Heegaard diagrams corresponding to Turaev surfaces

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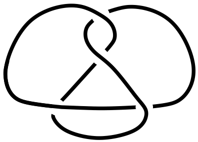
Main theorem (Armond, Druivenga, K)

There is a 1-to-1 correspondence between Turaev surfaces of connected link diagrams on $S^2 \subset S^3$ and diagrams $(\Sigma, \alpha, \beta, D)$ satisfying:

- (Σ, α, β) is a Heegaard diagram for S^3, and D is an alternating link diagram on Σ which cuts Σ into disks, with $D \pitchfork \alpha$, $D \pitchfork \beta$, $\alpha \pitchfork \beta$.
- $D \cap \alpha = D \cap \beta = \alpha \cap \beta$, none of these points being crossings of D.
- There is a checkerboard partition $\Sigma \setminus (\alpha \cup \beta) = \Sigma_\varnothing \cup \Sigma_K$, in which:
  - $\Sigma_\varnothing$ consists of disks disjoint from D,
  - D cuts Σ_K into disks, each of whose boundary contains at least one crossing point and at most two points of $\alpha \cap \beta$, and
  - $2g(\Sigma) + |\Sigma_\varnothing| = |\alpha| + |\beta|$.



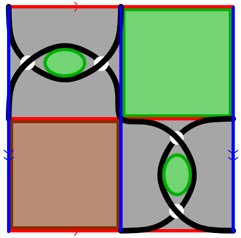
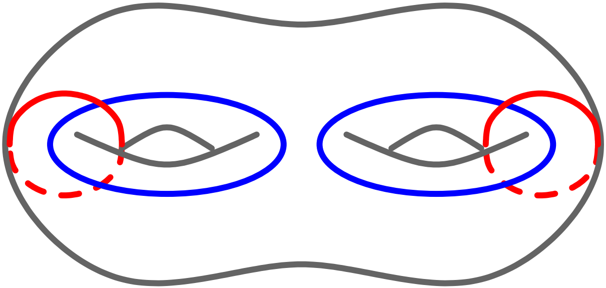


Figure: A link diagram on S^2, and the link-adapted Heegaard diagram $(\Sigma, \alpha, \beta, D)$ corresponding to its Turaev surface, a torus.

Heegaard splittings

A **Heegaard splitting** of an orientable 3-manifold M is a decomposition of M into two handlebodies H_α and H_β with common boundary. The surface $\partial H_\alpha = \partial H_\beta = \Sigma$ is called a **splitting surface** for M.

Describe a handlebody H by identifying a collection of disjoint s.c.c.'s $\alpha_1, \ldots, \alpha_k$ on its boundary $\partial H = \Sigma$, such that each α_i bounds a disk $\hat{\alpha}_i$ in H, and such that these disks together cut H into balls. The α_i are called **attaching circles** for H.



Heegaard diagrams

A **Heegaard diagram** (Σ, α, β) combines these ideas (Heegaard splitting and attaching circles) to blueprint a 3-manifold. The diagram consists of a splitting surface $\Sigma = \partial H_\alpha = \partial H_\beta$, together with a union $\alpha = \bigcup \alpha_i$ of attaching circles for H_α and a union $\beta = \bigcup \beta_i$ of attaching circles for H_β. Assume that $\alpha \pitchfork \beta$.

If (Σ, α, β) is a Heegaard diagram for S^3, then the circles of α and β together generate $H_1(\Sigma)$, and therefore cut Σ into disks. These disks are said admit a **checkerboard partition** if they can be partitioned into two classes so that no two disks in the same class are incident, except at the points of $\alpha \cap \beta$.

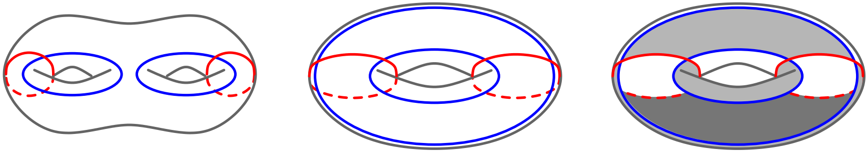


Figure: Two Heegaard diagrams, the second of which admits a checkerboard partition.

