Efficient multisections of odd-dimensional tori

THOMAS KINDRED

Rubinstein–Tillmann generalized the notions of Heegaard splittings of 3-manifolds and trisections of 4-manifolds by defining *multisections* of PL n-manifolds, which are decompositions into $k = \lfloor n/2 \rfloor + 1$ n-dimensional 1-handlebodies with nice intersection properties. For each odd-dimensional torus T^n, we construct a multisection which is *efficient* in the sense that each 1-handlebody has genus n, which we prove is optimal; each multisection is *symmetric* with respect to both the permutation action of S_n on the indices and the $\mathbb{Z}_k$ translation action along the main diagonal. We also construct such a trisection of T^4, lift all symmetric multisections of tori to certain cubulated manifolds, and obtain combinatorial identities as corollaries.

1 Introduction

Every closed 3-manifold[1] X admits a decomposition into two 3-dimensional 1-handlebodies[2] glued along their boundaries. Gay–Kirby extended this classical notion of *Heegaard splittings* by proving that every closed 4-manifold admits a *trisection*, i.e. a decomposition $X = \bigcup_{i\in\mathbb{Z}_3} X_i$ where each X_i is a 4-dimensional 1-handlebody, each $X_i \cap X_{i+1}$ is a 3-dimensional 1-handlebody, and $X_0 \cap X_1 \cap X_2$ is a closed surface. Rubinstein-Tillmann [RuTi20] then extended these decompositions to arbitrary dimension by proving that every closed (PL) manifold of arbitrary dimension admits a *PL multisection*:

Definition 1.1 A *PL multisection* of a closed manifold X of dimension $n = 2k-1$ (resp. $2k-2$) is a decomposition $X = \bigcup_{i\in\mathbb{Z}_k} X_i$, where:

- Each X_i is an n-dimensional 1-handlebody.
- $\bigcap_{i\in\mathbb{Z}_k} X_i$ is a closed $(n+1-k)$-dimensional submanifold.

[1] Unless stated otherwise, all **manifolds** are piecewise-linear (PL), compact, connected, and orientable. A manifold X is **closed** if $\partial X = \varnothing$. A general reference is [RoSa82].

[2] A d-dimensional h-**handlebody** is a d-manifold obtained by gluing d-dimensional r-handles for various $r = 0, \ldots, h$. Since we work in the PL category, the gluing maps must be PL and the attaching regions must be PL submanifolds.

- $\bigcap_{i\in I} X_i$ is an $(n+1-|I|)$-dimensional $|I|$- (resp. $(|I|-1)$-) handlebody for each $I \subset \mathbb{Z}_k$ with $2 \leq |I| \leq k-1$.[3]

One may define *smooth* multisections of *smooth* manifolds analogously: the only extra condition is that for each nonempty $I \subset \mathbb{Z}_k$, the inclusion of $X_I = \bigcap_{i\in I} X_i$ into X is a smooth embedding, with corners.[4] Lambert-Cole–Miller proved that every smooth 5-manifold admits a smooth trisection [LCMi21]. In dimensions $n \geq 6$, the topic is wide open. In particular:

Question 1 Does every closed smooth manifold of arbitrary dimension admit a smooth multisection?

The distinction between PL multisections and smooth ones comes down to that of PL and smooth handle decompositions.[5] This is because any PL multisection $X = \bigcup_{i\in\mathbb{Z}_k} X_i$ gives rise to a nice PL handle decomposition (see Proposition 2.5) coming from handle decompositions of the various X_I; requiring each inclusion $X_I \hookrightarrow X$ to be smooth (with corners) ensures that the gluings in this handle decomposition are smooth. Henceforth, unless stated otherwise, all multisections are PL.

The topology of a closed manifold X of dimension $n \neq 2$ bounds the *efficiency* of its multisection as follows. Let $g(X_i)$ denote the **genus** of X_i.[6]

Definition 1.2 The **efficiency** of a multisection $X = \bigcup_{i\in\mathbb{Z}_k} X_i$ is

$$\frac{1 + \operatorname{rank} \pi_1(X)}{1 + \max_i g(X_i)}.$$

A multisection is **efficient** if its efficiency is 1.

[3] Rubinstein-Tillmann state this condition differently, requiring that each $\bigcap_{i\in I} X_i$ is an $(n+1-|I|)$-dimensional submanifold with an $|I|$- (resp. $(|I|-1)$-) dimensional spine, where a spine of a manifold N is a subpolyhedron $P \subset \operatorname{int}(N)$ onto which N collapses. Certainly any h-handlebody has an h-dimensional spine. Conversely, given a spine P of N, we may assume that N is triangulated and P is a simplicial subcomplex which admits no elementary collapses; then N is PL homeomorphic to a regular neighborhood R of P in N, and R has handle decomposition consisting of one r-handle for each r-simplex in P.

[4] More precisely, for nonempty $I \subset \mathbb{Z}_k$, the set of corner points of X_I must be:
$$\operatorname{corners}(X_I) = \bigcup_{i,j\notin I;\ i\neq j} X_I \cap X_i \cap X_j.$$

[5] Note that, while any smooth structure determines a (smooth) handle decomposition, and conversely, a PL handle decomposition does not necessarily determine a smooth structure.

[6] X_i is an n-dimensional 1-handlebody, so we have $X_i \cong \natural^g(S^1 \times D^{n-1})$ for some $g = g(X_i)$. (Throughout, we denote PL homeomorphism by $\cong$.)

We will show:

Corollary 2.7 *In any dimension $n \neq 2$, no multisection of any manifold has efficiency greater than 1, and in any efficient multisection $X = \bigcup_{i \in \mathbb{Z}_k} X_i$, all X_i have the same genus, $g(X_i) = \operatorname{rank} \pi_1(X)$.*[7]

This notion of an *efficient* multisection generalizes a notion introduced by Lambert-Cole–Meier in [LCMi21]. They call a trisection of a simply-connected 4-manifold X *efficient* if the genus of the central surface Σ equals $b_2(X)$. Indeed, one always has $g(\Sigma) \leq b_2(X)$, and equality holds if and only if each piece of the trisection is a 4-ball.

We close the introduction with an outline of the paper.

- §2 establishes several general properties of multisections.
- §3 begins a detailed investigation of multisections of *odd-dimensional tori*, starting with detailed descriptions the multisections of T^n for $n = 3, 4, 5$. Roughly stated, the main result is:

  Theorem 7.10 *Each $n = (2k-1)$-torus admits an efficient multisection which is symmetric with respect to the S_n permutation action on the indices and the $\mathbb{Z}_k$ translation action along the main diagonal.*

  The full version of Theorem 7.10 gives a simple expression (1) for each piece X_i of this multisection. The hard part is describing a handle decomposition of $X_I = \bigcap_{i \in I} X_i$ for arbitrary n and $I \subsetneq \mathbb{Z}_k$.
- §4 introduces three types of building blocks; under our main construction, each handle of each X_I will be a product of such blocks.
- §5 describes further examples of X_I under our construction, each featuring a new complication in its handle decomposition.
- §6 proves several combinatorial facts about our main construction. In particular, §6.2 proves that $T^n = \bigcup_{i \in \mathbb{Z}_k} X_i$, and §6.4 establishes a closed expression (2) for arbitrary X_I. Also, §6.3 establishes two combinatorial corollaries, which may be of independent interest.
- §7 describes a handle decomposition of arbitrary X_I from our main construction, confirms the details of this decomposition, shows that the central intersection $\bigcap_{i \in \mathbb{Z}_k} X_i$ is a closed k-manifold, and puts everything together to prove Theorem 7.10

[7] In dimension two, efficiency is strictly bounded above by 2; this bound is sharp, since any surface of even genus g admits a multisection with efficiency $\frac{1+2g}{1+g}$.

- §8 extends Theorem 7.10 to certain cubulated manifolds.
- Appendix 1 features tables, several detailing follow-up examples for the complications introduced in §§3 and 5, others detailing aspects of the handle decomposition described in §7.1.
- Appendix 2 describes four other ways one might try to multisect T^n.

Thank you to Mark Brittenham, Charlie Frohman, Hugh Howards, Peter Lambert-Cole, and Maggie Miller for helpful discussions. Thank you to the anonymous referee for numerous suggestions to improve the clarity and exposition of the paper. Special thank you to Alex Zupan for helpful discussions throughout the project, especially during its early stages, when we collaborated to find efficient trisections of T^4 and T^5.

2 Multisections and their efficiency

In this section, we describe a way of obtaining a (PL) handle decomposition of a manifold given a multisection (see Proposition 2.5), and we deduce, with the exception of 2-manifolds, that no multisection has efficiency greater than 1 (see Corollary 2.7). We begin, however, by describing examples of multisections in arbitrary dimension.

