A simple proof of the Crowell–Murasugi theorem

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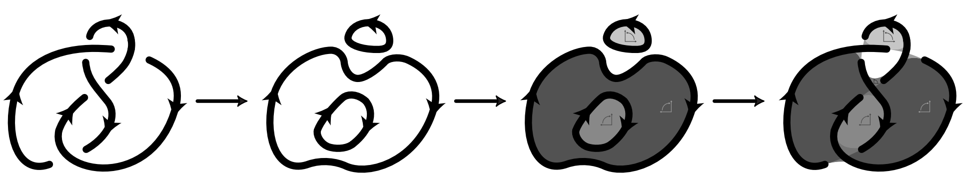
We give an elementary, self-contained proof of the theorem, proven independently in 1958-9 by Crowell and Murasugi, that the genus of any oriented non-split alternating link equals half the breadth of its Alexander polynomial (with a correction term for the number of link components), and that applying Seifert's algorithm to any oriented connected alternating link diagram gives a surface of minimal genus.

Every oriented link $K \subset S^3$ bounds a connected oriented surface F called a *Seifert surface*. Such F is homeomorphic to an ℓ-punctured surface of some genus $g(F)$, where $\ell = |K|$ (here and throughout, bars count components). The **link genus** $g(K)$ is the minimum genus among all Seifert surfaces for K.

An ordered basis $(a_1, \dots, a_n)$ for $H_1(F)$ determines an $n \times n$ *Seifert matrix* $V = (v_{ij})$, $v_{ij} = \text{lk}(a_i, a_j^+)$, where lk denotes linking number and a_j^+ is the pushoff of (an oriented multicurve representing) a_j in the positive normal direction determined by the orientations on F and S^3. The polynomial $\det(V - tV^T)$, denoted $\Delta_K(t)$, is called the *Alexander polynomial* of K. Up to degree shift, it is independent of Seifert surface and basis [Ka81, BFK98]. Writing $\Delta_K(t) = a_r t^r + a_{r+1} t^{r+1} + \cdots + a_{s-1} t^{s-1} + a_s t^s$ with $a_r, a_s \neq 0$, we call $s - r$ the **breadth** of $\Delta_K(t)$ and denote it by $\text{bth}(K)$.

Given any oriented connected diagram $D \subset S^2$ of a link $K \subset S^3$, **Seifert's algorithm** yields a Seifert surface for K as follows. First, "smooth" each crossing of D in the way that respects orientation: , . This gives a disjoint union of oriented circles on S^2 called the *Seifert state* of D; each circle is called a *Seifert circle*. Second, cap all the Seifert circles with disjoint, oriented disks, all on the same side of S^2. Third, attach an oriented half-twisted band at each crossing, so that the resulting surface F is oriented with $\partial F = K$, respecting orientation. Here is an example:



The purpose of this note is to give a short, elementary, self-contained proof of the following theorem, first proven independently in 1958-9 by Crowell and Murasugi:

Theorem 1 ([Cr59, Mu58]) *If F is a surface constructed via Seifert's algorithm from a connected alternating diagram D of an oriented ℓ-component link K, then*

$$g(F) = g(K) = \frac{1}{2}\left(bth(K) + 1 - \ell\right).$$

To prove Theorem 1, we will show that a Seifert matrix V for F is invertible. The next two results show that this indeed will suffice:

Proposition 2 *Let F be a Seifert surface for an oriented ℓ-component link K. If $bth(K) = 2g(F) + \ell - 1$, then $g(K) = g(F) = \frac{1}{2}\left(bth(K) + 1 - \ell\right)$.*

Proof Given an arbitrary Seifert surface F' for K, one may compute $\Delta_K(t)$ from any Seifert matrix for F', so $\mathrm{bth}(K) \leq \beta_1(F') = 2g(F') + 1 - \ell$. Hence $g(F) \leq g(F')$. $\square$

Proposition 3 *[Mu96] Let V be a real $n \times n$ matrix, and let $f(t) = \det(V - tV^T)$. If V is invertible, then the breadth of $f(t)$ equals n.*

Proof Denoting the transpose of V^{-1} by V^{-T},

$$f(t) = \det(V^T)\det(VV^{-T} - tI)$$

is a nonzero scalar multiple of the characteristic polynomial of the invertible matrix VV^{-T}, hence has breadth n.[1] $\square$

Next, suppose that $D \subset S^2$ is a connected oriented alternating link diagram such that applying Seifert's algorithm to D yields a *checkerboard* surface F.[2,3] Then, since D is alternating and connected, all of the crossing bands in F are identical: either they all positive, , or they are all negative, . Let V denote a Seifert matrix for F.