Exercise: The disks from (Σ, α, β) admit a checkerboard partition if and only if, for all attaching circles α_r, β_s, the quantities $|\alpha_r \cap \beta|$, $|\alpha \cap \beta_s|$ are even.

When do attaching circles generate $H_1(\Sigma)$?

Proposition

Let (Σ, α, β) be a Heegaard diagram for a 3-manifold $M = H_\alpha \cup_\Sigma H_\beta$. The attaching circles' homology classes $[\alpha_i]$, $[\beta_i]$ generate $H_1(\Sigma)$ iff $H_1(M) = 0$.

Proof.

($\Rightarrow$) In general, inclusion $\Sigma \hookrightarrow M$ induces a map $H_1(\Sigma) \twoheadrightarrow H_1(M)$ whose kernel contains all $[\alpha_i]$, $[\beta_i]$. If $H_1(M) = 0$, then $\{[\alpha_i], [\beta_i]\}$ cannot generate $H_1(\Sigma) \neq 0$.

($\Leftarrow$) Idea: If $\gamma \subset \Sigma$ is an oriented s.c.c. and $H_1(M) = 0$, then there is an oriented surface $S \subset S^3$ with $\partial S = \gamma$. Cut S along Σ and then along the disks bounded by α and β, and write:

$$[\gamma] = [\partial S] = \sum_{S_{\alpha,i} \subset S \cap H_\alpha} [\partial S_{\alpha,i}] + \sum_{S_{\beta,i} \subset S \cap H_\beta} [\partial S_{\beta,i}] = \sum_{i,j} a_{i,j}[\alpha_j] + \sum_{i,j} b_{i,j}[\beta_j]$$

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$(\Leftarrow)$ Idea: If $\gamma \subset \Sigma$ is an oriented s.c.c. and $H_1(M) = 0$, then there is an oriented surface $S \subset S^3$ with $\partial S = \gamma$. Cut S along Σ and then along the disks bounded by α and β, and write:

$$[\gamma] = [\partial S] = \sum_{S_{\alpha,i} \subset S \cap H_\alpha} [\partial S_{\alpha,i}] + \sum_{S_{\beta,i} \subset S \cap H_\beta} [\partial S_{\beta,i}] = \sum_{i,j} a_{i,j}[\alpha_j] + \sum_{i,j} b_{i,j}[\beta_j]$$

□

Link diagrams

A **link diagram** D on a closed surface $F \subset S^3$ is the image, in general position, of an immersion of one or more circles in F; each arc at any double-point is labeled with a direction normal to F near that point, so that under- and over-crossings are identified.

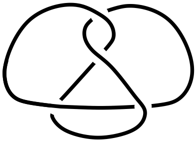


Figure: A link diagram on S^2

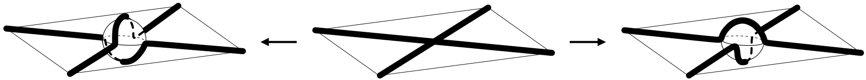


Figure: Double-points in link diagrams are labeled to identify over- and under-crossings.

Crossing balls (I)

By inserting small, mutually disjoint crossing balls $C = \bigcup C_i$ centered at the crossing points of D and pushing the two intersecting arcs of each $D \cap C_i$ off F to the appropriate hemisphere of $\partial C_i \setminus F$ as shown, one obtains a configuration of a link $K \subset (F \setminus C) \cup \partial C \subset S^3$. Call this a **crossing ball configuration** of the link K corresponding to the link diagram D.

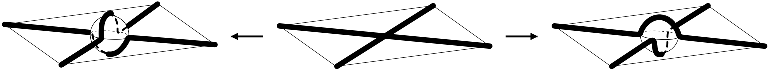


Figure: Each crossing in a link diagram is labeled in one of two ways. The label tells one how to adjust the link after inserting a crossing ball.