2.1 Simple examples of multisections

Example 2.1 For $n = 2k-1$, the n-sphere

$$S^n = \partial \prod_{i=0}^{k-1} D^2 = \bigcup_{i=0}^{k-1} \left(\prod_{j=0}^{i-1} D^2 \times S^1 \times \prod_{j=i+1}^{k-1} D^2 \right)$$

admits a multisection in which each

$$X_i = \prod_{j=0}^{i-1} D^2 \times S^1 \times \prod_{j=i+1}^{k-1} D^2$$

is an n-dimensional 1-handlebody of genus 1. In dimension 3, this is the genus 1 Heegaard splitting of $S^3 = D^2 \times D^2$ with central surface $S^1 \times S^1$. In arbitrary dimension n, the central intersection is the k-torus $\prod_{j=0}^{k-1} S^1$, and more generally, for each $I \subset \mathbb{Z}_k$ with $1 \leq |I| = \ell \leq k-1$, the intersection

$$X_I = \bigcap_{j\in I} X_i = \prod_{j=0}^{k-1} \begin{Bmatrix} S^1 & j \in I \\ D^2 & j \notin I \end{Bmatrix} \cong \prod_{j=0}^{\ell-1} S^1 \times \prod_{j=\ell}^{k-1} D^2 \cong T^\ell \times D^{2(k-\ell)}$$

is a thickened ℓ-torus. In dimension 5, Lambert-Cole–Miller use this construction and a second trisection of S^5, whose central intersection is a 3-sphere rather than a 3-torus, to show that, unlike Heegaard splittings of 3-manifolds and trisections of 4-manifolds, trisections of a given 5-manifold need not be stably equivalent [LCMi21].

Example 2.2 [RuTi20] Using homogeneous coordinates $[z_0 : \cdots : z_{k-1}]$ on $\mathbb{CP}^{k-1}$, one can define a multisection by

$$X_i = \{[z_0 : \cdots : z_{k-1}] \mid |z_i| \geq |z_j| \text{ for } j = 0, \ldots, k-1\} .$$

Then each X_I with $|I| = \ell$ is related by permutation to a thickened torus

$$\begin{aligned}\bigcap_{i=0}^{\ell-1} X_i &= \left\{ [1 : z_1 : \cdots : z_{k-1}] \;\middle|\; \begin{array}{l} |z_j| = 1 \text{ for } j = 1, \ldots, \ell-1, \\ |z_j| \leq 1 \text{ for } j = \ell, \ldots, k-1 \end{array} \right\} \\ &\cong T^{\ell-1} \times D^{2(k-\ell)}.\end{aligned}$$

In particular, the central intersection is the k-torus

$$\{[1 : z_1 : \cdots : z_{k-1}] : \ |z_1| = \cdots = |z_{k-1}| = 1\} .$$

These symmetric multisections are also efficient, since each X_i has genus 0.

2.2 General properties of multisections

Proposition 2.3 *Let Z_i be an n-dimensional h_i-handlebody, $i = 1, 2$, and let $\phi : Y_1 \to Y_2$ glue compact $Y_i \subset \partial Z_i$, such that $Y_1 \cong Y_2$ is an h-handlebody. Then $Z = Z_1 \cup_\phi Z_2$ is an h'-handlebody for $h' = \max\{h_1, h_2, h+1\}$.*

Proof By taking a regular neighborhood N of $Y = \phi(Y_1) = \phi(Y_2)$ in Z, where $N \equiv Y \times I$, we may identify $Z \setminus \mathrm{int}(N)$ with $Z_1 \sqcup Z_2$, which is a 2-component h''-handlebody where $h'' = \max\{h_1, h_2\}$. Then, for each i-handle $H \equiv D^i \times n-1-i$ in Y, $0 \leq i \leq h$, we can glue on $H \times I$ along $\partial(D^i \times I) \times D^{n-1-i} \cong S^i \times D^{n-1-i}$, and so attaching $H \times I$ is the same as attaching an $(i+1)$-handle, where $i+1 \leq h+1$. □

Proposition 2.4 *Let $X = \bigcup_{i \in \mathbb{Z}_k} X_i$ be a multisection of a closed manifold of dimension $n = 2k-1$ (resp. $n = 2k-2$). Then for each $1 \leq j \leq i \leq k-1$:*

$$\bigcup_{t=0}^{j-1} X_t \cap \bigcap_{t=j}^{i} X_t$$

is a $(2k+j-i-2)$-dimensional $(i+j)$-handlebody (resp. $(2k+j-i-3)$-dimensional $(i+j-1)$-handlebody).

Proof We address the odd-dimensional case, arguing by lexicographical induction on (i,j). The even-dimensional case follows analogously. When $(i,j) = (1,1)$, the proposition is true by definition, since $X_0 \cap X_1$ is a 2-handlebody.

Let $(i,j) > (1,1)$. Assume for each $(r,s) < (i,j)$ that $(X_0 \cup \cdots \cup X_{s-1}) \cap X_s \cap \cdots \cap X_r$ is a $(2k+s-r-2)$-dimensional $(r+s)$-handlebody. Let

$$Z_1 = \bigcup_{t=0}^{j-2} X_t \cap \bigcap_{t=j}^{i} X_t$$

and

$$Z_2 = \bigcap_{t=j-1}^{i} X_t,$$

so that

$$\bigcup_{t=0}^{j-1} X_t \cap \bigcap_{t=j}^{i} X_t = Z_1 \cup Z_2$$

Then, by induction, Z_1 is a $(2k+j-i-2)$-dimensional $(i+j-2)$-handlebody, and, by the definition of multisection, Z_2 is a $(2k+j-i-2)$-dimensional $(i+1-j)$-handlebody. Further,

$$Z_1 \cap Z_2 = \bigcup_{t=0}^{j-2} X_t \cap \bigcap_{t=j-1}^{i} X_t,$$

which, by induction, is a $(2k+j-i-3)$-dimensional $(i+j-1)$-handlebody. Therefore, by Proposition 2.3, $Z_1 \cup Z_2$ is a $(2k+j-i-2)$-dimensional h-handlebody, where

$$h = \max\{i+j-2, i+1-j, i+j\} = i+j. \qquad \square$$

Proposition 2.5 *Let $X = \bigcup_{i \in \mathbb{Z}_k} X_i$ be a multisection of a closed manifold of dimension $n = 2k-1$ (resp. $n = 2k-2$). Then X admits a handle decomposition in which each X_j contributes only r-handles for $r \leq 2j+1$ (resp. $r \leq 2j$).*

Proof We address the odd-dimensional case; the even-dimensional case follows analogously. Arguing by induction on i, we will show that $X_0 \cup \cdots \cup X_i$ admits a handle decomposition in which each X_j contributes only r-handles for $r \leq 2j+1$. The base case is trivial. For the induction step, consider

$$(X_0 \cup \cdots \cup X_{i-1}) \cup_{(X_0 \cup \cdots \cup X_{i-1}) \cap X_i} X_i.$$

By induction, $X_0 \cup \cdots \cup X_{i-1}$ admits a handle decomposition in which each X_j contributes only r-handles for $r \leq 2j+1$. Extend this to the required handle decomposition of $X_0 \cup \cdots \cup X_i$ as follows. Let N be a collared neighborhood of $(X_0 \cup \cdots \cup X_{i-1}) \cap X_i$ in X_i. As in the proof of Proposition 2.3, first construct the disjoint union

$$(X_0 \cup \cdots \cup X_{i-1}) \sqcup (X_i \setminus \operatorname{int}(N)),$$

thereby contributing 0- and 1-handles to X_i, as $X_i \setminus \operatorname{int}(N)$ is PL homeomorphic to X_i; second, glue in N, thereby contributing r-handles for $r = 1, \ldots, 2i+1$, since $(X_0 \cup \cdots \cup X_{i-1}) \cap X_i$ is a $2i$-handlebody by Proposition 2.4. □

2.3 Efficiency of multisections

Next, we consider the efficiency of multisections in light of Proposition 2.5. Recall Definition 1.2.

Proposition 2.6 *In dimension $n \neq 2$, any multisection $X = \bigcup_{i \in \mathbb{Z}_k} X_i$ obeys*

$$\min_{i \in \mathbb{Z}_k} g(X_i) \geq \operatorname{rank} \pi_1(X).$$

Proof Given a multisection of X, label the pieces so that $g(X_{k-1}) \leq g(X_i)$ for all i. Construct a handle structure on X as guaranteed by Proposition 2.5. All the n- and $(n-1)$-handles are in X_{k-1}, since $n \neq 2$. Flip X upside down. Now all the 0- and 1-handles are in X_{k-1}, so

$$\operatorname{rank} \pi_1(X) \leq \operatorname{rank} \pi_1(X_{k-1}) = g(X_{k-1}) = \min_{i \in \mathbb{Z}_k} g(X_i). \qquad \square$$

Corollary 2.7 *In any dimension $n \neq 2$, no multisection of any manifold has efficiency greater than 1, and in any efficient multisection $X = \bigcup_{i \in \mathbb{Z}_k} X_i$, all X_i have the same genus, $g(X_i) = \operatorname{rank} \pi_1(X)$.*

3 Motivating examples

In this section, we describe our multisections of T^3, T^4, and T^5 in detail. We also establish notation that will be used throughout the rest of the paper.