Lemma 4 *With the preceding setup, if the crossing bands in F are positive, then any nonzero $\mathbf{x} \in \mathbb{Z}^{\beta_1(F)}$ satisfies $\mathbf{x}^T V \mathbf{x} > 0$; if the crossing bands in F are negative, then any such $\mathbf{x}$ satisfies $\mathbf{x}^T V \mathbf{x} < 0$. Hence, in either case, V is invertible.*

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[1] The converse is also true. Indeed, if V is singular, then choose an invertible matrix P whose first column is in the nullspace of V. Then $\det(P^T VP - t(P^T VP)^T) = \det^2(P) \cdot f(t)$ has the same breadth as $f(t)$. Further, the first column of $P^T VP$ is $\mathbf{0}$, so only constants appear in the first row of $P^T VP - t(P^T VP)^T$. Hence, the breadth is less than n.

[2] That is, each Seifert circle bounds a disk in S^2 disjoint from the other Seifert circles.

[3] Such a diagram is either *positive* or *negative* and is called *special alternating*.

Here is a self-contained proof. A shorter argument, using [Gr17], follows.

Proof Assume without loss of generality that the crossing bands in F are positive. Among all oriented multicurves in F that represent $\mathbf{x}$, choose one, α, that intersects the crossing bands in F in the smallest possible number of components. Then, for each crossing band X in F, $\alpha \cap X$ will consist of a (possibly empty) collection of coherently oriented arcs. Therefore:

$$\mathbf{x}^T V\mathbf{x} = \mathrm{lk}(\alpha, \alpha^+) = \sum_{\text{crossing bands } X} \frac{|\alpha \cap X|^2}{2} \geq 0. \tag{1}$$

Moreover, the inequality in (1) is strict, or else α would be disjoint from all crossing bands, hence nullhomologous (since D is connected). It follows that V is nonsingular, or else we would have $V\mathbf{z} = \mathbf{0}$ for some nonzero vector $\mathbf{z}$, giving $\mathbf{z}^T V\mathbf{z} = 0$. □

Alternatively, denote the Gordon-Litherland pairing on F by $\langle\cdot,\cdot\rangle$ [GL78]. Since D is alternating and connected, this pairing is definite [Mu87ii, Gr17]. Thus:

$$\mathbf{x}^T V\mathbf{x} = \mathrm{lk}(\alpha, \alpha^+) = \frac{1}{2}\mathrm{lk}(\alpha, \alpha_+ \cup \alpha_-) = \langle \mathbf{x}, \mathbf{x}\rangle \neq 0.$$

To complete the proof of Theorem 1, we need one more definition and lemma. Murasugi sum, also called generalized plumbing, is a way of gluing together two spanning surfaces along a disk so as to produce another spanning surface. We will prove that if Seifert surfaces F_1 and F_2 have invertible Seifert matrices, then any Murasugi sum of F_1 and F_2 also has invertible Seifert matrix (and conversely).

Definition 5 For $i = 1, 2$, let F_i be a Seifert surface in a 3-sphere S_i^3, and choose a compact 3-ball $B_i \subset S_i^3$ that contains F_i such that (i) $F_i \cap \partial B_i$ is a disk U_i whose boundary consists alternately of arcs in ∂F_i and arcs in $\mathrm{int}(F_i)$, (ii) $|\partial U_1 \cap \partial F_1| = |\partial U_2 \cap \partial F_2|$, and (iii) the positive normal along U_1 (using the orientations on S_1^3 and F_1) points *into* B_1, whereas the positive normal along U_2 points *out of* B_2. Choose an orientation-reversing homeomorphism $h : \partial B_1 \to \partial B_2$ such that $h(U_1) = U_2$ and $h(\partial U_1 \cap \partial F_1) = \mathrm{cl}(\partial U_2 \cap \mathrm{int}(F_2))$.[4] Then $F = F_1 \cup_h F_2$ is a Seifert surface in the 3-sphere $B_1 \cup_h B_2$. It is a **Murasugi sum** or *generalized plumbing* of F_1 and F_2, denoted $F = F_1 * F_2$.

