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Crosscap numbers of alternating knots via unknotting splices

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Ito-Takimura recently defined a splice-unknotting number $u^-(D)$ for knot diagrams. They proved that this number provides an upper bound for the crosscap number of any prime knot, asking whether equality holds in the alternating case. We answer their question in the affirmative. (Ito has independently proven the same result.) As an application, we compute the crosscap numbers of all prime alternating knots through at least 13 crossings, using Gauss codes.

Keywords: knot, alternating, splice, crosscap number, state surface, Gauss code

Mathematics Subject Classification 2000: 57M25, 57M27

1. Introduction

Let $K \subset S^3$ be a knot. An embedded, compact, connected surface $F \subset S^3$ is said to *span* K if $\partial F = K$. The *crosscap number* of K, denoted $cc(K)$, is the smallest value of $\beta_1(F)$ among all 1-sided spanning surfaces for K.[a,b]

A theorem of Adams and the author [4] states that, given an alternating diagram D of a knot K, the crosscap number of K is realized by some state surface from D. (Section 2 reviews background.) Moreover, given such D and K, an algorithm in [4] finds a 1-sided state surface F from D with $\beta_1(F) = cc(K)$.

Ito-Takimura recently introduced a u^- *type* splice-unknotting move and used this move to define a splice-unknotting number $u^-(D)$ for knot diagrams [8]. Minimizing this number across all diagrams of a given knot K defines a knot invariant, $u^-(K)$. After proving that $u^-(D) \geq cc(K)$ holds for any diagram D of any nontrivial knot K, Ito-Takimura ask whether this inequality is ever strict in the case of prime alternating diagrams. The main theorem of this paper answers their question in the negative, and states that $u^-(D)$ is minimal among all diagrams of K:

Theorem 1.1. *If D is an alternating diagram of a prime knot K, then*

$$u^-(D) = u^-(K) = cc(K).$$

[a]Since S^3 is orientable, a spanning surface is 1-sided if and only if it contains a mobius band.
[b]$\beta_1(F) = \text{rank}(H_1(F)) = 1 - \chi(F)$ counts how many holes are in F.

The main idea behind Theorem 1.1 is that, when D is alternating, each splice-unknotting sequence that realizes $u^-(D)$ corresponds to a sequence of cuts (at vertical crossing arcs) which reduces some minimal-complexity state surface to a disk, via 1-sided spanning surfaces for other knots. The main difficulty in the proof is that for some diagrams, like the one in Fig. 1, any such sequence will include non-prime diagrams. The trouble this presents is that $u^-(D)$ is additive under diagrammatic connect sum, whereas crosscap number is not additive under connect sum. Addressing this issue requires some work. Lemmas addressing tangles appear in §3, with further technical lemmas in §4. The proof of Theorem 1.1 follows in §5.

Ito-Takimura have independently proven the same result [10]. Their proof uses generalized splice moves; unlike the u^- type move from [8] (see §2), each of these generalized moves either respects orientation or involves a new choice of orientation, and some of the moves change the number of link components. In [9], Ito-Takimura explore a related move of "type B_l," which generalizes splice moves in a different way and correspond to unoriented band sum operations on spanning surfaces. These B_l type moves lead to a knot invariant $B_l(K)$ which is closely related to $u^-(K)$, and when K is alternating, $B_l(K)$ equals its "overall" (orientable and nonorientable) genus $\beta_1(K)$ – see (2.1). Moreover, Ito-Takimura show that B_l is additive under connect sum of alternating knots. This allows them to determine the crosscap number of any (prime or non-prime) alternating knot K in terms of $B_l(K)$.

Section 6 describes how Theorem 1.1 enables an efficient computation of crosscap numbers for the table of prime alternating knots, using Gauss codes and data from the faces determined by the associated knot diagrams. An appendix lists the crosscap numbers for prime alternating knots through 12 crossings.[c] Previously, [4] determined all of these values in theory, listing them through 10 crossings, and knotinfo listed crosscap numbers for 174 of the 367 prime alternating knots with 11 crossings and for 316 of the 1288 with 12 crossings [1]. Most of these values,

[c]Crosscap numbers for prime alternating knots through at least 13 crossings are posted at [3], together with data regarding these knots and their diagrams.

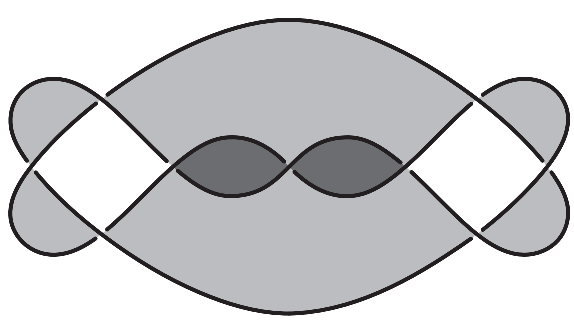


Fig. 1. This state surface for the 9_{10} knot realizes crosscap number, but cutting it at any crossing produces a state surface for either a 2-component link or a non-prime knot.

and the upper and lower bounds for the remaining 11- and 12-crossing knots, come from either Burton-Ozlen, using normal surfaces [5], or from Kalfagianni-Lee, using properties of the colored Jones polynomial [11]. Interestingly, every new crosscap number we compute through 12 crossings matches the upper bound previously given on knotinfo.

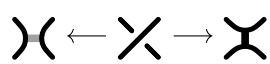
2. Background

2.1. *Splices, smoothings, and states*

Let $D \subset S^2$ be an n-crossing diagram of a knot $K \subset S^3$. Let c be a crossing of D, and let νc be a disk about c in S^2 such that $D \cap \nu c$ consists of two arcs which cross only at c. Up to isotopy, there are two ways to get an $(n-1)$-crossing knot diagram by replacing these two arcs within νc with a pair of disjoint arcs. These two replacements are called the *splices* of D at c:

Orient D arbitrarily. Of the two splices of D at a given crossing, one respects the orientation on D and yields a diagram of a two-component link; this splice is said to be of *Seifert type*. The other splice yields a knot diagram and does not respect orientation. If this *non-Seifert-type* splice has the same effect as a Reidemeister-I move (with planar isotopy), it is said to have *type* RI^-; otherwise this splice has *type* u^- (called type S^- in [9]). Note that splice types are independent of which orientation is chosen for D. See Fig. 2.

There are also two *smoothings* of D at any crossing c: these are the same as the splices of D at c, except with an extra A- or B-labeled arc in νc glued to the resulting diagram:



There are 2^n ways to smooth all the crossings in D, each of which results in a diagram x called a *state*. A state thus consists of a disjoint union of simple closed curves joined by A- and B- labeled arcs, one arc from each crossing in D. The arcs and circles in x are called *state arcs* and *state circles*, respectively.

2.2. *State surfaces*

Given a state x of a knot diagram, D, construct a *state surface* F_x from x as follows. (See Fig. 3.) First, as a preliminary step, perturb D near each crossing point

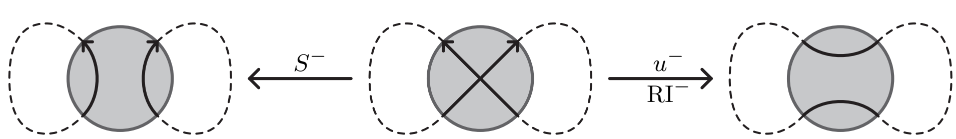


Fig. 2. Seifert (S^-) and non-Seifert (u^- and RI^-) type splices

to obtain an embedding of K in a thin neighborhood of S^2, such that projection $\pi : \nu S^2 \to S^2$ sends K to D. Note that the fiber over each crossing point c contains a properly embedded arc in the knot complement; call this arc the *vertical arc* associated to c.

Next, cap the state circles of x with disjoint disks on the same side of S^2. Then, near each state arc in x, glue on a half-twisted band (called a *crossing band*) which contains the associated vertical arc, such that the resulting surface F_x spans K, $\partial F_x = K$.

Given a state surface F_x from a reduced[d] knot diagram, partition the vertical arcs in F_x as $\mathcal{A}_x = \mathcal{A}_{x,S} \sqcup \mathcal{A}_{x,u}$, so $\mathcal{A}_{x,S}$ contains those of Seifert-type and $\mathcal{A}_{x,u}$ those of u^- type.

Observation 2.1. Given a state surface F_x from a reduced knot diagram, the following are equivalent:

(1) The state surface F_x is 2-sided.
(2) The state x has only Seifert-type smoothings, i.e. $\mathcal{A}_{x,u} = \varnothing$.
(3) The boundary of each disk of $S^2\backslash\backslash x$ contains an even number of state arcs.[e]

Regarding the last condition, note that the boundaries of the components of $S^2\backslash\backslash x$ give a generating set for $H_1(F_x)$, and each generator corresponds to an annulus or a mobius band in F_x according to whether it contains an even number of state arcs (see Fig. 3).

If F is a spanning surface for K, then one can increase the complexity of F by attaching a (positive or negative) crosscap or a handle. The inverses of these local moves, called compression and ∂-compression, are shown in Fig. 4. Note that attaching a $\pm$ crosscap increases $\beta_1(F)$ by 1 and changes slope(F) by ± 2, while attaching a handle increases $\beta_1(F)$ by 2 and does not change slope(F).[f]

[d] A knot diagram D is *reduced* if every crossing is incident to four distinct disks of $S^2\backslash\backslash D$.
[e] Notation: Whenever $Y \subset X$, $X\backslash\backslash Y$ denotes "X-cut-along-Y." This is the metric closure of $X \setminus Y$, which is homeomorphic to $X \setminus \nu Y$, where νY is a regular open neighborhood of Y in X.
[f] When F spans a knot K, slope(F) denotes the boundary slope of F, which is the linking number of K with a co-oriented pushoff of K in F.

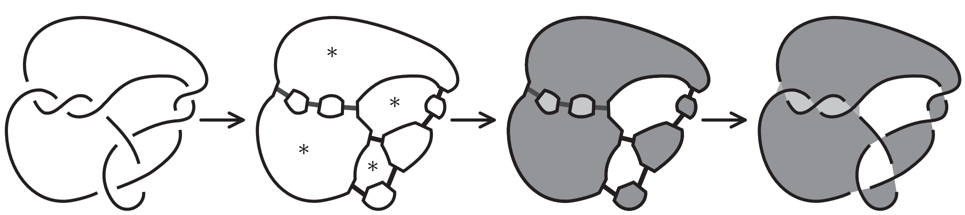


Fig. 3. Constructing a state surface F_x (right) from a state x (middle-left) of a knot diagram (left). Note regarding Observation 2.1 that each starred disk of $S^2\backslash\backslash x$ contains an odd number of state arcs and corresponds to a mobius band in F_x.

There are two traditional notions of essentiality for spanning surfaces; we will work with the weaker, "geometric" notion, defined as follows. If F admits (resp. does not admit) a compression move, then F is called *(in)compressible*. If F admits (resp. does not admit) a ∂-compression move, then F is called *geometrically ∂-(in)compressible*. If F is (resp. is not) incompressible and ∂-incompressible, then F is called *(in)essential*.[g]

Proposition 2.2. *Let F_x be a 1-sided state surface from a reduced alternating diagram D of a prime knot K, with $\beta_1(F_x) = cc(K)$. Then the following are equivalent:*

(1) The state surface F_x is essential.
(2) The state x is adequate (i.e. each state arc joins distinct state circles).
(3) The state x has more than one non-Seifert smoothing.

Proof. Any state surface F_x from an alternating diagram is a plumbing of checkerboard surfaces and is essential if and only if each checkerboard plumband is essential [6, 7, 16]. Moreover, since F_x comes from an alternating diagram, the checkerboard plumbands do as well, and so the checkerboard plumbands are all essential if and only if their underlying states are adequate; this is the case if and only if x is adequate. Thus (1) and (2) are equivalent.

If x is non-adequate, then it differs from the Seifert state at exactly one crossing, since $\beta_1(F_x) = cc(K)$, so there is exactly one non-Seifert smoothing. Conversely, if x has at most one non-Seifert smoothing, then x has exactly one non-Seifert smoothing, since F_x is 1-sided. Hence, x differs from the Seifert state at exactly one crossing, so x is non-adequate. Thus (2) and (3) are equivalent. □

The main theorem in [4] states that, when a knot K has an alternating diagram

[g]A standard application of the loop theorem implies that, with the exception of either mobius band spanning the unknot, if inclusion $\text{int}(F) \hookrightarrow S^3 \setminus K$ induces an injective map on fundamental groups, then F is essential. That is, if F is "algebraically essential," or "π_1-injective," then F is (geometrically) essential. The converse is true when F is 2-sided, but false in general.