Crossing Balls (II)

Conversely, given mutually disjoint crossing balls $C = \bigcup C_i$ centered at points on a closed surface $F \subset S^3$, and a link $K \subset (F \setminus C) \cup \partial C$ in which each crossing ball appears as shown, one may obtain a corresponding link diagram as follows.

Consider a regular neighborhood of F that contains C and is parameterized by an orientation-preserving homeomorphism with $F \times [-1, 1]$ which identifies F with $F \times \{0\}$. If $\pi : F \times [-1, 1] \to F$ denotes the natural projection, the link diagram corresponding to the crossing ball configuration $K \subset (F \setminus C) \cup \partial C \subset S^3$ is the projected image $\pi(K) \subset F$ with appropriate crossing labels.

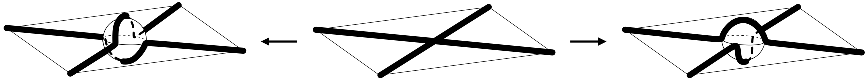


Figure: Crossing labels tell one how to adjust the link after inserting a crossing ball.

Overpasses, underpasses, and alternatingness

In such a crossing ball configuration, each arc of $K \cap \partial C$ lies either in $F \times [-1,0]$ or in $F \times [0,1]$. The former arcs are called **underpasses**, and the latter are called **overpasses**. A link diagram D is said to be **alternating** if each arc of $K \setminus C$ in a corresponding crossing ball configuration joins an underpass with an overpass. A link $K \subset S^3$ is alternating if it has an alternating diagram on S^2.

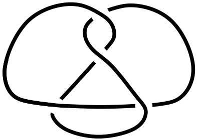


Figure: A non-alternating link diagram on S^2

Link-adapted Heegaard diagrams

In particular, any Heegaard diagram (Σ, α, β) for S^3 provides an embedding of the closed surface Σ in S^3. One may therefore superimpose a link diagram D on the Heegaard diagram to obtain a new type of diagram $(\Sigma, \alpha, \beta, D)$. This new diagram describes a Heegaard splitting of S^3 in which the splitting surface contains a link diagram.

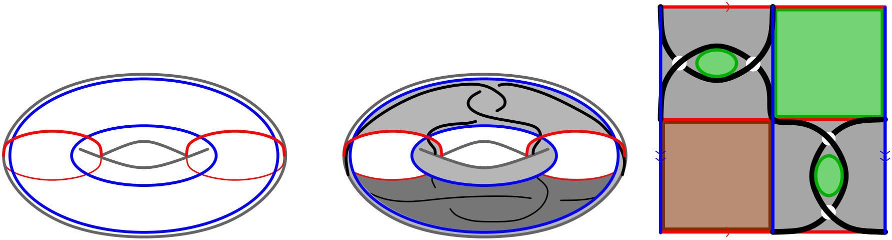


Figure: A Heegaard diagram (left) and a link-adapted Heegaard diagram (center, right).

A- and B-smoothings of crossings

Each crossing in a link diagram D on a surface F can be smoothed in two different ways, by inserting a crossing ball C_i and replacing $D \cap C_i$ with one of the two pairs of arcs of $(\partial C_i \cap F) \setminus D$ opposite to another. The two possibilities, called the **A-smoothing** and the **B-smoothing** of the crossing, are shown below. Making a choice of smoothing for each crossing in the diagram produces a disjoint union of circles on F, called a **state** of the diagram D. Two states of D are **dual** if they have opposite smoothings at each crossing.

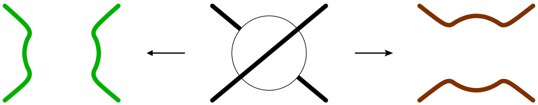


Figure: The A-smoothing (left) and B-smoothing (right) of a crossing.