3.1 Intuitive approach to T^3, T^4, and T^5

Figure 1 illustrates an efficient Heegaard splitting of the 3-torus, which suggests viewing T^3 as $(\mathbb{R}/2\mathbb{Z})^3$; then the splitting is determined by a partition of the eight unit cubes

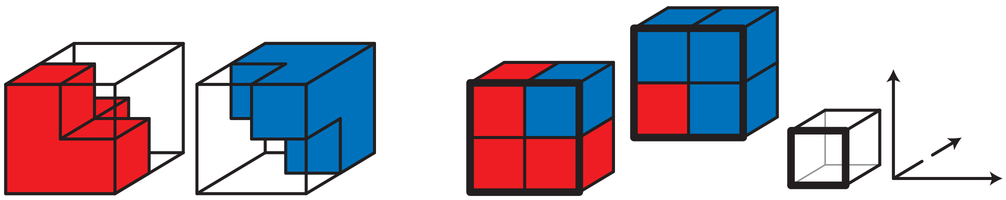

Figure 1: A Heegaard splitting of T^3.

with vertices in the lattice $(\mathbb{Z}/2\mathbb{Z})^3$. Moreover, *this* partition satisfies two symmetry properties: first, the permutation action of S_3 on the indices in T^3 fixes each piece of the splitting, and second, the $\mathbb{Z}_2$ translation action along the main diagonal of T^3 switches the two pieces: $X_i + (1,1,1) = X_{i+1}$.

How might one construct efficient trisections of T^n, $n = 4, 5$, with symmetry properties analogous to Figure 1's splitting of T^3? To begin, one might view these T^n as $(\mathbb{R}/3\mathbb{Z})^n$—rather than, say, $(\mathbb{R}/2\mathbb{Z})^n$, because we seek a trisection rather than a splitting—and seek an appropriate partition of the 3^n unit cubes with vertices in the lattice $(\mathbb{Z}/3\mathbb{Z})^n$. From now on, for brevity, we will refer to these unit cubes as *subcubes* of T^n.

To start forming this partition, one might assign each subcube $[i, i+1]^n$ to X_i (because of the translation action). Next, one might assign those subcubes of the forms $[i, i+1]^{n-1}[i+1, i+2]$ and $[i, i+1]^{n-1}[i-1, i]$ to X_i as well, and extend these assignments using the permutation action on the indices. At this point, each X_i is indeed an n-dimensional 1-handlebody, and so the rest of the partition should be constructed in a way that preserves this fact, while also giving rise to the needed intersection properties. Figure 2 illustrates this intermediate stage in the case of T^4.[8]

For T^4, the symmetry properties imply that the remaining partition is determined by the assignments of the subcubes $[0,1]^2[1,2][2,3]$ and $[0,1]^2[1,2]^2$. Assigning both subcubes to X_0 and extending symmetrically gives the decomposition of T^4 illustrated

[8]All combinatorial data conveyed in Figures 2–3 comes from the arrangements of the nine 3×3 squares outlined in bold; beyond this, the style of the illustration reflects the fact that each pictured subcube is a 4-cube. A model 4-cube is also drawn, next to coordinate axes. The solid axes represent directions in which abutting subcubes are shown in contact with each other (understanding that the interval that appears as $[0, 3]$ actually represents the circle $\mathbb{R}/3\mathbb{Z}$); the dashed axes represent directions in which abutting subcubes align at a distance in the figure. Similarly, Figure 1 shows a model 3-cube and coordinate axes, Figure 5 a model 5-cube and coordinate axes.

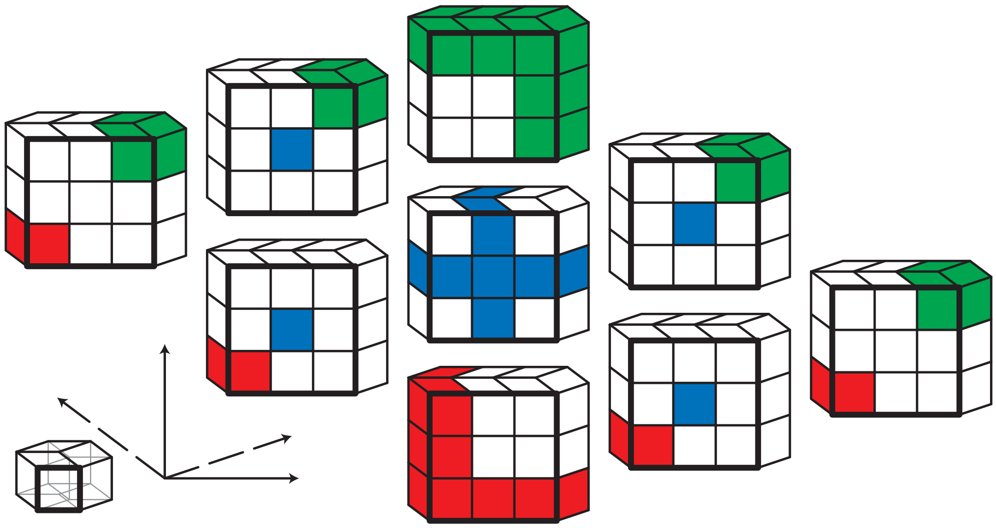

Figure 2: Start partitioning the subcubes of $T^4 = (\mathbb{R}/3\mathbb{Z})^4$ like this, giving three 4-dimensional 1-handlebodies.

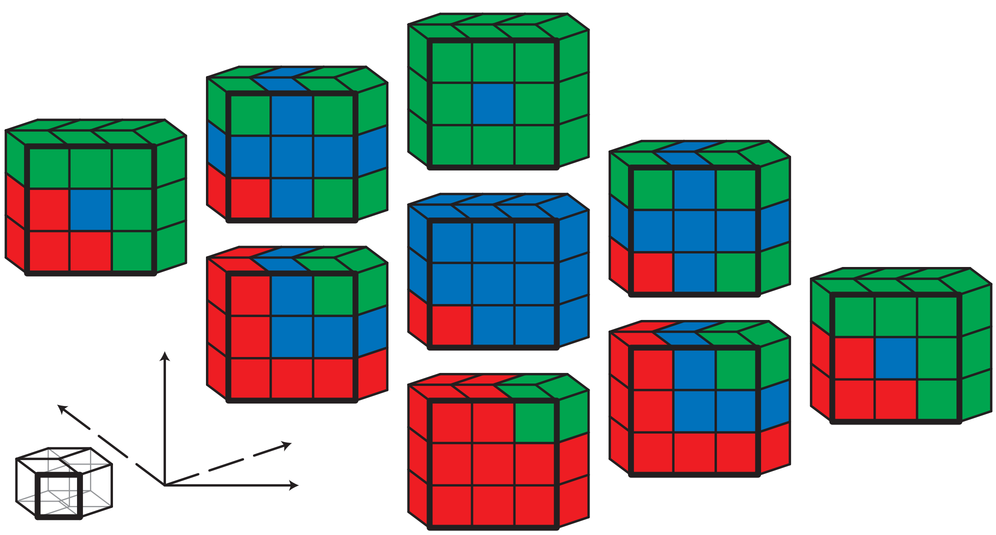

Figure 3: Partitioning the 3^4 subcubes of $T^4 = (\mathbb{R}/3\mathbb{Z})^4$ like this gives a symmetric efficient trisection.

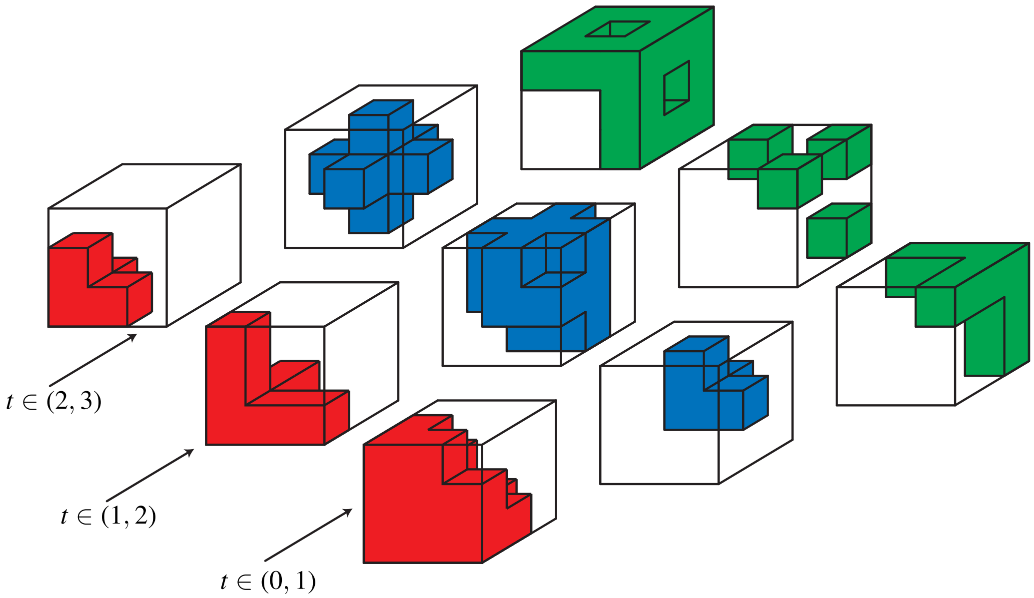

Figure 4: In the multisection of T^4 from Figure 3, each slice $T^3 \times \{t\}$, $t \in (\mathbb{R}/3\mathbb{Z}) \setminus \mathbb{Z}_3$, intersects X_0, X_1, X_2 like this.

in Figures 3 and 4. Section 3.3 will confirm that this decomposition is indeed a trisection.

A similar approach leads to the decomposition of T^5 shown in Figure 5. Section 3.4 will confirm that this, too, is a trisection.