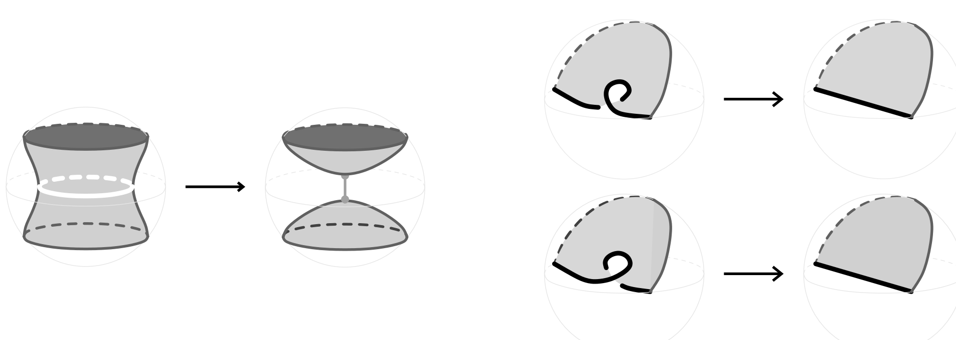


Fig. 4. Compressing and ∂-compressing a spanning surface

D, the state surfaces from D, stabilized with crosscaps and handles, classify the spanning surfaces of K up to homeomorphism type and boundary slope:

Theorem 2.3 (Adams-Kindred [4]). *Let D be an alternating diagram of a knot K, and let F be a spanning surface for K. Then, by choosing an appropriate state surface from D and attaching a (possibly empty) collection of crosscaps or handles, one can construct a spanning surface F' for K with the same number of sides (1 or 2) as F and with $\beta_1(F') = \beta_1(F)$ and slope(F') = slope(F).*[h]

In particular:

Corollary 2.4 (Adams-Kindred [4]). *If D is an alternating diagram of a nontrivial knot K, then $cc(K)$ is realized by a state surface from D. That is, D has a state x whose state surface F_x is 1-sided with $\beta_1(F_x) = cc(K)$.*[i]

Define the following invariant of any knot K:

$$\beta_1(K) := \min_{\text{surfaces } F \text{ spanning } K} \beta_1(F). \tag{2.1}$$

Note that $\beta_1(K) = \min\{cc(K), 2g(K)\}$, where $g(K)$ is the genus of K. Note also that $\beta_1(K) < cc(K)$ if and only if $\beta_1(K) = 2g(K) = cc(K) - 1$, i.e. iff all of the surfaces realizing $\beta_1(K)$ are 2-sided. Moreover, $\beta_1(K_1 \# K_2) = \beta_1(K_1) + \beta_1(K_2)$, by a standard argument. Therefore:

Proposition 2.5. *[Murakami-Yasuhara [15]] Any knots K_1, K_2 satisfy $cc(K_1 \# K_2) \leq cc(K_1) + cc(K_2)$. Equality holds if and only if $cc(K_i) = \beta_1(K_i)$ for $i = 1, 2$.*

Corollary 2.6. *A knot $K = \#_{i \in I} K_i$ satisfies $cc(K) = \sum_{i \in I} cc(K_i)$ if and only if K is prime or:*

$$\beta_1(K_i) = cc(K_i) \text{ for each } i \in I. \tag{†}$$

If D is an n-crossing knot diagram, and x is a state of D with ℓ state circles, then its state surface satisfies

$$\beta_1(F_x) = 1 - \chi(F_x) = 1 - (\ell - n) = n + 1 - \ell.$$

Thus, in order to compute $cc(K)$ when K is alternating, it suffices to find a non-Seifert state x of D with a maximal number of state circles. Although there are $2^n - 1$ possible states to choose from, [4] describes an algorithm that shortens the list of potentially optimal states to at most $2^{\lfloor n/3 \rfloor}$. A tricky question then arises as to how one might record and enumerate the states. Moreover, using this algorithm to compute the crosscap numbers of *all* alternating knots through a given number

[h]Theorem 2.3 extends to alternating links, by replacing "boundary slope" with "net" or "aggregate" slope, which is the sum of the boundary slopes of F along all the link components.
[i]Corollary 2.4 also holds for alternating links.

of crossings would unfortunately require a separate computation for each distinct alternating knot.

Ito-Takimura's splice-unknotting number $u^-(D)$, together with Theorem 1.1, will enable an alternate method (involving splice-unknotting sequences rather than states) for calculating crosscap numbers. In particular, this method will be well-suited to *tabulating* crosscap numbers of *all* alternating knots, by using crosscap numbers of lower-crossing knots when calculating those of with higher crossing numbers. In fact, by starting with sufficient data relating alternating knots with their DT codes and Gauss codes, it is possible to tabulate these crosscap numbers in such a way that the computation cost for each knot grows in polynomial time with respect to crossing number. Details follow in §6.

2.3. *Ito-Takimura's splice-unknotting number*

Let $D \subset S^2$ be an n-crossing diagram of a knot $K \subset S^3$. Ito-Takimura define the *splice-unknotting number* $u^-(D)$ as follows. Starting with D, there are $n!$ distinct sequences of non-Seifert splices, $D = D_n \to D_{n-1} \to \cdots \to D_1 \to D_0 = \bigcirc$, all of which terminate with the trivial diagram of the unknot. Each splice in each sequence is of either RI$^-$ type or u^- type. Ito-Takimura define $u^-(D)$ to be the minimum number of u^- splices among these *splice-unknotting sequences*.[j] They prove:

Theorem 2.7 (Ito-Takimura). *If D is a diagram of a nontrivial knot K, then*

$$cc(K) \leq u^-(D).$$

The point is this: if $D = D_n \to D_{n-1} \to \cdots \to D \to D_0 = \bigcirc$ is a splice-unknotting sequence that realizes $u^-(D)$, then one can construct a 1-sided state

[j]Since the over-under information at each crossing is immaterial in this definition, the splice-unknotting number $u^-(D)$ is most naturally defined on knot *projections*, rather than on knot diagrams, and indeed this is how Ito-Takimura defined it.

Fig. 5. Ito-Takimura's construction performs an isotopy for each RI$^-$-splice.

Fig. 6. Ito-Takimura's construction attaches a crossing band for each u^- splice.

surface F_n for D with $\beta_1(F_n) = u^-(D)$ as follows. For each D_i, let K_i be the underlying knot. Let F_0 be a disk spanning the unknot K_0. For each splice $D_i \to D_{i-1}$, construct F_i from F_{i-1} by:

- performing a local isotopy move, as in Fig. 5, if the splice has type RI$^-$; or
- gluing a crossing band to F_{i-1}, as in Fig. 6, if the splice has type u^-.

This sequence must include at least one gluing move, or else F_n would be a disk. Moreover, the first gluing move $F_{k-1} \to F_k$ produces a mobius band. Thus, all surfaces F_i with $i \geq k$ are 1-sided. Hence, the sequence $F_0 \to \cdots \to F_n$ terminates with a 1-sided surface F_n that spans K and has $\beta_1(F_n) = u^-(D)$. Therefore, $cc(K) \leq \beta_1(F_n) = u^-(D)$.

Define the *splice-unknotting number* of any knot $K \subset S^3$ to be:

$$u^-(K) = \min_{\text{diagrams } D \text{ of } K} u^-(D).$$

Observe that this is a knot invariant. Also note:

Corollary 2.8. *For any nontrivial knot K, $cc(K) \leq u^-(K)$.*

Proof. Theorem 2.7 gives:

$$cc(K) \leq \min_{\text{diagrams } D \text{ of } K} u^-(D) = u^-(K)$$

□

Ito-Takimura prove that $u^-(D)$ is additive under diagrammatic connect sum, although crosscap number is not additive under connect sum (see Proposition 2.5). With this in mind, Ito-Takimura ask:

Question 2.1 (Ito-Takimura). Does there exist an alternating diagram D of a prime knot K such that $u^-(D) > cc(K)$?

Theorem 1.1 will answer this question in the negative.

3. Boundary connect summands and tangle subsurfaces

Assume throughout §3, that D is an alternating diagram of a nontrivial knot K, and F_x is a 1-sided essential state surface from D. Also, given a u^- type vertical arc $\alpha \subset F_x$, denote $F_x \backslash\backslash \alpha = F_{x_\alpha}$ and $\partial F_{x_\alpha} = K_\alpha$.

Note that $x = x_\alpha \cup \beta$, where $\beta \subset x$ is the state arc that corresponds to the vertical arc $\alpha \subset F_x$, and that cutting F_x at α corresponds to performing a u^- splice on D at the associated crossing. This splice yields the underlying diagram D_α for x_α. Note also that D_α is alternating, but not necessarily prime or reduced.

Given a compact and connected subset $U \subset S^2$ whose boundary is disjoint from all state arcs in x, let x^U denote the union of all state circles and state arcs of x that intersect U, and let F_x^U denote the associated state surface, which is a subset of F_x. With this notation, we define diagrammatic notions of boundary connect sum and tangle decompositions for state surfaces, and characterize a few of their properties.

Although, strictly speaking, we will not need this fact, it is worth noting that these diagrammatic notions are more general than they seem a priori, because D is alternating. The basic point here is that, by work of Menasco [12], any 2- or 4-punctured sphere can be isotoped in the knot complement to intersect S^2 in a single circle; hence, every connect sum or tangle decomposition of the alternating knot K can be realized diagrammatically. When F_x is essential, every boundary connect sum or tangle decomposition of F_x can also be realized diagrammatically. For our purposes, however, it is more straightforward just to define these notions diagrammatically in the first place.

3.1. *Boundary connect summands*

A *boundary connect summand* of F_x is any F_x^U, where:

- each component of ∂U is disjoint from state arcs and intersects x transversally in two points,
- F_x^U is connected but not simply connected,
- for any simple closed curve $\gamma \subset U$ which is disjoint from state arcs and intersects x transversally in exactly two points, all of the non-nugatory state arcs in U lie on the same side of γ.[k]

Note that the last two conditions in the definition imply that any boundary connect summand F_x^U is *prime*, meaning that if $F_x^{U'}$ is a boundary connect summand of F_x^U, then F_x^U and $F_x^{U'}$ are isotopic in F_x.

Observation 3.1. Suppose that F_x is prime, but that, for some u^- type vertical arc α, F_{x_α} is not prime. Then every boundary connect summand of F_{x_α} has the form $F_{x_\alpha}^U$, where U is a disk or an annulus, and each component of ∂U intersects the state arc $\beta = x \setminus x_\alpha$. Moreover, when D is oriented, both points of $D \cap \partial U$ where D points out of U lie on the same state circle, and the orientation of one of the two strands of $D \cap U$ is reversed in $D_\alpha \cap U$.

See Fig. 7. In particular:

Observation 3.2. Suppose a u^- type splice at a crossing c in D produces a diagram D' of a non-prime knot K'. Then there is a simple closed curve $\gamma \subset S^2$ which intersects D transversally at c and two other points, both on edges of D not incident to c. Moreover, both disks of $S^2 \setminus \gamma$ contain non-nugatory crossings in D'.

3.2. *Tangle subsurfaces*

A *tangle subsurface* of F_x is any F_x^U, where:

- $U \subset S^2$ is compact and connected,

[k]A state arc β in x is *nugatory* if $x \setminus \text{int}(\beta)$ is disconnected.

- ∂U intersects x transversally in four points and is disjoint from all state arcs in x,
- F_x^U is connected but not simply connected.

Then F_x^U is the tangle subsurface of F_x determined by U. Note that $D \cap U$ is a (diagrammatic) *tangle* in the traditional sense.

Proposition 3.3. *Suppose that F_x^U is a 2-sided tangle subsurface of F_x which contains a u^- type vertical arc α. If $F_{x_\alpha}^U$ is connected, then F_{x_α} is 1-sided.*

Proof. Because $\alpha \subset U$, we have:

$$F_{x_\alpha} = \left(F_{x_\alpha}^{S^2\backslash\backslash U}\right) \cup \left(F_{x_\alpha}^U\right) = \left(F_x^{S^2\backslash\backslash U}\right) \cup \left(F_x^U\backslash\backslash\alpha\right).$$

Thus, if $F_x^{S^2\backslash\backslash U}$ is 1-sided, the result follows immediately. Otherwise, there exist properly embedded arcs $\rho_0 \subset F_x^{S^2\backslash\backslash U}$ and $\rho_1 \subset F_x^U$ with the same endpoints such that $\rho_0 \cup \rho_1$ is the core of a mobius band in F_x. Since $F_x^U\backslash\backslash\alpha$ is connected, there is a properly embedded arc $\rho_2 \subset F_x^U\backslash\backslash\alpha$ such that $\rho_1 \cap \rho_2 = \partial\rho_1 = \partial\rho_2$. The fact that F_x^U is 2-sided implies that $\rho_1 \cup \rho_2$ is the core of an annulus in F_x. Therefore, $\rho_0 \cup \rho_2$ is the core of a mobius band in $F_x\backslash\backslash\alpha$. □

Say that a tangle subsurface F_x^U is *minimal* if, for any tangle subsurface $F_x^{U'}$ with $U' \subset U$, every state arc in U' is also in U. Note that every tangle subsurface F_x^U contains a minimal one.

Observation 3.4. If F_x is prime and $\alpha \subset F_x$ is a u^- type vertical arc such that F_{x_α} is essential and non-prime, then each boundary connect summand $F_{x_\alpha}^U$ of F_{x_α} corresponds to a minimal tangle subsurface F_x^U of F_x.

(This extends Observation 3.1; see Fig. 7.)

Observation 3.5. If F_x^U is a minimal tangle subsurface of F_x, then:

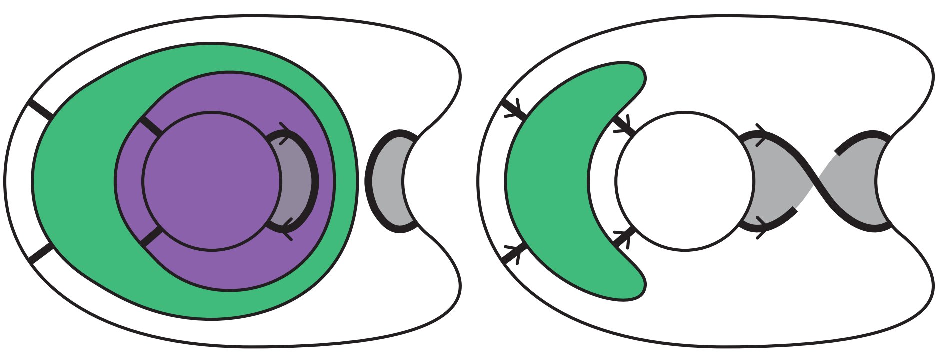


Fig. 7. If F_{x_α} is an essential boundary connect sum, then each of its summands appears as left (purple or green). Hence, F_x has an associated minimal tangle subsurface, shown right (green).