[4]It follows that $h(\mathrm{cl}(\partial U_1 \cap \mathrm{int}(F_1))) = \partial U_2 \cap \partial F_2$.

Note that there are generally many ways to form a Murasugi sum between two given surfaces. As an aside, we mention that the Murasugi sum construction extends easily to unoriented surfaces, and that both the oriented and unoriented notions of Murasugi sum are natural operations in many respects [Ga83, Ga85, Oz11, OP16, Ki18]. Here is one such respect:

Lemma 6 *Given a Murasugi sum $F = F_1 * F_2$ of Seifert surfaces with respective Seifert matrices V, V_1, and V_2, V is invertible if and only if both V_1 and V_2 are.*

Proof Denote $V = (v_{ij})$. We may assume that V is taken with respect to a basis $(a_1, \dots, a_r, b_1, \dots, b_s)$ for $H_1(F)$, where $(a_1, \dots, a_r)$ is a basis for $H_1(F_1)$ and $(b_1, \dots, b_s)$ is a basis for $H_1(F_2)$. Then V is a block matrix of the form $V = \begin{bmatrix} V_1 & A \\ B & V_2 \end{bmatrix}$. In fact, we claim that $B = 0$, i.e.

$$V = \begin{bmatrix} V_1 & A \\ 0 & V_2 \end{bmatrix}. \tag{2}$$

To see this, let $\alpha_j \subset F_1$ represent a_j and let $\beta_i \subset F_2$ represent b_i for arbitrary $1 \leq j \leq r$, $1 \leq i \leq s$. Then $v_{ij} = \mathrm{lk}(\beta_i, \alpha_j^+) = 0$ because, using the notation and setup from Definition 5, $\alpha_j^+ \subset \mathrm{int}(h(B_1))$ and $\beta_i \subset B_2$. From (2), we have $\det(V) = \det(V_1)\det(V_2)$,[5] so the result follows. □

Now we can prove Theorem 1:

Proof of Theorem 1 Let F be a surface constructed via Seifert's algorithm from an alternating diagram D of an oriented link K. Then F is a Murasugi sum of checkerboard Seifert surfaces from connected oriented alternating link diagrams.[6]

Lemma 4 implies that all of these checkerboard surfaces have invertible Seifert matrices, so Lemma 6 implies that F has an invertible Seifert matrix V. Since K has ℓ components, the size of V is $\beta_1(F) = 2g(F) + 1 - \ell$. Thus, by Propositions 2-3,

$$g(F) = g(K) = \frac{1}{2}\left(\mathrm{bth}(K) + 1 - \ell\right). \qquad \square$$

The preceding proof shows more generally:

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[5] This is due to the formula $\det(V) = \sum_{\sigma \in S_{r+s}} \mathrm{sign}(\sigma) \prod_{i=1}^{r+s} v_{i\sigma(i)}$ and the pigeonhole principle.

[6] Indeed, D is a $*$-*product* of special alternating diagrams: see Definition 2.37 and Remark 2.38 of [Ba13]. For an explicit construction, see page 98 of [QW04].