3.2 Notation

Notation 3.1 Let $X, Y \subset Z$ be compact subspaces of a topological space. Denote "X cut along Y" by $X \setminus\setminus Y$. In every example where we use this notation, $X \setminus\setminus Y$ equals the closure of $X \setminus Y$ in Z. (The general construction is somewhat more complicated.)

Given $n = 2k - 1, 2k - 2$, view the n-torus T^n as $(\mathbb{R}/k\mathbb{Z})^n$. Let S_n denote the permutation group on n elements.

Notation 3.2 Given $\vec{x} = (x_1, \ldots, x_n) \in T^n$ and $\sigma \in S_n$, denote

$$\vec{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

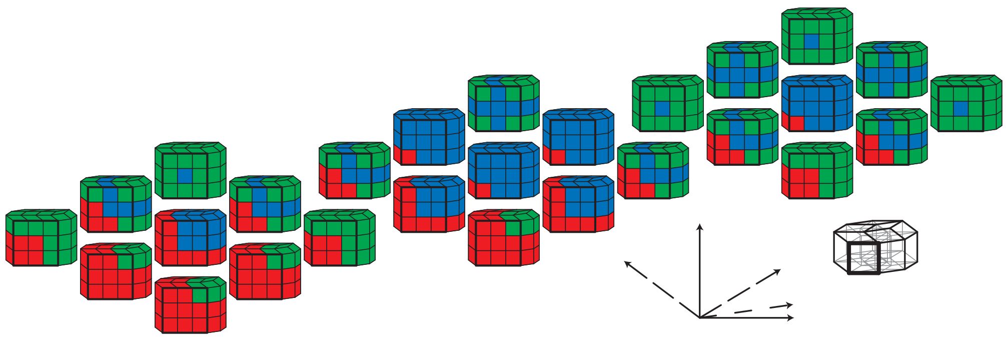

Figure 5: Partitioning the 3^5 subcubes of $T^5 = (\mathbb{R}/3\mathbb{Z})^5$ like this gives a symmetric efficient trisection.

Also, given $U \subset T^n$ and $\vec{v} \in T^n$, denote

$$U + \vec{v} = \{\vec{u} + \vec{v} : \ \vec{u} \in U\}.$$

The symmetric group S_n acts on T^n by permuting the indices, $\sigma : \vec{x} \mapsto \vec{x}_\sigma$. Because we are interested in subsets of T^n which are fixed by this action:

Notation 3.3 For any subset $U \subset T^n$, denote

$$\langle U \rangle = \{\vec{x}_\sigma : \vec{x} \in U, \ \sigma \in S_n\} \subset T^n.$$

Note, for any $U \subset T^n$, that $\langle U \rangle$ is fixed by the action of S_n on T^n. We can state our main result explicitly:

Theorem 7.10 *For $n = 2k - 1$, the n-torus $T^n = (\mathbb{R}/k\mathbb{Z})^n = [0, k]^n/\sim$ admits an efficient multisection $T^n = \bigcup_{i \in \mathbb{Z}_k} X_i$ defined by*

(1)
$$\begin{aligned} X_0 &= \left\langle [0,1]^2 \cdots [0, k-1]^2 [0, k] \right\rangle, \\ X_i &= X_0 + (i, \ldots, i), \ i \in \mathbb{Z}_k. \end{aligned}$$

By construction, the decomposition is symmetric with respect to the permutation action on the indices and the translation action on the main diagonal.

Anticipating the concrete and (somewhat) low-dimensional nature of the examples in §§3, 5 and Appendix 1, we give the first few intervals $[i, i+1]$, $i \in \mathbb{Z}_k$, special notations:

Notation 3.4 Denote

$$[0,1]=\alpha,\ [1,2]=\beta,\ [2,3]=\gamma,[3,4]=\delta,[4,5]=\varepsilon,\ [5,6]=\zeta,\ [6,7]=\eta.$$

To further abbreviate our notation, we often omit $\times$ symbols and use exponents to denote repeated factors. For example, we can describe the two pieces of the Heegaard splitting of T^3 from Figure 1 like this:

$$X_0=\alpha^3\cup\alpha^2\beta\cup\alpha\beta\alpha\cup\beta\alpha^2, \qquad X_1=\beta^3\cup\beta^2\alpha\cup\beta\alpha\beta\cup\alpha\beta^2.$$

Using Notation 3.3, we can further abbreviate this notation:

$$\begin{aligned} X_0&=\alpha^3\cup\left\langle\alpha^2\beta\right\rangle & X_1&=\beta^3\cup\left\langle\alpha\beta^2\right\rangle \\ &=\left\langle\alpha^2[0,2]\right\rangle & &=\left\langle[0,2]\beta^2\right\rangle. \end{aligned}$$

We often omit the braces around singleton factors. For example, in T^3:

$$\begin{aligned} X_0\cap X_1&=\langle[0,1]\times[1,2]\times\{0\}\rangle\cup\langle[0,1]\times[1,2]\times\{1\}\rangle \\ &=\langle\alpha\beta 0\rangle\cup\langle\alpha\beta 1\rangle. \end{aligned}$$

We also extend Notation 3.3 in the way suggested by the following example:

$$\langle 0\alpha\rangle\,\beta^2=\big(\{0\}\times\alpha\times\beta\times\beta\big)\cup\big(\alpha\times\{0\}\times\beta\times\beta\big).$$

More precisely, if we decompose T^n as a product $T^n=T^{n_1}\times\cdots\times T^{n_p}$ and $U_i\subset T^{n_i}$ for $i=1,\dots,p$, then

$$\langle U_1\rangle\cdots\langle U_p\rangle=\left\{(\vec{x}^1_{\sigma_1},\vec{x}^2_{\sigma_2},\dots,\vec{x}^p_{\sigma_p}):\ \vec{x}^i\in T^{n_i},\ \sigma_i\in S_{n_i},\ i=1,\dots,p\right\}$$

where, extending Notation 3.2 and denoting $\vec{x}^i=(x^i_1,\dots,x^i_{n_i})$, each

$$\vec{x}^i_{\sigma_i}=\left(x^i_{\sigma_i(1)},\dots,x^i_{\sigma_i(n_i)}\right).$$

Starting in dimension 7, some handle decompositions will require subdividing unit subintervals $\alpha,\beta,\gamma,\delta,\dots$ into halves or thirds. Anticipating this:

Notation 3.5 Denote

$$\alpha^-=\left[0,\tfrac{1}{2}\right],\ \alpha^+=\left[\tfrac{1}{2},1\right],\dots,\left[\eta^-\right]=\left[6,\tfrac{13}{2}\right],\eta^+=\left[\tfrac{13}{2},7\right]$$

and

$$\alpha_3^-=\left[0,\tfrac{1}{3}\right],\alpha_3^\circ=\left[\tfrac{1}{3},\tfrac{2}{3}\right],\ \alpha_3^+=\left[\tfrac{2}{3},1\right],\dots,\eta_3^\circ=\left[\tfrac{19}{3},\tfrac{20}{3}\right],\ \eta_3^+=\left[\tfrac{20}{3},7\right].$$

Because of the symmetry of our main construction under the $\mathbb{Z}_k$ translation action on T^n, it will suffice, when considering X_I from that construction, to allow I to be arbitrary *only up to cyclic permutation*. In order to utilize this convenience:

Notation 3.6 Given $I \subset \mathbb{Z}_k$ with $|I| = \ell > 0$, denote $X_I = \bigcap_{i \in I} X_i$, and denote $I = \{i_s\}_{s \in \mathbb{Z}_\ell}$ such that

$$0 \leq i_0 < i_1 < \cdots < i_{\ell-1} \leq k - 1.$$

Definition 3.7 Let $I = \{i_s\}_{s \in \mathbb{Z}_\ell}$ as in Notation 3.6. For each $r \in \mathbb{Z}_\ell$, define $I^r = \{i + r : \ i \in I\} \subset \mathbb{Z}_k$. Denote each $I^r = \{i^r_s\}_{s \in \mathbb{Z}_\ell}$ with $0 \leq i^r_0 < i^r_1 < \cdots < i^r_{\ell-1} \leq k - 1$. Say that I is **simple** if, for each $r \in \mathbb{Z}_\ell$, we have $I \leq I^r$ under the lexicographical ordering of their elements, i.e. if each $I_r \neq I$ has some $s \in \mathbb{Z}_\ell$ with $i_t = i^r_t$ for each $t = 0, \ldots, s-1$ and $i_s < i^r_s$.

Notation 3.8 Given simple $I = \{i_s\}_{s \in \mathbb{Z}_\ell} \subsetneqq \mathbb{Z}_k$ as in Notation 3.6, define

$$T = \{s \in \mathbb{Z}_\ell : \ i_s - 1 \notin I\}.$$

Denote $T = \{t_r\}_{r \in \mathbb{Z}_m}$ with $0 = t_0 < \cdots < t_m < \ell$ (see Observation 3.10). For each $r \in \mathbb{Z}_m$, denote $I_r = \{i_{t_r}, \ldots, i_{t_{r+1}-1}\}$. Then

$$I = I_1 \sqcup \cdots \sqcup I_m,$$

and for each $r = 0, \ldots, m-1$, we have $|I_r| = \max I_r + 1 - \min I_r$ (each block I_r is comprised of consecutive indices) and $\min I_{r+1} \geq \max I_r + 2$ (the blocks are nonconsecutive).