- no vertical arc $\alpha \subset F_x^U$ is parallel through F_x^U to ∂F_x^U, and
- for any properly embedded arc $\delta \subset U$ which intersects x transversally in two points, both on the same state circle of x, all of the non-nugatory state arcs of x in U lie on the same side of δ.

3.3. *Properties of 2-sided tangle subsurfaces*

Lemma 3.1. *Suppose that F_x^U is a prime 2-sided tangle subsurface of F_x; that when D is oriented, both points of $D \cap \partial U$ where D points out of U lie on the same state circle; and that, for some u^- type vertical arc $\alpha \subset F_x$, the orientation on one of the two strands of $D \cap U$ is reversed in $D_\alpha \cap U$. Then F_x^U contains a u^- type vertical arc.*

Figure 8 illustrates the situation.

Proof. The fact that both points of $D \cap \partial U$ where D points out of U lie on the same state circle implies that the underlying diagrams for both x^U and x_α^U represent *knots*, and that x_α^U is the Seifert state for its diagram. Thus, any crossing between the two strands of $D \cap U$ must have a u^- type smoothing in x^U. Moreover, these two strands must cross, since F_x^U is prime, in particular connected but not simply connected. Therefore, F_x^U must contain a u^- type vertical arc. □

In particular, using Observations 3.1 and 3.4 together with Lemma 3.1:

Corollary 3.6. *Suppose that F_x is prime and $F_{x_\alpha}^U$ is a 2-sided boundary connect summand of F_{x_α}. If necessary, adjust U so that it does not intersect the state arc $\beta = x \setminus x_\alpha$ or any other state arcs that join the same two state circles that β does. Then F_x^U is a 2-sided minimal tangle subsurface in F_x which contains a u^- type vertical arc.*

Lemma 3.2. *Suppose that F_x contains a 2-sided minimal tangle subsurface F_x^U which contains a u^- type vertical arc α. Then F_{x_α} is 1-sided, and K_α is prime.*

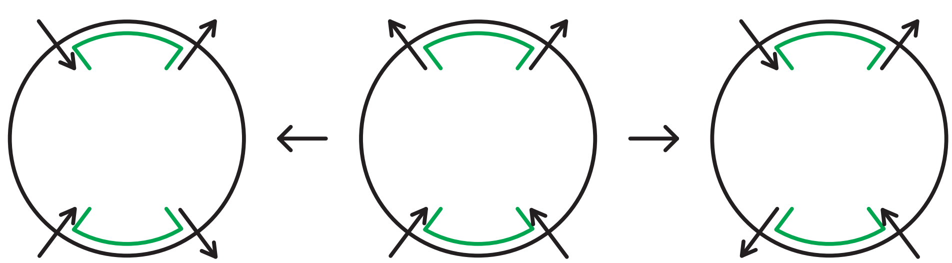


Fig. 8. The situation in Lemma 3.1: F_x^U (center), the two possibilities for $F_{x_\alpha}^U$ (left, right).

Proof. If $F_x^U \backslash\backslash \alpha$ is connected, then F_{x_α} is 1-sided, by Proposition 3.3. Assume instead that $F_x^U \backslash\backslash \alpha$ is not connected. Then $x_\alpha \cap U$ is not connected, so there is a properly embedded arc $\delta \subset U$ which separates the two components of $x_\alpha \cap U$. The fact that $x \cap U$ is connected implies that $|\delta \cap \beta| = 1$, where β is the state arc corresponding to α. The first part of Observation 3.5 implies that α is not parallel through F_x to ∂F_x^U. Hence, neither component of $F_x^U \setminus \alpha$ is simply connected. Thus, each component of $x_\alpha \cap U$ contains a non-nugatory state arc. This contradicts the second part of Observation 3.5. In all cases, therefore, F_{x_α} is 1-sided.

Assume for contradiction that K_α is not prime. Then there is a simple closed curve $\gamma \subset S^2$ which intersects D_α transversally in two points, neither of them crossings, such that both components of $D_\alpha \setminus \gamma$ contain non-nugatory crossings of D_α. The assumption that K is prime implies that γ must intersect β. Hence, there is a properly embedded arc $\delta \subset U$ which intersects x in a single point, which lies on β. Again, the first part of Observation 3.5 provides non-nugatory state arcs in both components of $x_\alpha \cap U$, contradicting the second part of Observation 3.5. Therefore, K_α is prime. □

4. Technical lemmas

Throughout §4, D will be a reduced alternating diagram of a prime knot K, and F_x will be a 1-sided state surface from D with $\beta_1(F_x) = cc(K)$.[l] Further, partitioning the vertical arcs in F_x as $\mathcal{A}_{x,S} \cup \mathcal{A}_{x,u}$ as in Observation 2.1, $\alpha \in \mathcal{A}_{x,u}$ will be a u^- type vertical arc in F_x.[m] As in §3, denote $F_x \backslash\backslash \alpha = F_{x_\alpha}$ and $\partial F_{x_\alpha} = K_\alpha$, with D_α the underlying diagram for x_α.

4.1. *Overview of cases*

The key step in Ito-Takimura's proof that $cc(K) \leq u^-(D)$ involves building up more complex state surfaces from simpler ones, often by gluing on crossing bands in a way that corresponds to undoing a u^- type splice. The key step in proving the reverse inequality is basically the opposite. Namely, the key is to show that there exist F_x and α such that F_{x_α} is 1-sided with $\beta_1(F_{x_\alpha}) = cc(K_\alpha)$, such that K_α either is prime or satisfies the condition (†) from Corollary 2.6.

This situation varies mainly according to whether or not $\beta_1(K) = cc(K)$. Subsection 4.2 addresses the case $\beta_1(K) < cc(K)$. For each of the states x which differs from the Seifert state y at a single crossing, F_x has a single u^- type vertical arc. Also $\beta_1(F_x) = cc(K) = \beta_1(K) + 1$. Lemma 4.1 establishes that, for at least one of these states x, F_{x_α} is 1-sided with $\beta_1(F_{x_\alpha}) = cc(K_\alpha)$, and K_α is prime.

Subsection 4.3 addresses the case $\beta_1(K) = cc(K)$. Given a 1-sided F_x from D with $\beta_1(F_x) = cc(K)$, Lemma 3.2 states that, if F_x has a 2-sided minimal tangle

[l] Such F_x exists by Theorem 2.3; sometimes this surface will be arbitrary, subject to these conditions; other times, we will choose a particular surface F_x of this type.

[m] Such α exists by Observation 2.1; as with F_x, we will sometimes take α to be arbitrary, and other times will we choose α.

subsurface which contains an arc $\alpha \in \mathcal{A}_{x,u}$, then F_{x_α} is 1-sided with $\beta_1(F_{x_\alpha}) = cc(K_\alpha)$, and K_α is prime. Otherwise, every 2-sided minimal tangle subsurface in F_x contains only Seifert-type vertical arcs. (This includes the case of the knot 9_{10}.) After some setup, this case follows easily from Corollary 3.6, using the condition (†) for K_α and an associated condition $(*)$ for F_{x_α}.

4.2. *Alternating knots with $\beta_1(K) < cc(K)$*

In addition to the assumptions stated at the beginning of §4, assume throughout §4.2 that $\beta_1(K) < cc(K)$, and that y is the Seifert state of D. Then the associated Seifert surface satisfies $\beta_1(F_y) = \beta_1(K) = cc(K) - 1 > 0$.

Proposition 4.1. *No two state arcs in y join the same two state circles.*

Proof. If two state arcs in y join the same two state circles, then reversing these two smoothings will produce a state $z \neq y$ with the same number of state circles as y. (See Fig. 9.) But then the state surface F_z will be 1-sided with $\beta_1(F_z) = \beta_1(F_y) = \beta_1(K) < cc(K)$.[n] □

[n] A similar argument proves more generally that if *any* knot K satisfies $cc(K) > \beta_1(K)$, then any minimal genus Seifert surface for K must have no Hopf band plumbands.

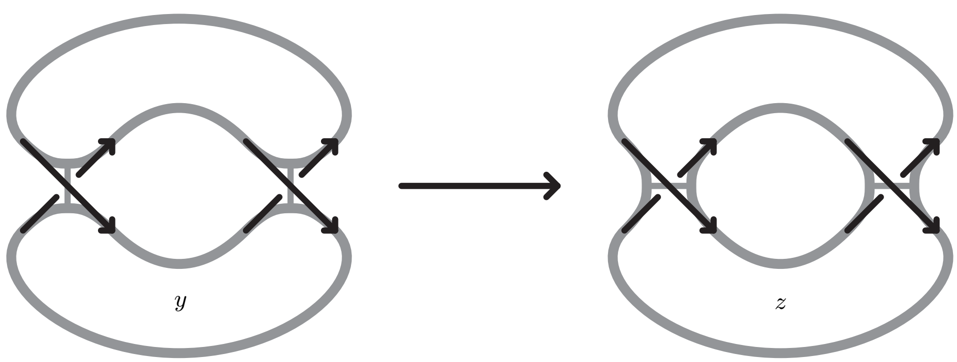


Fig. 9. Proposition 4.1 states that if a Seifert surface F_y for an alternating knot K satisfies $\beta_1(F_y) < cc(K)$, then no two state arcs in y join the same two state circles.

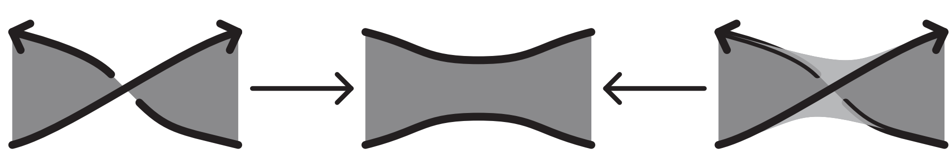


Fig. 10. If F_x differs from the Seifert surface F_y at a single crossing c, then cutting F_x at c gives the same surface as untwisting F_y at c.

Reversing any one smoothing of y produces a non-adequate state x whose associated state surface satisfies $\beta_1(F_x) = \beta_1(F_y) + 1 = cc(K)$. There is only one u^- type smoothing in x. Cutting F_x at the associated vertical arc yields the same surface as "untwisting" the associated crossing band in F_y. See Fig. 10.

Proposition 4.2. *Untwisting F_y at any crossing band gives a 1-sided state surface F_w from a reduced alternating knot diagram D'.*

Proof. To see that F_w is 1-sided, use the fact that D is reduced to obtain a simple closed curve $\gamma \subset F_y$ that passes exactly once through the given crossing band. This γ is the core of an annulus in F_y, and thus of a mobius band in F_w.

To see that D' is reduced, suppose otherwise. Then some state circle v in w either is incident to only one state arc or is incident to itself at a state arc, β_1. The former is impossible, since untwisting a crossing band merges two state circles, and all state circles in y are incident to at least two crossings. In the latter case, v must be the result of merging two state circles u_1, u_2 from y at the state arc β_2 that corresponds to the untwisted crossing band. Because no state circle in y is incident to itself at a state arc, it follows that both β_1 and β_2 join u_1 and u_2. This contradicts Proposition 4.1. □

Proposition 4.3. *Untwisting F_y at some crossing band yields a 1-sided state surface F_w from a prime reduced alternating knot diagram.*

Proof. Proposition 4.2 implies that, for each crossing c_i of D, untwisting F_y at the crossing band near c_i yields a 1-sided state surface from a reduced alternating knot diagram D_i. Assume for contradiction that *each* of these diagrams D_i is non-prime. Then Observation 3.2 implies that for every crossing c_i in D there is a simple closed curve $\gamma_i \subset S^2$ which intersects D transversally at c and two other points, both of which lie on edges of D which are not incident to c, such that $|\gamma_i \cap D'| = 2$ and both disks of $S^2 \setminus \gamma_i$ contain crossing points of D_i. See Fig. 11, left.

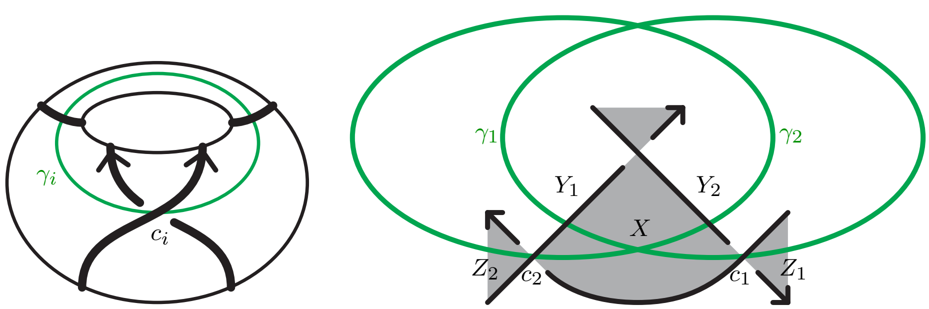


Fig. 11. If K is alternating and prime with $\beta_1(K) < cc(K)$, then there is a non-Seifert-type splice which yields a prime knot.

This, together with Proposition 4.1 and the fact that D is prime and reduced, implies that every disk of $S^2\backslash\backslash D$ is incident to *at least* three crossings. Yet, an euler characteristic argument shows that some disk of $S^2\backslash\backslash D$ is incident to *at most* three crossings. Hence, there is a disk X of $S^2\backslash\backslash D$ which is incident to *exactly* three crossings. Up to symmetry, there are two possible configurations around such a disk X in an arbitrary Seifert state; Proposition 4.1 rules out one of them. The only other possibility is that ∂X is a Seifert circle of y, as in Fig. 11, right.