The all-A and all-B states of a link diagram

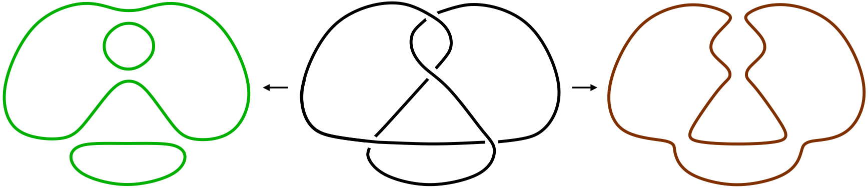


Figure: The all-A (left) and all-B (right) states for a link diagram.

Given a link diagram D on S^2, the two extreme states – the all-A and the all-B – are of particular interest, due in part to the bounds they give on the maximum and minimum degrees of the Jones polynomial. Kauffman's proof [5] that these bounds are sharp for reduced, alternating diagrams provided the impetus for Murasugi [7], Thistlethwaite [8], and Turaev [9] to prove Tait's conjecture on the crossing numbers of alternating links. Cromwell [2], Lickorish and Thistlethwaite [6] then extended these results to *adequate* link diagrams.

Turaev's cobordism between the all-A & all-B states

Viewing a bi-collar of S^2 as $S^2 \times [-1, 1]$, push the all-A and all-B states off S^2 to $S^2 \times \{1\}$ and $S^2 \times \{-1\}$, respectively, s.t. each state circle sweeps out an annulus to one side of S^2. Finally, glue together these annuli and the disks of $S^2 \cap C$.

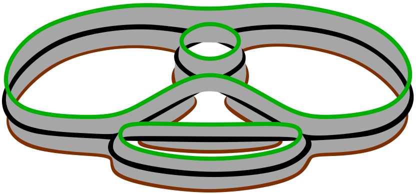


Figure: The cobordism between the all-A and all-B states from earlier.

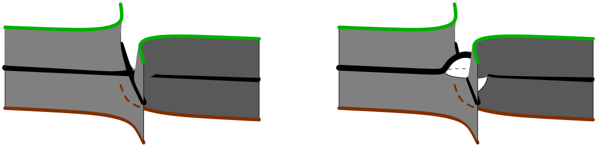


Figure: Near each crossing, the cobordism has a saddle, as shown.

Turaev surfaces

Having constructed the cobordism, Turaev caps off the all-A and all-B states with disjoint disks to form a closed surface Σ, called the **Turaev surface** of D. Since crossing information of D on S^2 translates to crossing information on the Turaev surface, D forms a link diagram on Σ.

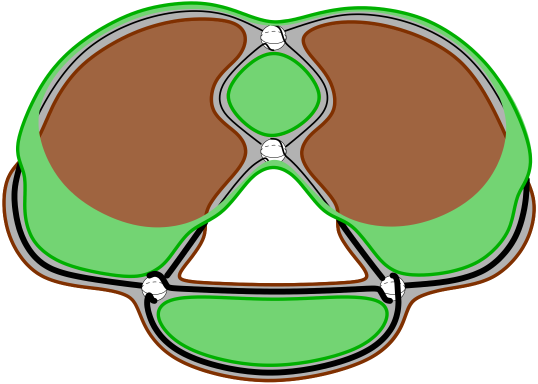


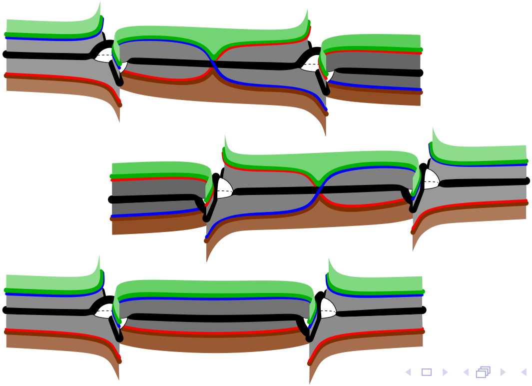
Figure: This torus is the Turaev surface of the link diagram from earlier.

Generalized Turaev surfaces

The construction of the Turaev surface generalizes to any pair of states s and $\tilde{s}$ dual to one another. By pushing s and $\tilde{s}$ to opposite sides of S^2 to sweep out annuli on opposite sides of S^2, gluing in disks near the crossings to obtain a cobordism between s and $\tilde{s}$, and capping off with disks, one obtains a closed surface Σ on which D forms a link diagram. Call this surface Σ the **generalized Turaev surface** of the dual states s and $\tilde{s}$.

