \qquad Kh_{\mathbb{F}_2}^{0,0} = (\mathbb{F}_2 \cdot ① \ ○) \oplus (\mathbb{F}_2 \cdot ○ \ ①), \qquad Kh_{\mathbb{F}_2}^{0,2} = \mathbb{F}_2 \cdot ○ \ ○.$$

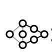
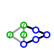
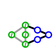
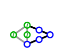
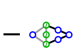
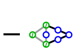
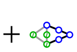
Now perform a Reidemeister-2 move to get the connected diagram . Each of the four homology generators can still be taken to be the trace of an enhancement of a single state, namely  or . Yet, these states are not essential, since their state surfaces are connected and span a split link.

Second, consider the enhancement $X =$  of the state $x =$  of the diagram $D =$  (c.f. Figure 6.10). This enhanced state X is a cycle with any coefficients. Moreover, taking $\mathcal{B}$ to be an $\mathbb{F}_2$-basis for $\mathcal{C}_{\mathbb{F}_2}^{i_X,j_X}(D)$, the map $\varepsilon \circ \pi_{\mathcal{B}} \circ d$ is identically zero. Thus, X represents a nonzero homology class in $Kh_{\mathbb{F}_2}^{i_X,j_X}(D)$, even though it enhances an *inessential* Kauffman state x.

Next, here is an idea for extending the main theorem: establish a class of essential checkerboard states which represent nonzero Khovanov homology classes in two distinct

j-gradings, say over $\mathbb{F}_2$, and then aim to extend by plumbing. The easiest class of essential checkerboard states consists of those which are *alternating*; plumbing these gives the homogeneously adequate states. To construct a new (non-alternating) class of essential checkerboard states, consider any (non-alternating) link diagram D which admits no $(n,0)$-pass moves [47]. This means that, whenever $\alpha \subset D$ is a smooth arc whose endpoints are away from crossings and whose interior contains n overpasses and no underpasses, or vice versa, every arc $\beta \subset S^2$ with the same endpoints as α intersects D in its interior. Construct either checkerboard surface F for D, and replace each of its half-twist crossing band s with a full-twist band in the same sense. (Any band with at least two half-twists in the same sense also suffices.)

The resulting link diagram D' has twice as many crossings as D, and the resulting surface F' is an essential, two-sided checkerboard surface with the same euler characteristic as F. (To prove essentiality, use Menasco's crossing ball structures, hypothesize a compressing disk Δ for F' which intersects the crossing ball structure minimally, characterize the outermost disks of $\Delta \setminus (S^2 \cup C)$, and observe that the only viable configuration for a height one component of $\Delta \cap (S^2 \cup C)$ implies that D admitted an $(n,0)$-pass move, contrary to assumption.) Does Khovanov homology detect the essential checkerboard states from this construction?

The simplest non-adequate diagram admitting no $(n,0)$-pass moves is a 4-crossing diagram of the trefoil. Following Figure 6.11, construct an 8-crossing diagram D from this one in the manner just described, and consider the traces of two enhancements of the state $x =$ : $X =$  and $Y =$ , with $\mathrm{tr}_R X =$  $-$  $-$  $+$  and $\mathrm{tr}_R Y = Y$. Both

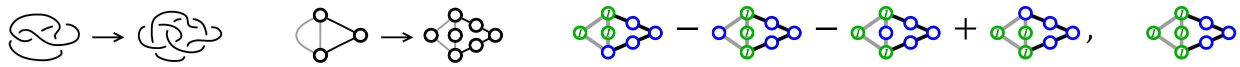


Figure 6.11: Doubling crossings yields essential states whose behavior is less obvious than in the adequate case, even in the case of the two cycles shown above-right.

traces are cycles over $R = \mathbb{F}_2$ and $R = \mathbb{Z}$, but exactness is not so easy. We have seen, when considering the exactness of a cycle $\mathrm{tr}_R Z$ from an enhancement Z of an homogeneously adequate state z of a link diagram D, that it suffices to consider $\pi_{[Z]_B} \circ d$, rather than, say, the entire map $d : \mathcal{C}_{\mathbb{F}_2}^{i_z-1, j_Z}(D) \to \mathcal{C}_{\mathbb{F}_2}^{i_z, j_Z}(D)$. In this example, although the map $\pi_{\mathcal{C}_{\mathbb{Z}}(x)} \circ d$ carries enough information to prove that $\mathrm{tr}_{\mathbb{Z}} X$ is not exact, testing $\mathrm{tr}_{\mathbb{F}_2} X$ and Y (over $\mathbb{F}_2$ and $\mathbb{Z}$) for exactness is more complicated, especially compared to the paper-and-pencil complexity of the methods that suffice in the homogeneously adequate case. We conclude by posing three questions:

Question 6.6.1. *If x is an essential state, does $\mathcal{C}_{\mathbb{F}_2}(x)$ always contain a nonzero homology class?*

Question 6.6.2. *Is there a general method for distinguishing those Khovanov homology classes that correspond to essential states from those that do not?*

Question 6.6.3. *Does every link have a diagram with an essential state? A homogeneously adequate state?*