Given $i_* \in I$ (denoted specifically as i_*), denote the block I_r containing i_* by I_*.

Convention 3.9 Throughout, reserve the notations n, k, $\alpha, \ldots, \eta$, $\alpha^-, \ldots, \eta^+$, $\alpha^-_3, \ldots, \eta^+_3$, I, X_I, ℓ, T, and m for the way they are used in Notations 3.4-3.8; assume, unless otherwise stated, that $I \subset \mathbb{Z}_k$ is simple; and reserve, for any $s \in \mathbb{Z}_\ell$ or $r \in \mathbb{Z}_m$, the notations i_s, t_r, I_r, i_*, and I_* for the way they are used in Notations 3.6 and 3.8.

Observation 3.10 *Given $I \subsetneqq \mathbb{Z}_k$, we have $i_0 = 0$, $i_{\ell-1} \leq k-2$, and $|I_0| \geq |I_r|$ for each $r \in \mathbb{Z}_m$; if $|I_0| = |I_r|$, then $|I_1| \geq |I_{r+1}|$.*

Given $I \subset \mathbb{Z}_k$ and $s \in \mathbb{Z}_\ell$, denote

$$(i_1, \ldots, \widehat{i_s}, \ldots, i_\ell) = (i_1, \ldots, i_{s-1}, i_{s+1}, \ldots, i_\ell) \subset T^{\ell-1}.$$

We now have enough notation to describe a closed formula for the X_I coming from our main construction (1):

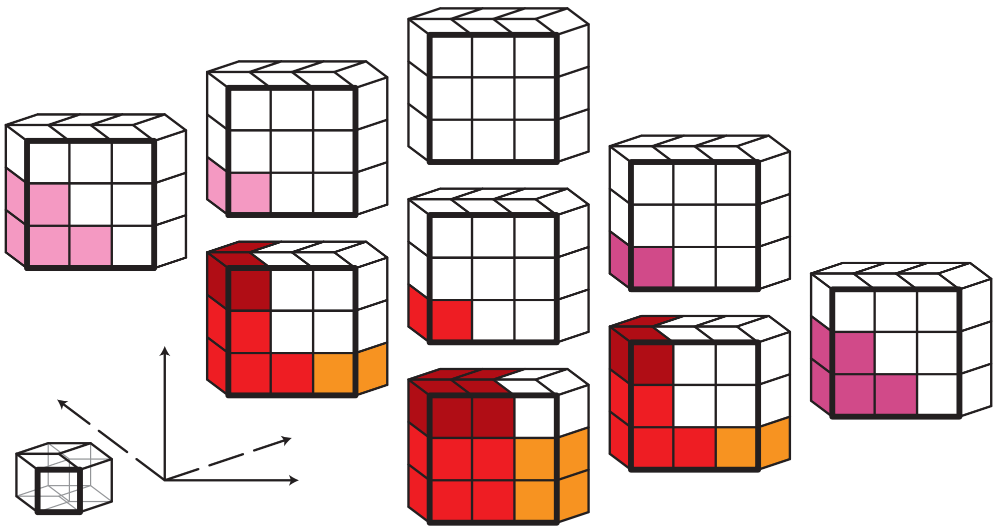

Figure 6: A handle decomposition of X_0 in Figure 3's trisection of T^4. Each of the five handles has a different color.

Lemma 6.13 *Given nonempty $I \subsetneqq \mathbb{Z}_k$, X_I is given by:*

$$\bigcup_{i_* \in I} \left\langle (i_1, \ldots, \widehat{i_*}, \ldots, i_\ell) \prod_{r \in \mathbb{Z}_\ell} [i_r, i_r + 1]^2 \cdots [i_r, i_{r+1} - 1]^2 [i_r, i_{r+1}] \right\rangle. \tag{2}$$

In particular,

$$\bigcap_{i \in \mathbb{Z}_k} X_i = \bigcup_{i_* \in \mathbb{Z}_k} \left\langle (0, \ldots, \widehat{i_*}, \ldots, k-1) \prod_{i \in \mathbb{Z}_k} [i, i+1] \right\rangle. \tag{3}$$

We will prove Lemma 6.13 in §6.4.

3.3 Trisection of T^4

The decomposition of T^4 from Figure 3 is given by

$$\begin{aligned} X_0 = \langle \alpha^2 [0,2][0,3] \rangle &= \langle \alpha^4 \rangle \cup \langle \alpha^3 \beta \rangle \cup \langle \alpha^3 \gamma \rangle \cup \langle \alpha^2 \beta^2 \rangle \cup \langle \alpha^2 \beta \gamma \rangle \\ X_i &= X_0 + (i,i,i,i). \end{aligned} \tag{4}$$

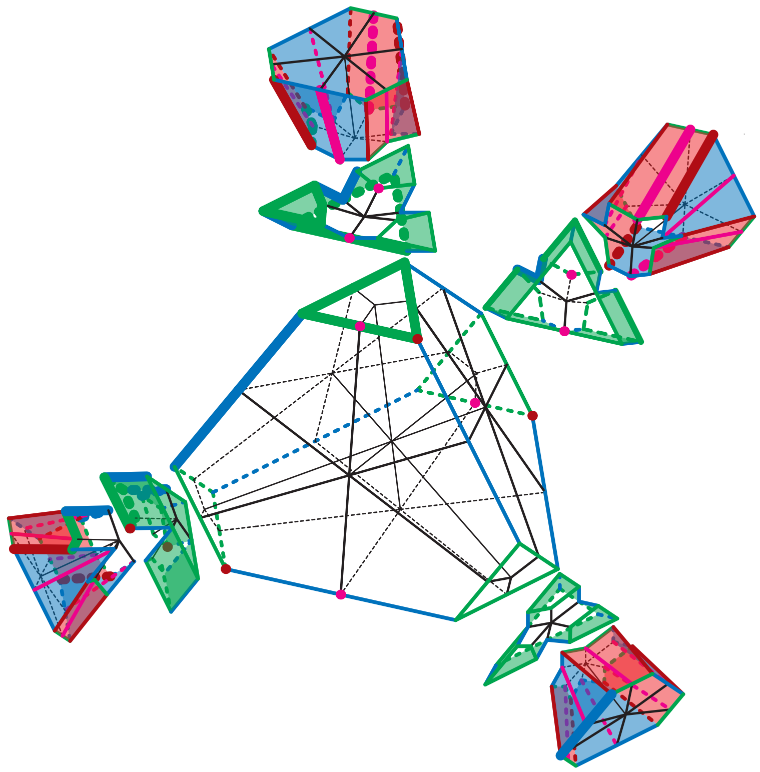

Figure 7: A handle decomposition of $X_0 \cap X_1$ in our trisection of T^4. The trisection diagram on $\partial(X_0 \cap X_1) = X_0 \cap X_1 \cap X_2 = \langle \alpha\beta 02 \rangle \cup \langle \alpha\gamma 12 \rangle \cup \langle \beta\gamma 01 \rangle$ has two types of red curves; one of each is bold. Same with blue and green.

It is evident from Figure 3 that $X_0 \cup X_1 \cup X_2 = T^4$. Also, $I = \{0\}$ and $I = \{0, 1\}$ are the only proper subsets of $\{0, 1, 2\}$ which are simple. Therefore, in order to check that (4) determines a trisection of T^4, it suffices to prove that X_0 is a 4-dimensional 1-handlebody and $X_0 \cap X_1$ is a 3-dimensional 1-handlebody with $\partial(X_0 \cap X_1) = X_0 \cap X_1 \cap X_2$.

Indeed, Figure 6 shows a handle decomposition of X_0 in which $\langle \alpha^2[0,2]^2 \rangle$ is a 0-handle and $\langle \alpha^2[0,2]\gamma \rangle$ supplies four 1-handles, each a permutation of $\langle \alpha^2[0,2] \rangle \gamma$. More precisely, each 1-handle is given, in terms of some permutation $\sigma \in S_4$ (using Notation 3.2), by

$$\{ \vec{x}_\sigma : \vec{x} \in \langle \alpha^2[0,2] \rangle \gamma \} . \tag{5}$$

Now consider

(6) $$X_0 \cap X_1 = \langle \alpha 1 \beta[1,3] \rangle \cup \left\langle 0 \alpha \beta^2 \right\rangle .$$

We claim that this is a 3-dimensional 1-handlebody in which:

- $Y_1 = \left\langle \alpha 1 \beta^2 \right\rangle$ is the 0-handle;
- $Y_2 = \left\langle 0 \alpha \beta^2 \right\rangle$ gives six 1-handles, all permutations of $Y_2^* = \langle 0\alpha \rangle \beta^2$;
- $Y_3 = \langle \alpha 1 \beta \gamma \rangle$ gives four 1-handles, all permutations of $Y_3^* = \langle \alpha 1 \beta \rangle \gamma$.