Let c_1, c_2 be two crossings on ∂X, and consider the arcs γ_1, γ_2 passing through them. Each γ_i passes through exactly three disks of $S^2\backslash\backslash D$, namely X and two others, Y_i and Z_i, where Z_i is incident to c_i. Since γ_1 and γ_2 intersect in a second point, outside of X, we must either have $Y_1 = Y_2$ or $Z_1 = Z_2$. The first possibility contradicts the assumptions that K is prime and D is reduced; the second contradicts Proposition 4.1. □

Therefore, with the assumptions and notation from the beginning of §4 and §4.2:

Lemma 4.1. *There exist F_x and α such that F_{x_α} is 1-sided with $\beta_1(F_{x_\alpha}) = \beta_1(K_\alpha) = cc(K_\alpha)$, and D_α is a reduced alternating diagram of the prime knot K_α.*

Proof. Use Proposition 4.3 to obtain a state x of D which differs from the Seifert state y of D at exactly one crossing, such that untwisting F_y at the associated crossing band yields a 1-sided state surface F_w from a prime reduced alternating knot diagram D_α. Then F_x contains only one u^- type vertical arc α, namely the one at the crossing where x differs from y, and $F_{x_\alpha} = F_w$. Hence, F_{x_α} is a 1-sided state surface from a prime reduced alternating knot diagram.

To see that $\beta_1(F_{x_\alpha}) = \beta_1(K_\alpha) = cc(K_\alpha)$, use Theorem 2.3 to obtain a state surface S' from D_α with $\beta_1(S') = \beta_1(K_\alpha)$. Attaching a crossing band to S' near α gives a state surface S for K with $\beta_1(S) = \beta_1(S') + 1$. If it were the case that $\beta_1(S') < \beta_1(F_{x_\alpha})$, then we would have the contradiction

$$\beta_1(K) = \beta_1(F_y) = \beta_1(F_{x_\alpha}) > \beta_1(S') = \beta_1(S) + 1.$$

The fact that F_{x_α} is 1-sided now gives $\beta_1(F_{x_\alpha}) = \beta_1(K_\alpha) = cc(K_\alpha)$. □

4.3. *Alternating knots with $\beta_1(K) = cc(K)$*

In addition to the assumptions stated at the beginning of §4, assume throughout §4.3 that $\beta_1(K) = cc(K)$.

Proposition 4.4. *For any $\alpha \in \mathcal{A}_{x,u}$, F_{x_α} is 1-sided and essential with*

$$\beta_1(F_{x_\alpha}) = \beta_1(K_\alpha) = cc(K_\alpha).$$

Proof. Assume for contradiction that some F_{x_α} is 2-sided. Then x_α is the Seifert state of D_α and, by Observation 2.1, the boundary of each component of $S^2\backslash\backslash x_\alpha$

contains an even number of state arcs from x_α. Therefore, the components of $S^2\backslash\backslash x$ incident to α were the *only two* that contained an odd number of state arcs. Since α was arbitrary in $\mathcal{A}_{x,u}$, all state arcs in $\mathcal{A}_{x,u}$ must be incident to the same two components of $S^2\backslash\backslash x$.

Hence, D consists of n crossings whose smoothing in x is non-Seifert-type, together with n diagrammatic tangles, each of which contains only crossings whose smoothing in x is Seifert-type. (Figure 12, left, shows the case $n = 3$.) Some of these tangles may be trivial, containing no crossings, but at least one of the tangles must contain crossings, since $\beta_1(F_x) > 1$. This situation is impossible, by Lemma 3.1. Thus, F_{x_α} is 1-sided.

Use Theorem 2.3 to obtain a state surface S' from D_α with $\beta_1(S') = \beta_1(K_\alpha)$. Attaching a crossing band to S' near α gives a state surface S for K with $\beta_1(S) = \beta_1(S') + 1$. If it were the case that $\beta_1(S') < \beta_1(F_{x_\alpha})$, then we would have the contradiction

$$\beta_1(K) = \beta_1(F_x) = \beta_1(F_{x_\alpha}) + 1 > \beta_1(S') + 1 = \beta_1(S).$$

The fact that F_{x_α} is 1-sided now implies that $\beta_1(F_{x_\alpha}) = \beta_1(K_\alpha) = cc(K_\alpha)$, and hence that F_{x_α} is essential. □

With the setup from the start of §4, suppose that $F_{x_\alpha} = \natural_{i\in I} F_i$ is a boundary connect sum decomposition of F_{x_α} associated to the connect sum decomposition $K_\alpha = \#_{i\in I} K_i$. Say that F_{x_α} satisfies $(*)$ if

$$F_i \text{ is 1-sided with } \beta_1(F_i) = \beta_1(K_i) \text{ for each } i \in I. \qquad (*)$$

Observation 4.5. Any F_{x_α} satisfying $(*)$ is 1-sided with $\beta_1(F_{x_\alpha}) = \beta_1(K_\alpha) = cc(K_\alpha)$.

Moreover, each F_i is essential, as is F_{x_α}. This further implies that the boundary connect sum decomposition of F_{x_α} is unique. Note additionally that, if F_{x_α} satisfies $(*)$, then K satisfies the property (†) defined in Corollary 2.6. Conversely, Theorem 2.3 implies:

Observation 4.6. Any alternating knot obeying (†) has a state surface obeying $(*)$.

Here is the main result of this subsection.

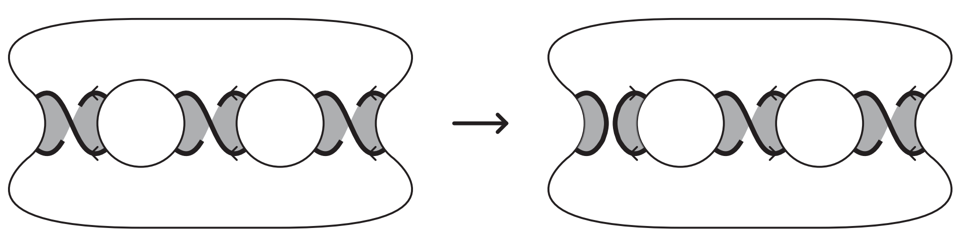


Fig. 12. The situation in the proof of Proposition 4.4.

Lemma 4.2. *Any 1-sided state surface F_x from D with $\beta_1(F_x) = \beta_1(K)$ contains a u^- type vertical arc α such that F_{x_α} satisfies $(*)$.*

Proof. Assume first that F_x contains a 2-sided minimal tangle subsurface which contains some $\alpha \in \mathcal{A}_{x,u}$. Then Lemma 3.2 implies that F_{x_α} is 1-sided and K_α is prime. Proposition 4.4 further implies that F_{x_α} is prime with $\beta_1(F_{x_\alpha}) = \beta_1(K_\alpha) = cc(K_\alpha)$. Therefore, F_{x_α} satisfies $(*)$.

Assume instead that every 2-sided minimal tangle subsurface of F_x contains *only* Seifert-type vertical arcs. Choose any $\alpha \in \mathcal{A}_{x,u}$. If F_{x_α} satisfies $(*)$, then we are done. Otherwise, some boundary connect summand of F_{x_α} is 2-sided. But then Corollary 3.6 implies that the corresponding minimal tangle subsurface in F_x is 2-sided and contains a u^- type vertical arc, contrary to assumption. □

5. Main theorem

Throughout §5, D will be a reduced alternating diagram of a nontrivial knot K, and F_x will be a 1-sided state surface from D with $\beta_1(F_x) = cc(K)$. (We no longer assume K is prime.) As in §4, denote $\mathcal{A}_x = \mathcal{A}_{x,S} \cup \mathcal{A}_{x,u}$, and given $\alpha \in \mathcal{A}_{x,u}$, denote $F_x \backslash\backslash \alpha = F_{x_\alpha}$ and $\partial F_{x_\alpha} = K_\alpha$. Now also let $F_x = \natural_{i\in I} F_i$ and $K = \#_{i\in I} K_i$ be corresponding (boundary) connect sum decompositions. Recall that F_x satisfies $(*)$ if each F_i is 1-sided with $\beta_1(F_i) = \beta_1(K_i)$. Recall also that, if K admits such a state surface, then K satisfies (†): $cc(K_i) = \beta_1(K_i)$ for each $i \in I$. Proposition 4.4 and Lemma 4.2 generalize to this setting as follows:

Observation 5.1. For any $\alpha \in \mathcal{A}_{x,u}$, F_{x_α} is 1-sided and essential with $\beta_1(F_{x_\alpha}) = \beta_1(K_\alpha) = cc(K_\alpha)$.

Observation 5.2. If F_x satisfies $(*)$, then F_{x_α} satisfies $(*)$ for some $\alpha \in \mathcal{A}_{x,u}$.

Before moving to the main theorem, we mention an application of Observation 5.2. Namely, given a reduced alternating diagram D of a prime alternating knot K satisfying $(*)$, every 1-sided state surface F_x from D with $\beta_1(F_x) = \beta_1(K)$ can be obtained from a minimal splice-unknotting sequence for D, using the construction behind Theorem 2.7. Thus, a list of all minimal-length splice-unknotting sequences for D conveys a list of all minimal-complexity 1-sided state surfaces from D. Unfortunately, the list of such sequences grows rather quickly with crossings. The data through 9 crossings is posted at [3].

Theorem 5.3. *Suppose that D is an alternating diagram whose underlying knot K is nontrivial and either is prime or satisfies (†). Then $u^-(D) = u^-(K) = cc(K)$.*

Proof. We argue by induction on $cc(K)$. In all cases, by Theorem 2.3, D has a 1-sided state surface F_x that satisfies $\beta_1(F_x) = cc(K)$. In the base case, F is a mobius band, which, cut at any crossing, becomes a disk; thus $u^-(D) = u^-(K) =$

$1 = cc(K)$.[o]

For the inductive step, let D be an alternating diagram of a knot K with $cc(K) \geq 2$, where K is prime or satisfies (†). Assume that whenever D' is an alternating diagram of a nontrivial knot K' with $cc(K') < cc(K)$, and K' is prime or satisfies (†), then $u^-(D') = u^-(K') = cc(K')$.

Assume first that $\beta_1(K) < cc(K)$. Then K does not obey (†), so by assumption K is prime. In this case, Lemma 4.1 provides a state surface F_x and a vertical arc $\alpha \in \mathcal{A}_{x,u}$ such that F_{x_α} is 1-sided with $\beta_1(F_{x_\alpha}) = cc(K_\alpha)$, and K_α is prime. Hence:

$$\begin{aligned} cc(K) = \beta_1(F_x) = \beta_1(F_{x_\alpha}) + 1 = cc(K_\alpha) + 1 = u^-(D_\alpha) + 1 \\ \geq u^-(D) \\ \geq u^-(K). \end{aligned} \tag{5.1}$$

Corollary 2.8 gives the reverse inequality, $cc(K) \leq u^-(K)$. Thus, $cc(K) = u^-(K)$. Also, $cc(K) \geq u^-(D) \geq u^-(K)$ by (5.1). Therefore, $u^-(D) = u^-(K) = cc(K)$.

Otherwise, $\beta_1(K) = cc(K)$. Then, if K is prime, K satisfies (†); also, by assumption, if K is not prime, then K satisfies (†). Thus, K satisfies (†). Use Observation 4.6 to obtain a state x of D such that F_x satisfies (∗). Then, by Observation 5.2, there exists $\alpha \in \mathcal{A}_{x,u}$ such that F_{x_α} satisfies (∗). Since F_{x_α} satisfies (∗), it follows that K_α satisfies (†). Therefore, by repeating the computation (5.1), with the subsequent application of Corollary 2.8 and squeeze argument, we can conclude in this final case that $u^-(D) = u^-(K) = cc(K)$. □

In particular, we have proven:

Theorem 5.4 (Theorem 1.1). *If D is a prime alternating diagram of a nontrivial knot K, then $u^-(D) = u^-(K) = cc(K)$.*

6. Computation

Using the fact that every prime alternating knot K satisfies $u^-(K) = cc(K)$, we will construct a list D_{cc} of dictionaries $D_{\text{cc}}[n]$, $n = 3, 4, 5, \ldots$, in which to look up prime alternating knots by name and crossing number and find their crosscap numbers. Everything is coded in python. All data is available at [3]. The basic idea for constructing D_{cc} is this.

First, using data imported from [1, 2], we construct a list D_{G} of dictionaries $D_{\text{G}}[n]$ in which to look up a prime alternating knot K by name and crossing number and find a Gauss code $G = D_{\text{G}}[n][K]$ for a reduced alternating diagram D of K.

Next, we write a list D_{splice} of dictionaries $D_{\text{splice}}[n]$ which associates to each n-crossing prime alternating knot K a list of n lists of knot names. For each knot K the dictionary $D_{\text{G}}[n][K]$ provides a Gauss code, which describes a diagram D.

[o]This uses the fact that any alternating diagram of the unknot can be reduced to the trivial diagram by RI moves.

Each of the n lists in $D_{\text{splice}}[n][K]$ describes the connect sum decomposition of the diagram obtained from D by the u^- type splice at one of the crossings of D.

We then define a list D_{u^-} of dictionaries $D_{u^-}[i]$ recursively, first setting $D_{u^-}[0][\text{`}1_0\text{'}] = 0$. Then for each K and n as above, we compute:

$$D_{u^-[n][K]} = 1 + \min_{i=1,\ldots,n} \sum_{j=0}^{\text{len}(D_{\text{splice}}[n][K][i])} D_{u^-}\left[\text{len}(D_{\text{splice}}[n][K][i][j])\right]\left[D_{\text{splice}}[n][K][i][j]\right]$$

Each new dictionary $D_{u^-}[n]$ records the invariant $u^-(K)$ for all prime alternating knots K with n crossings. Finally, using Theorem 1.1, we copy $D_{u^-}[i]$ for all $i \geq 3$ to construct a list D_{cc} of dictionaries $D_{\text{cc}}[i]$ which record the crosscap numbers of all prime alternating knots.