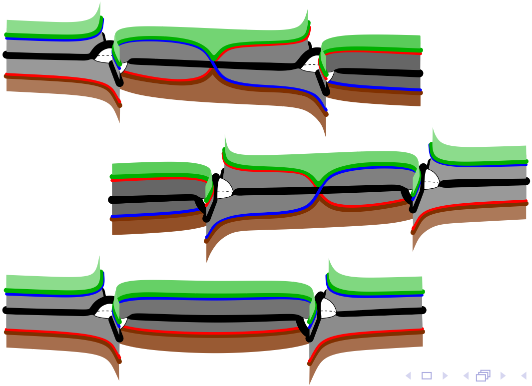
Given a Turaev surface Σ for D, define and adjust α, β

Define $\hat{\alpha} := (S^2 \setminus (C \cup K)) \cap H_\alpha$ and $\hat{\beta} := (S^2 \setminus (C \cup K)) \cap H_\beta$ to be the two checkerboard classes of $S^2 \setminus (C \cup K)$, with $\alpha := \partial\hat{\alpha}$ and $\beta := \partial\hat{\beta}$. From this setup, three modifications will complete the construction of the diagram $(\Sigma, \alpha, \beta, D)$. During these changes, Σ, D, S^2, C, and K will remain fixed. First, perturb α and β through the cobordism to appear as shown below, carrying along the disks of $\hat{\alpha}$ and $\hat{\beta}$.



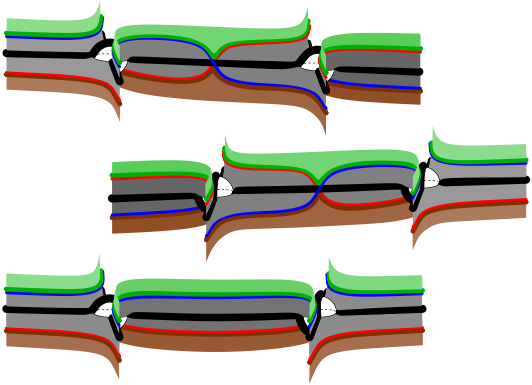
More precisely...

Let $X = \{x_1, \ldots, x_n\}$ (resp. $Y = \{y_1, \ldots, y_n\}$) consist of one point on each arc of $K \setminus C$ which joins two underpasses (resp. overpasses) on S^2. Each arc of $\alpha \setminus (X \cup Y)$ (resp. $\beta \setminus (X \cup Y)$) runs along a circle from either the all-A state or the all-B state. Isotope α (resp. β) through the cobordism so as to push arcs of the former type to $S^2 \times (0, 1)$ and arcs of the latter type to $S^2 \times (-1, 0)$. Now $\alpha \cap C = \varnothing = \beta \cap C$ and $\alpha \cap D = \beta \cap D = \alpha \cap D = \beta \cap D = X \cup Y$.



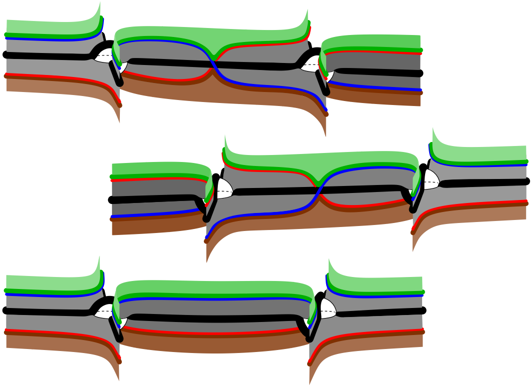
Next, adjust the state circles within Σ.

To further simplify the picture, push the state circles through the cobordism to align with $\alpha \cup \beta$, so that each state disk becomes a component of $\Sigma \setminus (\alpha \cup \beta)$. This causes the neighborhood of each arc of $K \setminus C$ to appear as shown below, possibly with red and blue reversed. Note that the state disks' interiors remain disjoint from D, in fact from S^2.