Figure 7 shows this decomposition of $X_0 \cap X_1$:

- The shape in the center (which looks like a truncated tetrahedron) is the 0-handle $\left\langle \alpha 1 \beta^2 \right\rangle$, comprised of 12 cubes, each a permutation of $\alpha 1 \beta^2$ (c.f. (5) and the paragraph before it). The interior lattice point is $(1,1,1,1)$, and each triangular-looking face is a permutation of $0 \left\langle 1 \beta^2 \right\rangle$ (again, c.f. (5)). Each blue segment on $\partial \left\langle \alpha 1 \beta^2 \right\rangle$ is a permutation of $\langle \alpha 1 \rangle 2^2$.
- Each of the four three-pronged pieces is a permutation of $0 \left\langle \alpha \beta^2 \right\rangle$, glued to the 0-handle along $0 \left\langle 1 \beta^2 \right\rangle$. The twelve cubes comprising these pieces are then glued in pairs: $0 \alpha \beta^2$ and $\alpha 0 \beta^2$, e.g., meet along the face $00\beta^2$, and the other pairs are permutations of this. The union of each pair of cubes, (a permutation of) $Y_2^* = \langle 0 \alpha \rangle \beta^2$, is a 1-handle which is glued to the 0-handle along (the corresponding permutation of) $\langle 01 \rangle \beta^2$. Note that Y_2^* intersects other permutations of Y_2^*, but only within $Y_2^* \cap Y_1$. Therefore, attaching Y_2^* to Y_1 amounts to attaching six 1-handles.
- Each of the four remaining pieces is a permutation of $Y_3^* = \langle \alpha 1 \beta \rangle \gamma$ and attaches to Y_1 and Y_2, respectively, along (the corresponding permutations of) $\langle \alpha 1 \beta \rangle 2 \subset \left\langle \alpha 1 \beta^2 \right\rangle$ and $\langle \alpha 1 \beta \rangle 0 \subset \left\langle \alpha \beta^2 \right\rangle 0$.

For emphasis, here are some key details of this decomposition which will be instructive toward the odd-dimensional case (we will justify some of these details in §4):

$$Y_1 = Y_1^* = \left\langle \alpha 1 \beta^2 \right\rangle \cong D^3,$$

so Y_1 is a 0-handle;

$$Y_2^* = \langle 0 \alpha \rangle \beta^2 \cong D^1 \times D^2 \text{ and}$$
$$Y_2^* \cap (Y_2 \setminus \setminus Y_2^*) \subset Y_2^* \cap Y_1 = \big(\partial \langle 0 \alpha \rangle\big) \times \beta^2 = \langle 01 \rangle \beta^2 \cong S^0 \times D^2,$$

so attaching Y_2 to Y_1 amounts to attaching a collection of 1-handles; and

$$Y_3^* = \langle \alpha 1 \beta \rangle \gamma \cong D^2 \times D^1 \text{ and}$$
$$Y_3^* \cap (Y_3 \setminus \setminus Y_3^*) \subset Y_3^* \cap (Y_1 \cup Y_2) = \langle \alpha 1 \beta \rangle \times \partial \gamma \cong D^2 \times S^0,$$

so attaching Y_3 to $Y_1 \cup Y_2$ amounts to attaching a collection of 1-handles. Thus, $X_0 \cap X_1$

is a 4-dimensional 1-handlebody. Note in Figure 7 that $\partial(X_0 \cap X_1)$ is the central surface

$$X_0 \cap X_1 \cap X_2 = \langle \alpha\beta 02 \rangle \cup \langle \alpha\gamma 12 \rangle \cup \langle \beta\gamma 01 \rangle \,, \tag{7}$$

which is colored in Figure 7 according to the color scheme from (7). Moreover, the red (resp. blue, green) line segments in Figure 7 comprise the "red (resp. blue, green) curves" in a trisection diagram for this trisection, and so Figure 7 *is*, in fact, a trisection diagram (see [GaKi16, MeScZu16]).

Note that what we have actually shown is that Figures 3, 4, and 7 give a combinatorial description of an efficient trisection of T^4. Thus, since the PL and smooth categories coincide in dimension 4, T^4 has a smooth structure for which we have described a trisection. Most likely, this is the standard smooth structure on T^4, but we have not yet proven this, nor will we in this paper.

One way to prove this would be to describe a (smooth=PL) isotopy (i.e. a sequence of handleslides on the central surface) between our trisection and another trisection of the standard T^4, such as either of those due to Koenig or Williams, the former obtained by viewing T^4 as $T^3 \times S^1$ [Ko21], the latter by viewing T^4 as $T^2 \times T^2$ [Wi20]. There may well be isotopies between our constructions are theirs, but attempting to construct such isotopies explicitly is messy, in part because the central surface has genus 10, and so it remains an open question as to whether or not *all* efficient trisections of T^4 are mutually isotopic. In other words does the following theorem, proven using minimal surface theory, extend to dimension four?

Theorem 3.11 (Frohman [Fr86]) *Up to isotopy, T^3 has a unique minimal genus Heegaard splitting.*

Question 2 Up to isotopy, does T^4 have a unique efficient trisection?

Question 3 Does T^4 admit exotic smooth structures? If it does, then which of these exotic structures are compatible with efficient trisections?

3.4 Trisection of T^5

The decomposition of T^5 from Figure 5 is given by

$$X_0 = \left\langle \alpha^2[0,2]^2[0,3] \right\rangle \,, \; X_i = X_0 + (i,i,i,i,i). \tag{8}$$

The handle decompositions of X_I, $I = \{0\}, \{0,1\}$, are quite similar to those from T^4. Focus first on $I = \{0\}$, i.e. on the handle decomposition of X_0. Note the single factor of $[0,3]$ in (8). As we will explain shortly, the handle decomposition of X_0 here comes from the decomposition of the interval

$$[0,3] = [0,2] \cup \gamma,$$

and likewise for X_0 from the trisection of T^4. These handle decompositions appear in Tables 1 and 2. These and subsequent tables are organized as follows. In each z^{th} row, Y_z is a union of handles of index h, Y_z^* is an *example* of such an h-handle, and the entry in the column **glue to** lists those indices z' for which Y_z^* glues to $Y_{z'}$ along at least one face of codimension 1. The other handles from Y_z are related to Y_z^* by permutation; for details, see §7.1.5.

Y_z	Y_z^*	h	z	glue to
$\langle \alpha^2[0,2]^2 \rangle$	$\langle \alpha^2[0,2]^2 \rangle$	0	1	
$\langle \alpha^2[0,2]\gamma \rangle$	$\langle \alpha^2[0,2] \rangle \gamma$	**1**	2	1

Table 1: X_0 from the trisection of T^4.

J	Y_z	Y_z^*	h	z	glue to
$\varnothing$	$\langle \alpha^2[0,2]^3 \rangle$	$\langle \alpha^2[0,2]^3 \rangle$	0	1	
$\{0\}$	$\langle \alpha^2[0,2]^2\gamma \rangle$	$\langle \alpha^2[0,2]^2 \rangle \gamma$	**1**	2	1

Table 2: X_0 from the trisection of T^5

Note in both Tables 1 and 2 that $Y_1 = Y_1^*$ is star-shaped in a particularly nice way (more detail to come in §4), hence is a ball which we may view as a 0-handle. Then Y_2^* is the product of the same sort of star-shaped ball with the interval γ and glues to Y_1 along the product of that ball with $\partial\gamma$. The red γ here, and all red henceforth, indicates a positive contribution to the handle index h.

Next, consider X_I, $I = \{0,1\}$ from T^4 and T^5. Similarly to the former (recall (6)), the latter is given by

$$X_0 \cap X_1 = \langle \alpha 1 \beta^2[1,3] \rangle \cup \langle 0\alpha\beta^2[1,3] \rangle . \tag{9}$$

Handle decompositions are summarized in Tables 3 and 4, which are organized largely the same way as Tables 1 and 2.

Regarding the first columns of Table 4, each Y_z there corresponds to a pair (J, i_*), where $J \subset \{\min I_r\} = \{0\}$[9] and $i_* \in I = \{0,1\}$. For details on this correspondence,

[9] Recall from Notation 3.8 that $\{\min I_r\} = \{i_t : t \in T\} = \{i_t \in I : i_t - 1 \notin I\}$, so e.g. $\{\min I_r\} = \{0\}$ if $I = \{0\}$, $I = \{0,1\}$ or $I = \{0,1,2\}$, and $\{\min I_r\} = \{0,2\}$ if $I = \{0,2\}$.

Y_z	Y_z^*	h	z	glue to
$\langle \alpha 1 \beta^2 \rangle$	$\langle \alpha 1 \beta^2 \rangle$	0	1	
$\langle 0 \alpha \beta^2 \rangle$	$\langle 0 \alpha \rangle \beta^2$	1	2	1
$\langle \alpha 1 \beta \gamma \rangle$	$\langle \alpha 1 \beta \rangle \gamma$	1	3	1,2

Table 3: From the trisection of T^4: $X_0 \cap X_1 = \langle \alpha 1 \beta [1,3] \rangle \cup \langle 0 \alpha \beta^2 \rangle$.

J	i_*	Y_z	Y_z^*	h	z	glue to
$\varnothing$	0	$\langle \alpha 1 \beta^3 \rangle$	$\langle \alpha 1 \beta^3 \rangle$	0	1	
	1	$\langle 0 \alpha \beta^3 \rangle$	$\langle 0 \alpha \rangle \beta^3$	1	2	1
$\{0\}$	0	$\langle \alpha 1 \beta^2 \gamma \rangle$	$\langle \alpha 1 \beta^2 \rangle \gamma$	1	3	1,2
	1	$\langle \gamma 0 \alpha \beta^2 \rangle$	$\langle \gamma 0 \alpha \rangle \beta^2$	**2**	4	2,3

Table 4: From the trisection of T^5: $X_0 \cap X_1 = \langle \alpha 1 \beta^2 [1,3] \rangle \cup \langle 0 \alpha \beta^2 [1,3] \rangle$.

see §7.1.2.