The main technical challenge is that a given alternating knot can have many distinct alternating diagrams, each of which has its own unique reduced Gauss code. Thus, given a Gauss code (say, resulting from a u^- type splice) its reduced form may or may not appear in D_{G}; it may not be obvious which knot the code represents. In order to solve this problem, we construct a list D_{DT} of dictionaries $D_{\text{DT}}[n]$ in which to look up certain DT codes (one for each prime alternating diagram) and find the name of the associated knot.

After some background, we give more details regarding the construction of D_{G}, D_{DT}, D_{splice}, D_{u^-}, and D_{cc}. Of these constructions, the most computationally expensive is that of D_{DT}. These lists of dictionaries are among the data posted at [3].

6.1. *Basics of Gauss and DT codes*

For an arbitrary knot diagram D, one obtains a Gauss code G as follows. First, choose an orientation and a starting point (away from crossings). Then, moving along D accordingly, label the crossings of D as $1, \ldots, n$, where n is the number of crossings in D, according to the order in which they first appear along D. Also, record all crossings of D, in order, as a word of length $2n$ in which each character $-n, \ldots, -1, 1, \ldots, n$ appears exactly once: the entry in the Gauss code corresponding to the overpass (resp. underpass) at the crossing with label i is i (resp. $-i$). Note that D is reduced if and only if any Gauss code from D has no cyclically consecutive entries i, $-i$.

Working exclusively with alternating knots and regarding mirror images as equivalent renders the signs in the Gauss code redundant. Thus, it makes sense to omit these signs, as we will do from now on.

If $G = [c_1, c_2, \ldots, c_{2n}]$ is a Gauss code, then for each $r = 1, \ldots, n$ there exist odd i and even j with $c_i = r = c_j$. Thus, for each $s = 1, \ldots, n$, there is a unique even integer $2 \leq j(s) \leq 2n$ with $c_{j(s)} = c_{2s-1}$. The Dowker-Thistlethwaite code associated to G is $[j(1), j(2), \ldots, j(n)]$. For example, the DT code abbreviating the Gauss code $[1, 2, 3, 1, 2, 3]$ is $[4, 6, 2]$, since $c_1 = 1 = c_4$, $c_3 = 3 = c_6$, and $c_5 = 2 = c_2$. The main advantage of DT codes over Gauss codes is their length; DT codes are useful when writing dictionaries.

Given a Gauss code G of length $2n$, one can determine all the Gauss codes from the same diagram, but with different choices of starting point and/or orientation, by permuting and/or reversing the $2n$ characters in the Gauss code arbitrarily, and then permuting the n crossing labels so that smaller labels always precede larger ones. (That is, *act dihedrally* on G and then *relabel*.) Among the resulting codes, one, say Y, is lexicographically minimal. Call Y the *reduced form* of G. Say that G is *reduced* if its underlying diagram is reduced and if G is its own reduced form.

For any reduced Gauss code G which represents a prime alternating knot diagram, there is, up to isotopy and reflection, a unique knot diagram D whose reduced Gauss code is G. (There may be several choices of basepoint and orientation on D that give G.)

A reduced Gauss code G of a knot K represents a connect sum if and only if $G = w_1w_2w_3$, where w_2 is a nonempty proper subword of G that shares no characters with w_1 nor w_3. After relabeling (so that smaller labels always precede larger ones), w_2 and w_1w_3 give Gauss codes for two, not necessarily prime, connect summands of K. Continuing in this way eventually gives the connect sum decomposition of K.

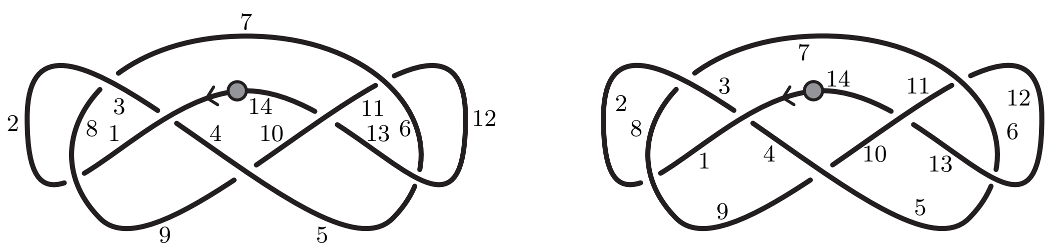


Fig. 13. Face data from the diagram of 7_7 with Gauss code $[1,2,3,1,4,5,6,3,2,4,7,6,5,7]$: Edges around A-faces: $\big[[14,4,10],[8,1,3],[12,5,9,2,7],[13,6,11]\big]$. Edges around B-faces: $\big[[4,1,9],[2,8],[10,5,13],[6,12],[11,7,3,14]\big]$.

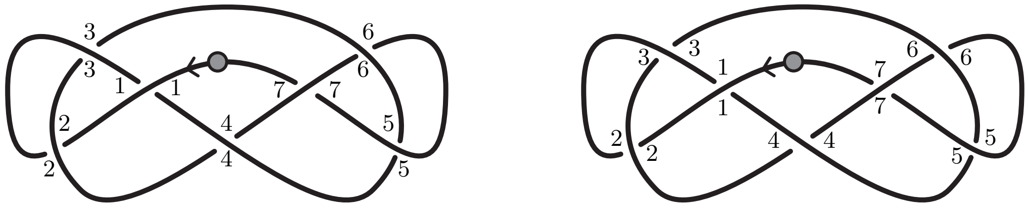


Fig. 14. Face data from the diagram of 7_7 with Gauss code $[1,2,3,1,4,5,6,3,2,4,7,6,5,7]$. Crossings around A-faces: $\big[[1,4,7],[2,1,3],[5,4,2,3,6],[5,6,7]\big]$. Crossings around B-faces: $\big[[1,2,4],[2,3],[4,5,7],[5,6],[6,3,1,7]\big]$.

6.2. *Face data and flypes*

We have imported Gauss codes from [1, 2], one for each prime alternating knot through n crossings. We we have organized this data as a list, D_{G}, of dictionaries, $D_{\mathrm{G}}[n]$, so that one can look up the name (e.g. '7_4') of any n-crossing prime alternating knot K in $D_{\mathrm{G}}[n]$ and find a Gauss code $D_{\mathrm{G}}[n][K]$ for a reduced alternating diagram of K. Then we clean up this data by replacing each Gauss code with its reduced form. Finally, we augment this data by replacing each entry in each dictionary, a Gauss code G, with the list $[G, S]$: here, S lists the signs of the crossings of the diagram associated to G, with the convention that the first crossing is an overpass with a positive sign. Although these signs are encoded by G, they take some time to compute; recording them now ensures that we only need to compute them this once.

We now set about constructing a list D_{DT} of dictionaries $D_{\mathrm{DT}}[n]$ in which to look up certain DT codes (one code for each prime alternating diagram with n crossings) and find the name of the associated knot. The key is to find a list $D_0, \ldots, D_k$ of all reduced alternating diagrams of each prime alternating knot K. To do so, we need to use the flyping theorem, conjectured by Tait [17] and proven by Menasco-Thistlethwaite [13, 14]. Here is how to do this.

Let G_0 be a reduced Gauss code of a prime alternating knot. If G_0 has length $2n$, then the associated projection has n crossings, which are joined by $2n$ *edges* (in the sense that the projection is a 4-valent graph). Also, the projection cuts S^2 into $n + 2$ *black* and *white* disks, or *faces*. The *face data* from G_0 records which edges and crossings are incident to each face, proceeding counterclockwise around the boundary of the face.[p] It is convenient to partition this data into four sets, two for crossings and two for edges, each split between data from the black faces and from the white. Figure 14 shows an example.

This face data allows one to identify possible flype moves on the diagram. To do this, define four sets as follows. The first two sets, EE_B and EE_W, consist of pairs of distinct edges which lie on the boundary of the same (black or white, resp.) face and which do not share any endpoints. The other two sets, ECE_B and ECE_W, consist of triples, each triple consisting of two edges and a crossing, such that neither edge is incident to the crossing and the two edges abut the (two black or two white, resp.) faces incident to the crossing. Associate to each element of EE_B (EE_W, resp.) an arc whose interior lies in a black (white) face of $S^2\backslash\backslash D$ and whose endpoints lie on non-incident edges of D. Likewise, associate to each element of ECE_B (ECE_W, resp.) an arc whose interior intersects D in a single point, a crossing, and otherwise lies entirely in two black (white) faces of $S^2\backslash\backslash D$, and whose endpoints lie on edges of D which are not incident to this crossing. Thus, associated to each element of $EE_B \cap ECE_W$ ($EE_W \cap ECE_B$, resp.) is a simple closed curve which intersects

[p]For edges, the data at [3] also records the orientation of the edge with a sign: + if the edge runs counterclockwise along the boundary of the face, − if it runs clockwise.

one black (white) face of $S^2 \backslash\backslash D$ and two white (black) faces of $S^2 \backslash\backslash D$, and which intersects D transversally in two edges e_1, e_2 and one crossing c, none of them incident. In this way, each element of $EE_B \cap ECE_W$ identifies a possible flype move on D, as does each element of $EE_W \cap ECE_B$.

The flype move changes the Gauss code by removing both c terms, re-inserting them in the intervals of the Gauss code associated to e_1 and e_2, and then relabeling. More precisely, with $G = (c_1, \dots, c_{2n})$, there exist indices $1 \leq i_1, i_2 \leq 2n - 1$ such that e_1 joins c_{i_1} and c_{i_1+1}, while e_2 joins c_{i_2} and c_{i_2+1}. Assume without loss of generality that $i_1 < i_2$. There are also two indices $1 \leq j_1 < j_2 \leq 2n$ such that $c_{j_1} = c = c_{j_2}$. There are two explicit possibilities for the Gauss code resulting from the flype. If $i_1 < j_1 < i_2 < j_2$, then the new Gauss code is

$$(c_1, \dots, c_{i_1}, c, c_{i_1+1}, \dots, \widehat{c_{j_1}}, \dots, c_{i_2}, c, c_{i_2+1}, \dots, \widehat{c_{j_2}}, \dots, c_{2n}),$$

after relabeling. (The hats indicate entries to delete from the Gauss code.) Otherwise, $j_1 < i_1 < j_2 < i_2$, and the new Gauss code is

$$(c_1, \dots, \widehat{c_{j_1}}, \dots, c_{i_1}, c, c_{i_1+1}, \dots, \widehat{c_{j_2}}, \dots, c_{i_2}, c, c_{2_1+1}, \dots, c_{2n}),$$

after relabeling. See Fig. 15. This is how we construct, for each element of $EE_B \cap ECE_W$ and $EE_W \cap ECE_B$, a Gauss code for the diagram produced by the associated flype move on D.

Given a Gauss code G for an alternating diagram D_0 of a prime knot K, we are now ready to compute a list L of DT codes, one from each reduced alternating diagram of K. (Each DT code will correspond to the reduced Gauss code of some diagram of K.) Begin by computing the reduced form G_0 of G, let T_0 be its DT code, and let $L = [T0]$. Then compute $EE_B \cap ECE_W$ and $EE_W \cap ECE_B$ from G_0 to identify possible flype moves on D_0. Compute the reduced form of the Gauss code resulting from each flype move. If L does not already contain the DT code for this reduced Gauss code, then append that DT code. After doing this for each possible flype move on D_0, repeat the process for each of the other diagrams described by the DT codes in L, appending any new DT codes to L. The flyping theorem implies that this process will produce a list L consisting of one DT code for each reduced alternating diagram of K.

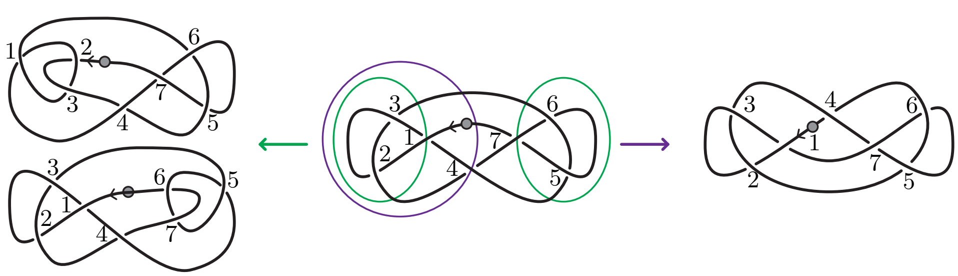


Fig. 15. Four flype moves on the same diagram of 7_7.

Now we can build the dictionary D_{DT}: for each knot type K, say with Gauss code G, we compute the list L as above from G, and then for each T_i in L we update the dictionary D_{DT} with the entry $T_i : K$. For example, for knots with seven crossings, D_{DT} looks like:

DT code	knot	DT code	knot
$[8, 10, 12, 14, 2, 4, 6]$	7_1	$[4, 10, 14, 12, 2, 8, 6]$	7_2
$[6, 10, 12, 14, 2, 4, 8]$	7_3	$[6, 12, 10, 14, 2, 4, 8]$	7_4
$[4, 10, 12, 14, 2, 8, 6]$	7_5	$[4, 10, 14, 12, 2, 6, 8]$	7_5
$[4, 8, 12, 2, 14, 6, 10]$	7_6	$[4, 8, 12, 10, 2, 14, 6]$	7_6
$[4, 8, 10, 12, 2, 14, 6]$	7_7	$[4, 8, 12, 14, 2, 6, 10]$	7_7

The dictionary list D_{DT} through at least 13 crossings is available at [3].