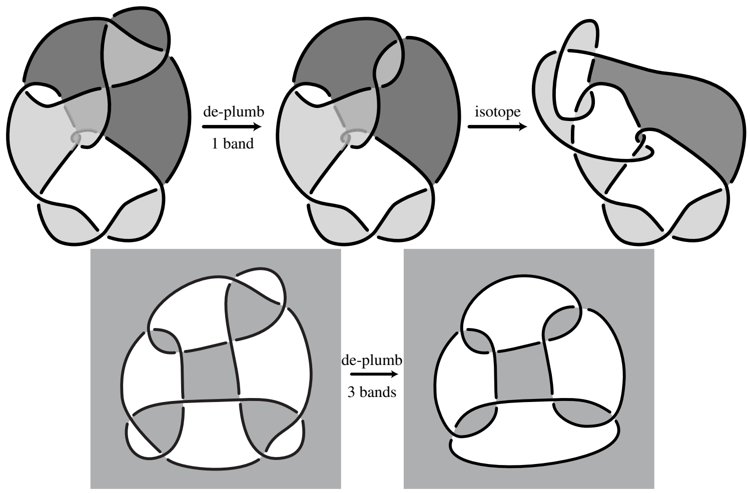

Figure 1: De-plumbing Hopf bands from minimal genus Seifert surfaces for the knots $11n67$ and $11n73$

Theorem 7 *Let F be a Seifert surface for an oriented ℓ-component link K. If F is a Murasugi sum of checkerboard surfaces from connected oriented alternating link diagrams, then $g(K) = g(F) = \frac{1}{2}(bth(K) + 1 - \ell)$.*

In particular, an oriented connected link diagram is called **homogeneous** if it is a $*$*-product*, i.e. *diagrammatic Murasugi sum*, of special alternating link diagrams. By definition, Theorem 7 applies to all such diagrams (c.f. [Cr89] Corollary 4.1):

Corollary 8 *If F is constructed via Seifert's algorithm from a homogeneous diagram of an ℓ-component oriented link K, then $g(F) = g(K) = \frac{1}{2}(bth(K) + 1 - \ell)$.*

We note another consequence of Lemma 6, in combination with:

Theorem 9 (Harer's conjecture [Ha82]; Corollary 3 of [GG06]) *Any fiber surface in S^3 can be constructed by plumbing and de-plumbing Hopf bands.*

Corollary 10 *If F is an oriented fiber surface spanning an ℓ-component link $K \subset S^3$ and K has the boundary orientation from F, then*

$$g(F) = g(K) = \frac{1}{2}(bth(K) + 1 - \ell).$$

We close by considering knots K with $g(K) > \frac{1}{2}\mathrm{bth}(K)$. The simplest such knots have 11 crossings. There are seven of them [1]: the Conway knot $11n34$ has genus three, as do $11n45$, $11n73$, and $11n152$, while the Kinoshita-Terasaka knot $11n42$ has genus two, as do $11n67$ and $11n97$. Lemma 6 implies that if one takes a minimal genus Seifert surface for any one of these knots and de-plumbs (i.e. decomposes it as a nontrivial Murasugi sum),[7] then at least one of the resulting factors will have a singular Seifert matrix. Also, by Theorem 1 of [Ga85], all of these surfaces will have minimal genus. This raises the following natural problem:

Problem 11 Characterize or tabulate those Seifert surfaces F which (i) have minimal genus, (ii) do not de-plumb,[8] and (iii) have singular Seifert matrices.

Interestingly, for each of the four aforementioned 11-crossing knots of genus three, de-plumbing a minimal genus Seifert surface gives three Hopf bands and the planar pretzel surface $P_{2,2,-2,-2}$, which has Seifert matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix},$$

and doing this for any of the three aforementioned 11-crossing knots of genus two gives one Hopf band and a surface of genus one that has Seifert matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & -1 & 0 \end{bmatrix}.$$

See Figure 1. Another simple example of the type of surface referenced in Problem 11 is the planar pretzel surface $P_{4,4,-2}$, which has Seifert matrix

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}.$$

In particular, each of these simplest examples spans a link of multiple components.

Question 12 Does there exist a *knot* K that satisfies $g(K) > \frac{1}{2}\mathrm{bth}(K)$ and has a minimal genus Seifert surface F that does not de-plumb?

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[7] Beware: surfaces may admit distinct de-plumbings [Ki18]. Still, Lemma 6 implies that this sentence is true for *any* de-plumbing of such a surface.

[8] That is, any decomposition of F as a Murasugi sum $F = F_1 * F_2$ has F_1 or F_2 as a disk.

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