3.5 The difficulty with T^6

Suppose we try to quadrisect T^6 in the same way, viewing T^6 as $(\mathbb{R}/4\mathbb{Z})^6 = [0,4]^6/\sim$ and partitioning the 4^6 resulting subcubes into four classes. The first problem is that no such partition is symmetric with respect to both the permutation action of $\mathbb{Z}_6$ on the indices and the translation action of $\mathbb{Z}_4$ along the main diagonal. To see this, consider the subcube $\alpha^3\gamma^3$. The problem is that

$$\alpha^3\gamma^3 + (2,2,2,2,2,2) = \gamma^3\alpha^3 \subset \langle \alpha^3\gamma^3 \rangle .$$

Fundamentally, the problem is that $k = 4$ and $n = 2k-2 = 6$ are not relatively prime. (In odd dimensions, this trouble does not arise, since k and $2k-1$ are relatively prime.) Perhaps there is a less symmetric way to partition the subcubes of $[0,4]^6/\sim$ which gives a quadrisection of T^6, but trial and error suggests to the author that this is unlikely.

Conjecture 4 No partition of the subcubes of $[0,4]^6/\sim$ gives a quadrisection of T^6.

Question 5 Does the 6-dimensional torus admit an *efficient* quadrisection?

4 Star-shaped building blocks

This section introduces three types of building blocks, each of which is PL homeomorphic to a ball.[10] In §7, when we describe and then justify the handle decomposition of arbitrary X_I in arbitrary odd dimension, this will be particularly helpful. The main idea is that we will decompose arbitrary X_I into many pieces. Each piece will be a product of such building blocks, hence PL homeomorphic to a ball (see Lemma 7.6). Of course, we will still need to describe how all these balls are glued together and explain why this gives a handle decomposition.

In fact, we saw all three types of building blocks in §3. For example, denoting PL homeomorphism by $\cong$, the factors $\alpha^2[0,2]\cong D^3$, $\alpha^2[0,2]^2\cong D^4$, and $\alpha^2[0,2]^3\cong D^5$ from Tables 1 and 2 are examples of the first type of building block; see (10). The factor $\langle 0\alpha\rangle\cong D^1$ of Y_2^* in Tables 3 and 4 is an example of the second type of building block; see (11). The factor $\langle\gamma 0\alpha\rangle\cong D^2$ from Y_4^* in Table 4 is an example of the third type, as are those factors $\langle\alpha 1\beta^r\rangle\cong D^{r+1}$, which appear four places in Tables 3 and 4.

Given $\vec{p},\vec{q}\in\mathbb{R}^n$, denote the convex hull of $\{\vec{p},\vec{q}\}$ by

$$\left[\vec{p},\vec{q}\right]=\{t\vec{p}+(1-t)\vec{q}:\ 0\leq t\leq 1\}\,.$$

Let $\vec{p}\in Y\subset\mathbb{R}^n$. Define the *scope* of $\vec{p}$ in Y to be the largest star of $\vec{p}$ in Y:

$$\mathrm{scope}(Y;\vec{p})=\{\vec{q}\in Y:\ \left[\vec{p},\vec{q}\right]\subset Y\}.$$

Say that Y is *star-shaped* about $\vec{p}$ if $Y=\mathrm{scope}(Y;\vec{p})$. The *link* of $\vec{p}$ in Y is

$$\mathrm{lk}_Y(\vec{p})=\left\{\vec{v}\in\mathbb{R}^n:\ |\vec{v}|=1,\ \left[\vec{p},\vec{p}+\varepsilon\vec{v}\right]\subset Y\text{ for some }\varepsilon>0\right\}.$$

Thus, Y is a d-dimensional PL submanifold of $\mathbb{R}^n$ near $\vec{p}$ if and only if either

- $\mathrm{lk}_Y(\vec{p})\cong S^{d-1}$, in which case $\vec{p}$ is in the *interior* of Y; or
- $\mathrm{lk}_Y(\vec{p})\cong D^{d-1}$, in which case $\vec{p}\in\partial Y$.

Suppose $Y=\mathrm{scope}(Y;\vec{p})$ and $\mathrm{lk}_Y(\vec{p})\cong S^{d-1}$, so Y is star-shaped about $\vec{p}$ and is a PL d-submanifold of $\mathbb{R}^n$ near $\vec{p}$. In this situation, we say Y is *strongly star-shaped* about $\vec{p}$ if moreover, for every point $\vec{q}\in Y$, every point $\vec{x}\in\left[\vec{p},\vec{q}\right]\setminus\{\vec{q}\}$ satisfies $\mathrm{lk}_Y(\vec{x})\cong S^{d-1}$. This extra requirement implies that, for each $\vec{q}\in\mathrm{link}_Y(\vec{p})$, the ray from $\vec{p}$ through $\vec{q}$ contains at most one point of ∂Y. Moreover:

[10] Note that, in the PL category, an n-ball D^n is any manifold PL homeomorphic to the standard n-simplex, and an n-sphere S^n is any manifold PL homeomorphic to ∂D^n.

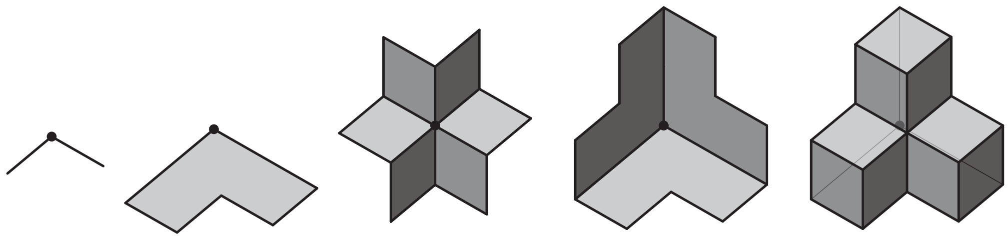

Figure 8: Left to right: $\langle 0\alpha\rangle$, $\langle\alpha[0,2]\rangle$, $\langle\alpha 1\beta\rangle$, $\langle 0\alpha[0,2]\rangle$, $\left\langle\alpha^2[0,2]\right\rangle$.

Proposition 4.1 *If $Y \subset \mathbb{R}^n$ is compact and strongly star-shaped about $\vec{p} \in Y$, then Y is PL homeomorphic to a compact ball.*

Proof By definition, there is a PL homeomorphism $\phi : S^{d-1} \to \mathrm{lk}_Y(\vec{p})$. There is also a map $\psi : Y \setminus \{\vec{p}\} \to \mathrm{lk}_Y(\vec{p})$ given by $\psi : \vec{q} \mapsto \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|}$.[11] Denote the restriction $\psi|_{\partial Y}$ by Ψ. The assumptions that Y is compact and *strongly* star-shaped about $\vec{p}$ imply that Ψ has a well-defined, continuous inverse map, hence is a PL homeomorphism. Define a polar coordinate system $\Phi : Y \to D^d$ by $\Phi : \vec{p} \mapsto \vec{0}$ and, for $\vec{q} \neq \vec{p}$,

$$\Phi : \vec{q} \mapsto \frac{|\vec{q}-\vec{p}|}{|\Psi^{-1} \circ \psi(\vec{q}) - \vec{p}|} \cdot \phi^{-1} \circ \psi(\vec{q}).$$

This map Φ is a PL homeomorphism, because the inverse map $D^d \to Y$ is

$$\Phi^{-1} : r\vec{\theta} \mapsto \vec{p} + r|\Psi^{-1} \circ \phi(\vec{\theta}) - \vec{p}| \cdot \phi(\vec{\theta}). \qquad \square$$

In $T^n = (\mathbb{R}/k\mathbb{Z})^n$, for $d \leq n-1$, identify $T^d = (\mathbb{R}/k\mathbb{Z})^d$ with $(\mathbb{R}/k\mathbb{Z})^d \times \{\vec{0}\} \subset T^n$, and likewise for T^{d+1}. For any $0 < a_1 \leq \cdots \leq a_d < k$ (not necessarily integers), define

$$C_1 = \left\langle \prod_{r=1}^{d} [0, a_r] \right\rangle \subset T^d, \tag{10}$$

$$C_2 = \left\langle \{0\} \times \prod_{r=1}^{d} [0, a_r] \right\rangle \subset T^{d+1}, \text{ and} \tag{11}$$

$$C_3 = \left\langle [0, a_1] \times \{a_1\} \times \prod_{r=2}^{d} [a_1, a_r] \right\rangle \subset T^{d+1}. \tag{12}$$

[11] We use the product metric on $\mathbb{R}^n$: if $\vec{p} = (p_1, \ldots, p_n)$ and $\vec{q} = (q_1, \ldots, q_n)$, then $|\vec{q}-\vec{p}| = \max_i |q_i - p_i|$.