6.3. *Splices from face data*

The next step is to construct a dictionary D_{splice} in which one can look up any prime alternating knot K, say with crossing number n, and find n lists of knot types, where each list describes the connect sum decomposition of the knot which results from splicing a given diagram for K (the one described by its imported Gauss code) at one of its n crossings.

Recall that we have used our imported data to construct a list D_{G} of dictionaries $D_{\text{G}}[n]$ which give us, for every prime alternating knot K with crossing number n, the reduced Gauss code G of some reduced alternating diagram D of K (and a list of the signs of the crossings in D). Given any $i = 1, \dots, n$, let $c = c_i$. We can write $G = w_1 c w_2 c w_3$, where w_2 is nonempty, as is at least one of w_1 or w_3. After relabeling, $w_1 \overline{w_2} w_3$ is a Gauss code for the diagram obtained from D via a u^- type splice at c; $\overline{w_2}$ denotes the reverse of w_2. Let G_i be the reduced form of this Gauss code.

The Gauss codes $G_1, \dots, G_n$ constructed in this way from G are the reduced Gauss codes which describe the knot diagrams which result from each of the possible u^- type splices on D. For each $i = 1, \dots, n$, decompose G_i into its connect summands, as described in §6.1. Then compute the reduced Gauss code of each summand, look up the associated DT code in D_{DT}, and record the knot type. For

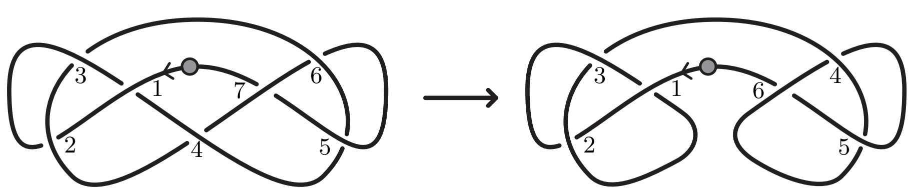


Fig. 16. Given the diagram of the knot 7_7 with Gauss code $(1, 2, 3, 1, 4, 5, 6, 3, 2, 4, 7, 6, 5, 7)$, splicing at crossing 4 gives the diagram of $3_1 \# 3_1$ with Gauss code $(1, 2, 3, 1, 2, 3, 4, 5, 6, 4, 5, 6)$.

example, for knots with seven crossings, D_{splice} looks like:

knot	splice	splice	splice	splice	splice	splice	splice
7_1	0_1	0_1	0_1	0_1	0_1	0_1	0_1
7_2	6_1	5_1	5_1	6_1	6_1	6_1	6_1
7_3	6_1	3_1	3_1	3_1	3_1	6_1	6_1
7_4	6_2	$3_1, 3_1$	6_2	6_2	6_2	6_2	6_2
7_5	6_2	5_2	5_2	4_1	4_1	4_1	6_2
7_6	6_1	5_2	5_2	6_2	6_2	6_3	6_3
7_7	6_2	6_3	6_3	$3_1, 3_1$	6_3	6_3	6_2

6.4. *Crosscap numbers from splice data*

Finally, we are ready to construct a list D_{u^-} of dictionaries $D_{u^-}[n]$, each listing $u^-(K)$ for the unknot and all prime alternating knots K with n crossings. Because all prime alternating knots K satisfy $u^-(K) = cc(K)$ by Theorem 1.1, we can then copy these dictionaries to obtain the list D_{cc} of dictionaries $D_{\text{cc}}[n]$ recording the crosscap numbers of all prime alternating knots with n crossings, for $n \geq 3$.

First, let $D_{u^-}[0] = \{`0_1' : 0\}$, with $D_{u^-}[1] = [] = D_{u^-}[2]$. Then starting with crossing number $n = 3$ and increasing from there, compute $D_{u^-}[n]$ as follows. For each K in $D_{\text{splice}}[n]$ and each $i = 1, \dots, n$, consider $D_{\text{splice}}[n][K][i] = [K'_{i,1}, \dots, K'_{i,m_i}]$. Each $K'_{i,j}$ has fewer crossings $n_{i,j}$ than K, so we can look up each $D_{u^-}[n_{i,j}][K'_{i,j}]$. This gives:

$$cc(K) = u^-(K) = D_{u^-}[n](K) = 1 + \min_{i=1,\dots,n} \sum_{j=1}^{m_i} u^-(K'_{i,j}) = 1 + \min_{i=1,\dots,n} \sum_{j=1}^{m_i} D_{u^-}[n_{i,j}][K'_{i,j}].$$

In other words, we build the dictionary D_{u^-} of splice-unknotting numbers inductively, by looking at the connect summands of the diagrams obtained by u^--splices on a given diagram, looking up these summands' crosscap numbers in D_{u^-}, summing, minimizing, and adding 1.

6.5. *A note about computational efficiency*

Tabulating the list D_{DT} of dictionaries $D_{\text{DT}}[n]$ is admittedly computationally expensive. Yet, this work has nothing to do with splices or crosscap numbers per se. Starting from the data D_{DT} and D_{G}, we can justify the claim from the end of §2.2 regarding computational efficiency.

Recall that, given an n-crossing alternating diagram D, the main theorem of [4] states that one of the $2^n - 1$ non-Seifert states of D realizes $cc(D)$, and the minimal genus algorithm from [4] shortens this list of potentially optimal states from $2^n - 1$ to at most $2^{\lfloor n/3 \rfloor}$. Thus, in order to compute $cc(D)$, one might find a way to enumerate these states and select one of minimal complexity. Putting aside the question of how in fact to record and enumerate the states, this computation promises to grow, like the number of potentially optimal states, exponentially with crossing number.

Yet, supplied with the lists D_{G} and D_{DT} of dictionaries $D_{\mathrm{G}}[n]$ and $D_{\mathrm{DT}}[n]$, the computational cost of each entry in $D_{\mathrm{splice}}[n]$ grows in polynomial time with respect to crossing number, n. Indeed, for each of the n crossings of a given diagram, computing the spliced Gauss code $w_1\overline{w_2}w_3$, computing the reduced form of that Gauss code, and computing the Gauss codes of the resulting connect summands are all polynomial-time computations.

Supplied further with D_{splice} and $D_{\mathrm{cc}}[m]$ for all $m < n$, the computational cost of each entry in $D_{\mathrm{cc}}[n]$ also grows in polynomial time, since this simply involves looking up and adding the crosscap numbers coming from the n different splices of the given diagram.

Therefore, the computation cost for calculating the crosscap number of an alternating knot grows, too, in polynomial time with respect to crossing number.

Note that the dictionary $D_{\mathrm{cc}}[n]$ relies on the dictionaries $D_{\mathrm{cc}}[m]$ with $m < n$, and so this improved efficiency relies heavily on the fact that we are computing crosscap numbers for the *entire alternating knot tables*, rather than for individual knots, as well as on the fact that we have already tabulated D_{G} and D_{DT}.

Using Theorem 1.1 and the facts about Gauss codes and splices from §6.3, but no further data (such as the lists of dictionaries D_{G} and D_{DT}), one can compute the crosscap number of any particular alternating knot diagram, given its Gauss code, by computing each possible splice-unknotting sequence and finding a sequence of minimal length. The crosscap numbers tabulated in the appendix were double-checked with this sort of computation. This computation, however, grows exponentially with crossing number.

Appendix A: Tables of crosscap numbers

Table 1. Crosscap numbers $n = cc(K)$ of 11-crossing prime alternating knots K

K	n	K	n	K	n	K	n	K	n	K	n	K	n	K	n
11_{1}	5	11_{2}	5	11_{3}	5	11_{4}	4	11_{5}	5	11_{6}	5	11_{7}	4	11_{8}	4
11_{9}	3	11_{10}	4	11_{11}	5	11_{12}	5	11_{13}	4	11_{14}	5	11_{15}	4	11_{16}	5
11_{17}	5	11_{18}	5	11_{19}	5	11_{20}	5	11_{21}	4	11_{22}	4	11_{23}	5	11_{24}	5
11_{25}	5	11_{26}	5	11_{27}	5	11_{28}	5	11_{29}	4	11_{30}	5	11_{31}	5	11_{32}	5
11_{33}	4	11_{34}	5	11_{35}	5	11_{36}	5	11_{37}	4	11_{38}	5	11_{39}	4	11_{40}	4
11_{41}	5	11_{42}	5	11_{43}	5	11_{44}	5	11_{45}	4	11_{46}	4	11_{47}	5	11_{48}	5
11_{49}	5	11_{50}	4	11_{51}	5	11_{52}	5	11_{53}	4	11_{54}	5	11_{55}	4	11_{56}	5
11_{57}	4	11_{58}	4	11_{59}	3	11_{60}	4	11_{61}	4	11_{62}	3	11_{63}	4	11_{64}	5
11_{65}	4	11_{66}	5	11_{67}	4	11_{68}	4	11_{69}	5	11_{70}	5	11_{71}	5	11_{72}	5
11_{73}	5	11_{74}	3	11_{75}	4	11_{76}	5	11_{77}	5	11_{78}	5	11_{79}	5	11_{80}	5
11_{81}	4	11_{82}	4	11_{83}	4	11_{84}	5	11_{85}	5	11_{86}	4	11_{87}	5	11_{88}	4
11_{89}	5	11_{90}	4	11_{91}	5	11_{92}	4	11_{93}	4	11_{94}	5	11_{95}	4	11_{96}	5
11_{97}	3	11_{98}	4	11_{99}	4	11_{100}	5	11_{101}	5	11_{102}	4	11_{103}	4	11_{104}	5
11_{105}	5	11_{106}	4	11_{107}	4	11_{108}	4	11_{109}	5	11_{110}	4	11_{111}	4	11_{112}	5
11_{113}	4	11_{114}	5	11_{115}	4	11_{116}	5	11_{117}	5	11_{118}	4	11_{119}	4	11_{120}	5
11_{121}	5	11_{122}	5	11_{123}	4	11_{124}	5	11_{125}	5	11_{126}	5	11_{127}	4	11_{128}	5
11_{129}	4	11_{130}	5	11_{131}	5	11_{132}	5	11_{133}	4	11_{134}	5	11_{135}	5	11_{136}	5
11_{137}	4	11_{138}	5	11_{139}	4	11_{140}	3	11_{141}	4	11_{142}	3	11_{143}	4	11_{144}	4
11_{145}	4	11_{146}	5	11_{147}	5	11_{148}	4	11_{149}	5	11_{150}	5	11_{151}	5	11_{152}	4
11_{153}	4	11_{154}	4	11_{155}	5	11_{156}	4	11_{157}	5	11_{158}	4	11_{159}	5	11_{160}	5
11_{161}	3	11_{162}	5	11_{163}	4	11_{164}	5	11_{165}	4	11_{166}	3	11_{167}	5	11_{168}	5
11_{169}	4	11_{170}	5	11_{171}	5	11_{172}	5	11_{173}	5	11_{174}	4	11_{175}	5	11_{176}	5
11_{177}	4	11_{178}	5	11_{179}	3	11_{180}	4	11_{181}	4	11_{182}	4	11_{183}	5	11_{184}	4
11_{185}	4	11_{186}	5	11_{187}	5	11_{188}	3	11_{189}	5	11_{190}	4	11_{191}	4	11_{192}	4
11_{193}	4	11_{194}	4	11_{195}	3	11_{196}	5	11_{197}	5	11_{198}	4	11_{199}	4	11_{200}	4
11_{201}	4	11_{202}	5	11_{203}	3	11_{204}	4	11_{205}	4	11_{206}	3	11_{207}	4	11_{208}	5
11_{209}	5	11_{210}	4	11_{211}	4	11_{212}	5	11_{213}	5	11_{214}	4	11_{215}	4	11_{216}	5
11_{217}	5	11_{218}	5	11_{219}	4	11_{220}	4	11_{221}	4	11_{222}	4	11_{223}	3	11_{224}	4
11_{225}	3	11_{226}	4	11_{227}	5	11_{228}	5	11_{229}	4	11_{230}	3	11_{231}	4	11_{232}	4
11_{233}	5	11_{234}	3	11_{235}	4	11_{236}	5	11_{237}	4	11_{238}	4	11_{239}	5	11_{240}	3
11_{241}	4	11_{242}	3	11_{243}	4	11_{244}	5	11_{245}	4	11_{246}	3	11_{247}	2	11_{248}	5
11_{249}	4	11_{250}	3	11_{251}	5	11_{252}	4	11_{253}	5	11_{254}	4	11_{255}	5	11_{256}	4
11_{257}	4	11_{258}	3	11_{259}	3	11_{260}	3	11_{261}	4	11_{262}	4	11_{263}	3	11_{264}	5
11_{265}	4	11_{266}	5	11_{267}	5	11_{268}	4	11_{269}	4	11_{270}	5	11_{271}	5	11_{272}	5
11_{273}	5	11_{274}	5	11_{275}	5	11_{276}	5	11_{277}	5	11_{278}	4	11_{279}	3	11_{280}	4
11_{281}	4	11_{282}	4	11_{283}	5	11_{284}	5	11_{285}	5	11_{286}	4	11_{287}	5	11_{288}	5
11_{289}	5	11_{290}	4	11_{291}	4	11_{292}	5	11_{293}	3	11_{294}	4	11_{295}	4	11_{296}	4
11_{297}	5	11_{298}	5	11_{299}	4	11_{300}	5	11_{301}	5	11_{302}	4	11_{303}	4	11_{304}	4
11_{305}	4	11_{306}	4	11_{307}	4	11_{308}	3	11_{309}	4	11_{310}	3	11_{311}	4	11_{312}	4
11_{313}	3	11_{314}	5	11_{315}	5	11_{316}	4	11_{317}	4	11_{318}	5	11_{319}	5	11_{320}	4
11_{321}	5	11_{322}	5	11_{323}	3	11_{324}	4	11_{325}	4	11_{326}	5	11_{327}	5	11_{328}	5
11_{329}	5	11_{330}	3	11_{331}	4	11_{332}	5	11_{333}	3	11_{334}	3	11_{335}	4	11_{336}	3
11_{337}	4	11_{338}	3	11_{339}	3	11_{340}	4	11_{341}	3	11_{342}	2	11_{343}	3	11_{344}	5
11_{345}	4	11_{346}	3	11_{347}	4	11_{348}	4	11_{349}	5	11_{350}	5	11_{351}	5	11_{352}	4
11_{353}	5	11_{354}	4	11_{355}	3	11_{356}	4	11_{357}	4	11_{358}	2	11_{359}	3	11_{360}	3
11_{361}	3	11_{362}	3	11_{363}	3	11_{364}	2	11_{365}	3	11_{366}	4	11_{367}	1		