Finally, remove any attaching circles disjoint from D.

Also remove the corresponding disks of $\hat{\alpha}$ and $\hat{\beta}$, and let α, β, $\hat{\alpha}$ and $\hat{\beta}$ retain their names. Because each removed circle lies in some disk of $\Sigma \setminus D$, each removed disk is parallel to Σ.



Some facts about Turaev surfaces

Lemma (Dasbach, Futer, Kalfagianni, Lin, Stolzfus)

The Turaev surface Σ of any connected link diagram D on $S^2 \subset S^3$ is a splitting surface for S^3.

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Lemma (DFKLS [3])

Any connected link diagram D on $S^2 \subset S^3$ forms an alternating link diagram on its Turaev surface Σ.

One defines the *Turaev genus* $g_T(K)$ of a link $K \subset S^3$ to be the minimum genus among the Turaev surfaces of all diagrams of K on S^2. The resulting invariant, surveyed in [1], measures how far a link is from being alternating. In particular, Turaev genus provides the crux of Turaev's proof of Tait's conjecture:

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Corollary (Turaev [9], DFKLS [3])

A link K is alternating if and only if $g_T(K) = 0$.

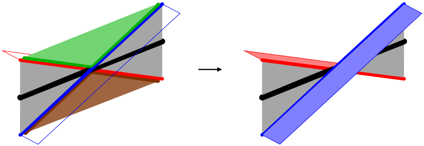
Statement of main theorem

There is a 1-to-1 correspondence between Turaev surfaces of connected link diagrams on $S^2 \subset S^3$ and diagrams $(\Sigma, \alpha, \beta, D)$ satisfying:

- (Σ, α, β) is a Heegaard diagram for S^3, and D is an alternating link diagram on Σ which cuts Σ into disks, with $D \pitchfork \alpha$, $D \pitchfork \beta$, $\alpha \pitchfork \beta$.
- $D \cap \alpha = D \cap \beta = \alpha \cap \beta$, none of these points being crossings of D.
- There is a checkerboard partition $\Sigma \setminus (\alpha \cup \beta) = \Sigma_\varnothing \cup \Sigma_K$, in which:
  - $\Sigma_\varnothing$ consists of disks disjoint from D,
  - D cuts Σ_K into disks, each of whose boundary contains at least one crossing point and at most two points of $\alpha \cap \beta$, and
  - $2g(\Sigma) + |\Sigma_\varnothing| = |\alpha| + |\beta|$.

Proof of main theorem (idea)

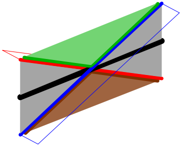
The theorem from DKFLS is one direction of this correspondence. It remains to prove the converse.



Given such $(\Sigma, \alpha, \beta, D)$, remove $\Sigma_\varnothing$ from Σ and glue in $\hat{\alpha} \cup \hat{\beta}$ to obtain a closed surface—a sphere, since $2g(\Sigma) + |\Sigma_\varnothing| = |\alpha| + |\beta|$—on which D forms a link diagram. The only work is to show that Σ is this diagram's Turaev surface. The key is that, because D is alternating, $\Sigma \setminus D$ admits a checkerboard partition. One class consists of the green disks, the other the brown.

Proof of generalized theorem (idea)

Without D alternating on Σ, it is less obvious that the disks of $\Sigma \setminus D$ admit a checkerboard partition. Still, it's true.



Here's why: for each arc of $\alpha \setminus \beta$, one endpoint appears as above, the other as the mirror image. The same holds for arcs of $\beta \setminus \alpha$. Thus, each α_i intersects D in an even number of points, as does each β_i. Since the $[\alpha_i], [\beta_i]$ generate $H_1(\Sigma)$, a generic s.c.c. on Σ intersects D in an even number of points. This is why the disks of $\Sigma \setminus D$ admit a checkerboard partition.

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Thank you!