Table 2. Crosscap numbers $n = cc(K)$ of 12-crossing prime alternating knots K

K	n	K	n	K	n	K	n	K	n	K	n	K	n	K	n
12_1	5	12_2	4	12_3	5	12_4	6	12_5	6	12_6	5	12_7	6	12_8	5
12_9	4	12_{10}	6	12_{11}	5	12_{12}	5	12_{13}	5	12_{14}	6	12_{15}	5	12_{16}	5
12_{17}	5	12_{18}	4	12_{19}	5	12_{20}	5	12_{21}	6	12_{22}	4	12_{23}	5	12_{24}	4
12_{25}	6	12_{26}	5	12_{27}	5	12_{28}	6	12_{29}	6	12_{30}	5	12_{31}	4	12_{32}	5
12_{33}	5	12_{34}	4	12_{35}	5	12_{36}	4	12_{37}	5	12_{38}	4	12_{39}	5	12_{40}	6
12_{41}	5	12_{42}	5	12_{43}	6	12_{44}	6	12_{45}	5	12_{46}	5	12_{47}	6	12_{48}	6
12_{49}	5	12_{50}	5	12_{51}	5	12_{52}	4	12_{53}	5	12_{54}	5	12_{55}	5	12_{56}	4
12_{57}	5	12_{58}	6	12_{59}	6	12_{60}	6	12_{61}	5	12_{62}	5	12_{63}	6	12_{64}	6
12_{65}	5	12_{66}	5	12_{67}	5	12_{68}	5	12_{69}	6	12_{70}	5	12_{71}	5	12_{72}	5
12_{73}	6	12_{74}	6	12_{75}	5	12_{76}	4	12_{77}	6	12_{78}	5	12_{79}	5	12_{80}	5
12_{81}	5	12_{82}	6	12_{83}	6	12_{84}	5	12_{85}	5	12_{86}	5	12_{87}	5	12_{88}	6
12_{89}	5	12_{90}	6	12_{91}	5	12_{92}	5	12_{93}	4	12_{94}	5	12_{95}	5	12_{96}	4
12_{97}	4	12_{98}	5	12_{99}	6	12_{100}	5	12_{101}	5	12_{102}	6	12_{103}	6	12_{104}	5
12_{105}	4	12_{106}	5	12_{107}	6	12_{108}	6	12_{109}	5	12_{110}	5	12_{111}	5	12_{112}	5
12_{113}	6	12_{114}	6	12_{115}	5	12_{116}	5	12_{117}	6	12_{118}	5	12_{119}	5	12_{120}	6
12_{121}	5	12_{122}	5	12_{123}	4	12_{124}	5	12_{125}	6	12_{126}	6	12_{127}	5	12_{128}	4
12_{129}	5	12_{130}	5	12_{131}	5	12_{132}	6	12_{133}	5	12_{134}	5	12_{135}	5	12_{136}	5
12_{137}	5	12_{138}	5	12_{139}	6	12_{140}	5	12_{141}	5	12_{142}	5	12_{143}	4	12_{144}	5
12_{145}	5	12_{146}	3	12_{147}	4	12_{148}	5	12_{149}	6	12_{150}	5	12_{151}	5	12_{152}	4
12_{153}	4	12_{154}	6	12_{155}	5	12_{156}	4	12_{157}	5	12_{158}	4	12_{159}	5	12_{160}	4
12_{161}	5	12_{162}	6	12_{163}	5	12_{164}	5	12_{165}	4	12_{166}	5	12_{167}	5	12_{168}	4
12_{169}	3	12_{170}	5	12_{171}	5	12_{172}	4	12_{173}	5	12_{174}	5	12_{175}	6	12_{176}	4
12_{177}	5	12_{178}	4	12_{179}	5	12_{180}	5	12_{181}	6	12_{182}	5	12_{183}	4	12_{184}	6
12_{185}	5	12_{186}	5	12_{187}	5	12_{188}	5	12_{189}	5	12_{190}	5	12_{191}	6	12_{192}	5
12_{193}	4	12_{194}	5	12_{195}	4	12_{196}	5	12_{197}	4	12_{198}	6	12_{199}	6	12_{200}	5
12_{201}	4	12_{202}	5	12_{203}	5	12_{204}	5	12_{205}	4	12_{206}	4	12_{207}	4	12_{208}	5
12_{209}	6	12_{210}	6	12_{211}	5	12_{212}	5	12_{213}	5	12_{214}	6	12_{215}	5	12_{216}	4
12_{217}	5	12_{218}	5	12_{219}	5	12_{220}	5	12_{221}	5	12_{222}	6	12_{223}	4	12_{224}	5
12_{225}	6	12_{226}	6	12_{227}	6	12_{228}	5	12_{229}	6	12_{230}	5	12_{231}	5	12_{232}	6
12_{233}	6	12_{234}	5	12_{235}	5	12_{236}	4	12_{237}	5	12_{238}	5	12_{239}	4	12_{240}	5
12_{241}	5	12_{242}	5	12_{243}	5	12_{244}	5	12_{245}	5	12_{246}	4	12_{247}	5	12_{248}	4
12_{249}	5	12_{250}	4	12_{251}	5	12_{252}	4	12_{253}	5	12_{254}	4	12_{255}	4	12_{256}	5
12_{257}	6	12_{258}	5	12_{259}	4	12_{260}	4	12_{261}	5	12_{262}	4	12_{263}	6	12_{264}	5
12_{265}	6	12_{266}	5	12_{267}	5	12_{268}	6	12_{269}	5	12_{270}	4	12_{271}	5	12_{272}	5
12_{273}	6	12_{274}	5	12_{275}	5	12_{276}	4	12_{277}	5	12_{278}	5	12_{279}	5	12_{280}	5
12_{281}	5	12_{282}	6	12_{283}	5	12_{284}	5	12_{285}	5	12_{286}	5	12_{287}	6	12_{288}	6
12_{289}	5	12_{290}	5	12_{291}	4	12_{292}	5	12_{293}	6	12_{294}	5	12_{295}	6	12_{296}	6
12_{297}	5	12_{298}	5	12_{299}	4	12_{300}	5	12_{301}	5	12_{302}	5	12_{303}	5	12_{304}	4
12_{305}	5	12_{306}	5	12_{307}	5	12_{308}	5	12_{309}	5	12_{310}	6	12_{311}	5	12_{312}	4
12_{313}	5	12_{314}	6	12_{315}	6	12_{316}	6	12_{317}	5	12_{318}	5	12_{319}	5	12_{320}	4
12_{321}	4	12_{322}	5	12_{323}	6	12_{324}	5	12_{325}	5	12_{326}	5	12_{327}	5	12_{328}	5
12_{329}	5	12_{330}	4	12_{331}	5	12_{332}	5	12_{333}	6	12_{334}	5	12_{335}	5	12_{336}	6
12_{337}	6	12_{338}	6	12_{339}	4	12_{340}	6	12_{341}	6	12_{342}	6	12_{343}	5	12_{344}	5
12_{345}	4	12_{346}	5	12_{347}	5	12_{348}	6	12_{349}	5	12_{350}	6	12_{351}	5	12_{352}	6
12_{353}	5	12_{354}	5	12_{355}	4	12_{356}	4	12_{357}	5	12_{358}	5	12_{359}	6	12_{360}	5
12_{361}	6	12_{362}	5	12_{363}	5	12_{364}	6	12_{365}	4	12_{366}	5	12_{367}	4	12_{368}	5
12_{369}	3	12_{370}	4	12_{371}	4	12_{372}	5	12_{373}	4	12_{374}	5	12_{375}	4	12_{376}	4
12_{377}	5	12_{378}	4	12_{379}	3	12_{380}	3	12_{381}	5	12_{382}	4	12_{383}	5	12_{384}	5
12_{385}	5	12_{386}	5	12_{387}	5	12_{388}	6	12_{389}	6	12_{390}	6	12_{391}	5	12_{392}	4
12_{393}	6	12_{394}	5	12_{395}	5	12_{396}	5	12_{397}	5	12_{398}	4	12_{399}	5	12_{400}	5
12_{401}	5	12_{402}	5	12_{403}	5	12_{404}	4	12_{405}	5	12_{406}	6	12_{407}	5	12_{408}	6
12_{409}	4	12_{410}	5	12_{411}	5	12_{412}	5	12_{413}	5	12_{414}	4	12_{415}	6	12_{416}	5
12_{417}	6	12_{418}	5	12_{419}	5	12_{420}	4	12_{421}	4	12_{422}	3	12_{423}	4	12_{424}	5

Table 3. Table 2 Continued. Crosscap numbers $n = cc(K)$ of 12-crossing prime alternating knots K

K	n	K	n	K	n	K	n	K	n	K	n	K	n	K	n
12_{425}	4	12_{426}	6	12_{427}	6	12_{428}	5	12_{429}	5	12_{430}	5	12_{431}	6	12_{432}	6
12_{433}	6	12_{434}	5	12_{435}	6	12_{436}	4	12_{437}	5	12_{438}	5	12_{439}	6	12_{440}	5
12_{441}	5	12_{442}	4	12_{443}	4	12_{444}	4	12_{445}	5	12_{446}	5	12_{447}	4	12_{448}	4
12_{449}	5	12_{450}	5	12_{451}	5	12_{452}	6	12_{453}	5	12_{454}	4	12_{455}	5	12_{456}	6
12_{457}	5	12_{458}	6	12_{459}	5	12_{460}	6	12_{461}	6	12_{462}	5	12_{463}	4	12_{464}	5
12_{465}	6	12_{466}	5	12_{467}	6	12_{468}	5	12_{469}	5	12_{470}	6	12_{471}	4	12_{472}	6
12_{473}	5	12_{474}	6	12_{475}	5	12_{476}	4	12_{477}	6	12_{478}	5	12_{479}	5	12_{480}	6
12_{481}	4	12_{482}	4	12_{483}	6	12_{484}	6	12_{485}	5	12_{486}	6	12_{487}	6	12_{488}	4
12_{489}	5	12_{490}	5	12_{491}	5	12_{492}	5	12_{493}	4	12_{494}	5	12_{495}	5	12_{496}	6
12_{497}	6	12_{498}	5	12_{499}	6	12_{500}	5	12_{501}	5	12_{502}	4	12_{503}	4	12_{504}	5
12_{505}	5	12_{506}	5	12_{507}	4	12_{508}	5	12_{509}	6	12_{510}	6	12_{511}	5	12_{512}	5
12_{513}	5	12_{514}	5	12_{515}	5	12_{516}	6	12_{517}	4	12_{518}	5	12_{519}	4	12_{520}	4
12_{521}	4	12_{522}	5	12_{523}	5	12_{524}	5	12_{525}	4	12_{526}	6	12_{527}	5	12_{528}	5
12_{529}	5	12_{530}	5	12_{531}	5	12_{532}	4	12_{533}	5	12_{534}	5	12_{535}	5	12_{536}	4
12_{537}	5	12_{538}	4	12_{539}	5	12_{540}	5	12_{541}	4	12_{542}	4	12_{543}	6	12_{544}	5
12_{545}	5	12_{546}	6	12_{547}	6	12_{548}	4	12_{549}	4	12_{550}	5	12_{551}	4	12_{552}	4
12_{553}	5	12_{554}	5	12_{555}	5	12_{556}	5	12_{557}	4	12_{558}	5	12_{559}	5	12_{560}	5
12_{561}	5	12_{562}	5	12_{563}	4	12_{564}	4	12_{565}	5	12_{566}	5	12_{567}	5	12_{568}	5
12_{569}	5	12_{570}	5	12_{571}	6	12_{572}	5	12_{573}	4	12_{574}	4	12_{575}	5	12_{576}	3
12_{577}	4	12_{578}	5	12_{579}	5	12_{580}	3	12_{581}	4	12_{582}	4	12_{583}	5	12_{584}	5
12_{585}	5	12_{586}	5	12_{587}	4	12_{588}	6	12_{589}	5	12_{590}	4	12_{591}	4	12_{592}	6
12_{593}	5	12_{594}	4	12_{595}	4	12_{596}	3	12_{597}	4	12_{598}	5	12_{599}	5	12_{600}	4
12_{601}	4	12_{602}	5	12_{603}	5	12_{604}	6	12_{605}	4	12_{606}	5	12_{607}	5	12_{608}	5
12_{609}	5	12_{610}	4	12_{611}	6	12_{612}	4	12_{613}	5	12_{614}	6	12_{615}	6	12_{616}	5
12_{617}	5	12_{618}	5	12_{619}	4	12_{620}	5	12_{621}	5	12_{622}	5	12_{623}	5	12_{624}	5
12_{625}	5	12_{626}	6	12_{627}	6	12_{628}	5	12_{629}	6	12_{630}	5	12_{631}	6	12_{632}	4
12_{633}	5	12_{634}	4	12_{635}	5	12_{636}	3	12_{637}	5	12_{638}	5	12_{639}	5	12_{640}	4
12_{641}	3	12_{642}	4	12_{643}	4	12_{644}	4	12_{645}	6	12_{646}	5	12_{647}	4	12_{648}	5
12_{649}	4	12_{650}	5	12_{651}	4	12_{652}	5	12_{653}	4	12_{654}	5	12_{655}	5	12_{656}	5
12_{657}	5	12_{658}	5	12_{659}	6	12_{660}	4	12_{661}	5	12_{662}	6	12_{663}	4	12_{664}	4
12_{665}	5	12_{666}	6	12_{667}	4	12_{668}	5	12_{669}	3	12_{670}	5	12_{671}	5	12_{672}	6
12_{673}	5	12_{674}	6	12_{675}	5	12_{676}	5	12_{677}	5	12_{678}	5	12_{679}	4	12_{680}	5
12_{681}	4	12_{682}	4	12_{683}	4	12_{684}	5	12_{685}	6	12_{686}	6	12_{687}	6	12_{688}	5
12_{689}	4	12_{690}	4	12_{691}	4	12_{692}	5	12_{693}	4	12_{694}	4	12_{695}	6	12_{696}	5
12_{697}	6	12_{698}	5	12_{699}	5	12_{700}	5	12_{701}	5	12_{702}	4	12_{703}	6	12_{704}	5
12_{705}	6	12_{706}	5	12_{707}	5	12_{708}	4	12_{709}	5	12_{710}	6	12_{711}	5	12_{712}	6
12_{713}	5	12_{714}	4	12_{715}	5	12_{716}	3	12_{717}	4	12_{718}	5	12_{719}	5	12_{720}	4
12_{721}	5	12_{722}	2	12_{723}	3	12_{724}	4	12_{725}	4	12_{726}	4	12_{727}	5	12_{728}	5
12_{729}	5	12_{730}	5	12_{731}	4	12_{732}	4	12_{733}	3	12_{734}	6	12_{735}	4	12_{736}	5
12_{737}	5	12_{738}	4	12_{739}	4	12_{740}	4	12_{741}	6	12_{742}	4	12_{743}	4	12_{744}	3
12_{745}	3	12_{746}	5	12_{747}	5	12_{748}	4	12_{749}	4	12_{750}	4	12_{751}	5	12_{752}	4
12_{753}	3	12_{754}	5	12_{755}	6	12_{756}	5	12_{757}	5	12_{758}	4	12_{759}	3	12_{760}	4
12_{761}	5	12_{762}	3	12_{763}	4	12_{764}	5	12_{765}	6	12_{766}	5	12_{767}	4	12_{768}	5
12_{769}	5	12_{770}	6	12_{771}	5	12_{772}	4	12_{773}	4	12_{774}	4	12_{775}	4	12_{776}	5
12_{777}	4	12_{778}	6	12_{779}	5	12_{780}	6	12_{781}	5	12_{782}	4	12_{783}	5	12_{784}	5
12_{785}	5	12_{786}	5	12_{787}	4	12_{788}	6	12_{789}	4	12_{790}	5	12_{791}	3	12_{792}	4
12_{793}	6	12_{794}	4	12_{795}	5	12_{796}	3	12_{797}	4	12_{798}	6	12_{799}	5	12_{800}	4
12_{801}	4	12_{802}	3	12_{803}	2	12_{804}	5	12_{805}	4	12_{806}	5	12_{807}	5	12_{808}	4
12_{809}	5	12_{810}	5	12_{811}	4	12_{812}	5	12_{813}	4	12_{814}	5	12_{815}	4	12_{816}	5
12_{817}	4	12_{818}	4	12_{819}	5	12_{820}	4	12_{821}	5	12_{822}	4	12_{823}	4	12_{824}	4
12_{825}	4	12_{826}	3	12_{827}	3	12_{828}	5	12_{829}	4	12_{830}	5	12_{831}	5	12_{832}	4
12_{833}	4	12_{834}	4	12_{835}	3	12_{836}	4	12_{837}	4	12_{838}	2	12_{839}	3	12_{840}	4
12_{841}	4	12_{842}	3	12_{843}	3	12_{844}	5	12_{845}	3	12_{846}	5	12_{847}	4	12_{848}	5

Table 4. Table 2 Continued. Crosscap numbers $n = cc(K)$ of 12-crossing prime alternating knots K

K	n	K	n	K	n	K	n	K	n	K	n	K	n	K	n
12_{849}	5	12_{850}	4	12_{851}	5	12_{852}	5	12_{853}	4	12_{854}	4	12_{855}	4	12_{856}	5
12_{857}	5	12_{858}	4	12_{859}	3	12_{860}	4	12_{861}	5	12_{862}	5	12_{863}	4	12_{864}	5
12_{865}	5	12_{866}	6	12_{867}	6	12_{868}	6	12_{869}	4	12_{870}	5	12_{871}	5	12_{872}	4
12_{873}	4	12_{874}	6	12_{875}	5	12_{876}	4	12_{877}	4	12_{878}	3	12_{879}	4	12_{880}	5
12_{881}	3	12_{882}	4	12_{883}	4	12_{884}	5	12_{885}	5	12_{886}	5	12_{887}	6	12_{888}	5
12_{889}	4	12_{890}	5	12_{891}	5	12_{892}	5	12_{893}	6	12_{894}	5	12_{895}	6	12_{896}	4
12_{897}	5	12_{898}	5	12_{899}	5	12_{900}	6	12_{901}	5	12_{902}	5	12_{903}	6	12_{904}	5
12_{905}	4	12_{906}	6	12_{907}	5	12_{908}	5	12_{909}	4	12_{910}	5	12_{911}	5	12_{912}	4
12_{913}	4	12_{914}	5	12_{915}	6	12_{916}	5	12_{917}	5	12_{918}	5	12_{919}	5	12_{920}	4
12_{921}	5	12_{922}	6	12_{923}	4	12_{924}	5	12_{925}	5	12_{926}	4	12_{927}	4	12_{928}	5
12_{929}	4	12_{930}	4	12_{931}	5	12_{932}	4	12_{933}	5	12_{934}	6	12_{935}	5	12_{936}	5
12_{937}	3	12_{938}	4	12_{939}	5	12_{940}	4	12_{941}	4	12_{942}	4	12_{943}	5	12_{944}	5
12_{945}	5	12_{946}	4	12_{947}	4	12_{948}	5	12_{949}	5	12_{950}	4	12_{951}	5	12_{952}	4
12_{953}	5	12_{954}	5	12_{955}	4	12_{956}	5	12_{957}	5	12_{958}	5	12_{959}	5	12_{960}	6
12_{961}	6	12_{962}	5	12_{963}	4	12_{964}	5	12_{965}	6	12_{966}	5	12_{967}	5	12_{968}	5
12_{969}	4	12_{970}	3	12_{971}	4	12_{972}	4	12_{973}	5	12_{974}	5	12_{975}	4	12_{976}	5
12_{977}	4	12_{978}	4	12_{979}	5	12_{980}	5	12_{981}	4	12_{982}	5	12_{983}	5	12_{984}	3
12_{985}	4	12_{986}	5	12_{987}	5	12_{988}	4	12_{989}	5	12_{990}	5	12_{991}	4	12_{992}	6
12_{993}	5	12_{994}	6	12_{995}	5	12_{996}	5	12_{997}	5	12_{998}	6	12_{999}	5	12_{1000}	4
12_{1001}	4	12_{1002}	5	12_{1003}	5	12_{1004}	6	12_{1005}	5	12_{1006}	5	12_{1007}	4	12_{1008}	5
12_{1009}	4	12_{1010}	5	12_{1011}	4	12_{1012}	4	12_{1013}	4	12_{1014}	5	12_{1015}	4	12_{1016}	5
12_{1017}	3	12_{1018}	4	12_{1019}	6	12_{1020}	5	12_{1021}	6	12_{1022}	5	12_{1023}	4	12_{1024}	4
12_{1025}	5	12_{1026}	4	12_{1027}	3	12_{1028}	4	12_{1029}	3	12_{1030}	3	12_{1031}	3	12_{1032}	4
12_{1033}	4	12_{1034}	4	12_{1035}	4	12_{1036}	4	12_{1037}	5	12_{1038}	5	12_{1039}	4	12_{1040}	4
12_{1041}	5	12_{1042}	5	12_{1043}	5	12_{1044}	5	12_{1045}	4	12_{1046}	5	12_{1047}	5	12_{1048}	5
12_{1049}	5	12_{1050}	5	12_{1051}	4	12_{1052}	5	12_{1053}	5	12_{1054}	5	12_{1055}	5	12_{1056}	6
12_{1057}	5	12_{1058}	5	12_{1059}	4	12_{1060}	5	12_{1061}	6	12_{1062}	4	12_{1063}	4	12_{1064}	5
12_{1065}	5	12_{1066}	5	12_{1067}	6	12_{1068}	4	12_{1069}	6	12_{1070}	5	12_{1071}	5	12_{1072}	5
12_{1073}	5	12_{1074}	4	12_{1075}	4	12_{1076}	6	12_{1077}	5	12_{1078}	5	12_{1079}	6	12_{1080}	4
12_{1081}	5	12_{1082}	4	12_{1083}	4	12_{1084}	4	12_{1085}	5	12_{1086}	5	12_{1087}	5	12_{1088}	6
12_{1089}	4	12_{1090}	5	12_{1091}	5	12_{1092}	5	12_{1093}	5	12_{1094}	4	12_{1095}	3	12_{1096}	5
12_{1097}	5	12_{1098}	6	12_{1099}	5	12_{1100}	5	12_{1101}	5	12_{1102}	6	12_{1103}	5	12_{1104}	5
12_{1105}	6	12_{1106}	4	12_{1107}	3	12_{1108}	4	12_{1109}	5	12_{1110}	5	12_{1111}	4	12_{1112}	5
12_{1113}	5	12_{1114}	3	12_{1115}	4	12_{1116}	5	12_{1117}	6	12_{1118}	4	12_{1119}	5	12_{1120}	4
12_{1121}	5	12_{1122}	5	12_{1123}	6	12_{1124}	6	12_{1125}	4	12_{1126}	4	12_{1127}	4	12_{1128}	3
12_{1129}	4	12_{1130}	4	12_{1131}	3	12_{1132}	4	12_{1133}	5	12_{1134}	3	12_{1135}	4	12_{1136}	5
12_{1137}	4	12_{1138}	3	12_{1139}	4	12_{1140}	4	12_{1141}	5	12_{1142}	3	12_{1143}	5	12_{1144}	4
12_{1145}	3	12_{1146}	4	12_{1147}	4	12_{1148}	3	12_{1149}	2	12_{1150}	5	12_{1151}	4	12_{1152}	6
12_{1153}	4	12_{1154}	5	12_{1155}	6	12_{1156}	4	12_{1157}	2	12_{1158}	3	12_{1159}	4	12_{1160}	4
12_{1161}	3	12_{1162}	3	12_{1163}	4	12_{1164}	4	12_{1165}	3	12_{1166}	3	12_{1167}	6	12_{1168}	5
12_{1169}	4	12_{1170}	4	12_{1171}	3	12_{1172}	5	12_{1173}	5	12_{1174}	4	12_{1175}	5	12_{1176}	4
12_{1177}	5	12_{1178}	4	12_{1179}	3	12_{1180}	5	12_{1181}	4	12_{1182}	5	12_{1183}	4	12_{1184}	5
12_{1185}	5	12_{1186}	5	12_{1187}	6	12_{1188}	6	12_{1189}	5	12_{1190}	5	12_{1191}	4	12_{1192}	5
12_{1193}	6	12_{1194}	4	12_{1195}	5	12_{1196}	5	12_{1197}	5	12_{1198}	5	12_{1199}	5	12_{1200}	4
12_{1201}	5	12_{1202}	5	12_{1203}	4	12_{1204}	4	12_{1205}	3	12_{1206}	6	12_{1207}	5	12_{1208}	5
12_{1209}	5	12_{1210}	4	12_{1211}	6	12_{1212}	5	12_{1213}	5	12_{1214}	2	12_{1215}	4	12_{1216}	4
12_{1217}	5	12_{1218}	4	12_{1219}	4	12_{1220}	3	12_{1221}	5	12_{1222}	5	12_{1223}	4	12_{1224}	4
12_{1225}	6	12_{1226}	4	12_{1227}	5	12_{1228}	5	12_{1229}	6	12_{1230}	5	12_{1231}	5	12_{1232}	4
12_{1233}	3	12_{1234}	4	12_{1235}	4	12_{1236}	4	12_{1237}	5	12_{1238}	4	12_{1239}	5	12_{1240}	3
12_{1241}	4	12_{1242}	2	12_{1243}	3	12_{1244}	4	12_{1245}	5	12_{1246}	4	12_{1247}	3	12_{1248}	5
12_{1249}	6	12_{1250}	5	12_{1251}	6	12_{1252}	6	12_{1253}	5	12_{1254}	4	12_{1255}	4	12_{1256}	4
12_{1257}	5	12_{1258}	5	12_{1259}	4	12_{1260}	5	12_{1261}	5	12_{1262}	4	12_{1263}	5	12_{1264}	4
12_{1265}	5	12_{1266}	4	12_{1267}	4	12_{1268}	5	12_{1269}	5	12_{1270}	6	12_{1271}	5	12_{1272}	5
12_{1273}	3	12_{1274}	4	12_{1275}	5	12_{1276}	3	12_{1277}	4	12_{1278}	2	12_{1279}	3	12_{1280}	6
12_{1281}	4	12_{1282}	3	12_{1283}	3	12_{1284}	4	12_{1285}	3	12_{1286}	2	12_{1287}	3	12_{1288}	4

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