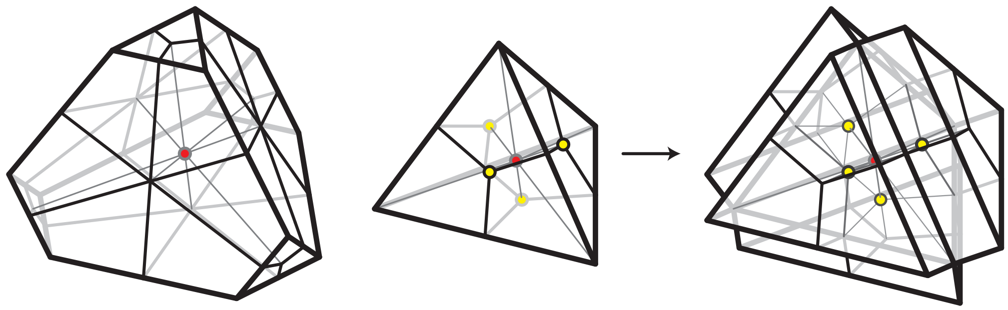

Figure 9: Left to right: $\left\langle\alpha 1\beta^2\right\rangle$ and $\left\langle 0\alpha^3\right\rangle \to \left\langle 0\alpha^2[0,2]\right\rangle$.

Figures 8 and 9 show low-dimensional examples of these *building blocks*. In Figure 8, $\langle\alpha[0,2]\rangle$ and $\left\langle\alpha^2[0,2]\right\rangle$ are examples of C_1, $\langle 0\alpha\rangle$ and $\langle 0\alpha[0,2]\rangle$ are examples of C_2, and $\langle\alpha 1\beta\rangle$ is an example of C_3. In Figure 9, $\left\langle 0\alpha^3\right\rangle$ and $\left\langle 0\alpha^2[0,2]\right\rangle$ are examples of C_2, and $\left\langle\alpha 1\beta^2\right\rangle$ is an example of C_3.

Lemma 4.2 *C_1, C_2, and C_3 from (10)-(12) are PL homeomorphic to D^d.*

Proof Let $b = \frac{1}{2}(k + a_d)$. Then $C_1 \subset [0,b]^d$ and $C_2, C_3 \subset [0,b]^{d+1}$, where $b < k$, so we may view C_1 as a subset of $\mathbb{R}^d$ and C_2, C_3 as subsets of $\mathbb{R}^{d+1}$. Let $a = \frac{a_1}{2}$, $\vec{p}_1 = (a, \ldots, a) \in \mathbb{R}^d$, $\vec{p}_2 = \vec{0} \in \mathbb{R}^{d+1}$, and $\vec{p}_3 = (a_1, \ldots, a_1) \in \mathbb{R}^{d+1}$. Then, for $i = 1, 2, 3$, C_i is compact and strongly star-shaped about $\vec{p}_i$, with $\mathrm{link}_{C_i}(\vec{p}_i) \cong S^{d-1}$, hence PL homeomorphic to D^d by Proposition 4.1. $\square$

5 Further examples

As noted in the introduction, the hardest part of verifying our multisection of T^n, in arbitrary odd dimension n, is describing the handle decomposition of X_I for arbitrary $I \subset \mathbb{Z}_k$. That task will follow three main steps. First, Lemma 6.13 will establish a closed formula (2) for arbitrary X_I. Second, §7.1 will describe how (in several steps) to decompose X_I into pieces, each of which is a product of the building blocks from §4, and will describe an order on these pieces. Third, §7.2 will establish several properties of the resulting decomposition, eventually proving that it is an appropriate handle decomposition of X_I and thus verifying Theorem 7.10.

To prepare, this section describes a few more examples, each of which confronts and resolves an additional complication in the handle decomposition of some X_I in some dimension. This section contains no proofs and little narration. Instead, the reader is encouraged to peruse the tables that follow in order to build intuition for the denser sections that follow. Indeed, assuming only the correctness of the formula (2), the reader should now be able to use their understanding of the building blocks from §4 to check the correctness of the handle decompositions, as detailed in the last five columns of the tables (starting with Y_z).

The harder part will be understanding how each handle decomposition has been constructed. This is the purpose of the columns in each table which precede Y_z, which we do not attempt to describe in detail until §7.

5.1 Quadrisection of T^7

The next several examples come from the decomposition of T^7 given by $X_0 = \left\langle \alpha^2[0,2]^2[0,3]^2[0,4] \right\rangle$ and $X_i = X_0 + (i,i,i,i,i,i,i)$. The handle decompositions of X_I, $I = \{0\}, \{0,1\}$, summarized in Tables 5 and 6, respectively, follow the same pattern in dimension seven (and all higher odd dimensions) as in dimension five (recall Tables 2 and 4 and the attending discussions). More instructive examples follow.

J	Y_z	Y_z^*	h	z	glue to
$\varnothing$	$\left\langle \alpha^2[0,2]^2[0,3]^2[0,4]^3 \right\rangle$	$\left\langle \alpha^2[0,2]^2[0,3]^2[0,4]^3 \right\rangle$	0	1	
$\{0\}$	$\left\langle \alpha^2[0,2]^2[0,3]^2[0,4]^2\varepsilon \right\rangle$	$\left\langle \alpha^2[0,2]^2[0,3]^2[0,4]^2 \right\rangle \varepsilon$	**1**	2	1

Table 5: X_0 from the quadrisection of T^7

J	i_*	Y_z	Y_z^*	h	z	glue to
$\varnothing$	0	$\left\langle \alpha 1\beta^2[1,3]^3 \right\rangle$	$\left\langle \alpha 1\beta^2[1,3]^3 \right\rangle$	0	1	
	1	$\left\langle 0\alpha\beta^2[1,3]^3 \right\rangle$	$\langle 0\alpha \rangle \left\langle \beta^2[1,3]^3 \right\rangle$	1	2	1
$\{0\}$	0	$\left\langle \alpha 1\beta^2[1,3]^2\delta \right\rangle$	$\left\langle \alpha 1\beta^2[1,3]^2 \right\rangle \delta$	1	3	1,2
	1	$\left\langle \delta 0\alpha\beta^2[1,3]^2 \right\rangle$	$\langle \delta 0\alpha \rangle \left\langle \beta^2[1,3]^2 \right\rangle$	**2**	4	2,3

Table 6: X_I, $I = \{0,1\}$ from the quadrisection of T^7

5.1.1 X_I when $I = \{0,2\}$

From the quadrisection of T^7, consider

$$X_0 \cap X_2 = \left\langle \alpha^2[0,2]\gamma^2[2,4] \right\rangle \cup \left\langle \alpha^2[0,2]2\gamma^2[2,4] \right\rangle.$$

J	i_*	V	V^-	Y_z	Y_z^*	h	z	glue to
$\varnothing$	0	$\varnothing$	$\varnothing$	$\langle\alpha^3 2\gamma^3\rangle$	$\alpha^3\langle 2\gamma^3\rangle$	0	1	
	2	$\varnothing$		$\langle 0\alpha^3\gamma^3\rangle$	$\langle 0\alpha^3\rangle\gamma^3$	0	2	
$\{0\}$	0	$\varnothing$	$\varnothing$	$\langle\alpha^3 2\gamma^2\delta\rangle$	$\alpha^3\langle 2\gamma^2\rangle\delta$	1	3	1,2
	2	$\{0\}$	$\varnothing$	$\langle\delta^+ 0\alpha^3\gamma^2\rangle$	$\langle\delta^+ 0\alpha^3\rangle\gamma^2$	0	4	
			$\{0\}$	$\langle\delta^- 0\alpha^3\gamma^2\rangle$	$\delta^-\langle 0\alpha^3\rangle\gamma^2$	1	5	2,4
$\{2\}$	0	$\{2\}$	$\varnothing$	$\langle\alpha^2\beta^+ 2\gamma^3\rangle$	$\alpha^2\langle\beta^+ 2\gamma^3\rangle$	0	6	
			$\{2\}$	$\langle\alpha^2\beta^- 2\gamma^3\rangle$	$\alpha^2\beta^-\langle 2\gamma^3\rangle$	1	7	1,6
	2	$\varnothing$	$\varnothing$	$\langle 0\alpha^2\beta\gamma^3\rangle$	$\langle 0\alpha^2\rangle\beta\gamma^3$	1	8	1,2
$\{0,2\}$	0	$\{2\}$	$\varnothing$	$\langle\alpha^2\beta^+ 2\gamma^2\delta\rangle$	$\alpha^2\langle\beta^+ 2\gamma^2\rangle\delta$	1	9	6,8
			$\{2\}$	$\langle\alpha^2\beta^- 2\gamma^2\delta\rangle$	$\alpha^2\beta^-\langle 2\gamma^2\rangle\delta$	**2**	10	3,7,8,9
	2	$\{0\}$	$\varnothing$	$\langle\delta^+ 0\alpha^2\beta\gamma^2\rangle$	$\langle\delta^+ 0\alpha^2\rangle\beta\gamma^2$	1	11	3,4
			$\{0\}$	$\langle 0\alpha^2\beta\gamma^2\delta^-\rangle$	$\langle 0\alpha^2\rangle\beta\gamma^2\delta^-$	**2**	12	3,5,8,11

Table 7: X_I, $I=\{0,2\}$ from the quadrisection of T^7

Table 7 summarizes a handle decomposition $X_I = Y_1\cup\cdots\cup Y_{12}$. As with X_I, $I=\{0,1\}$, the decomposition of X_I, $I=\{0,2\}$ is organized largely according to $\{(J,i_*) : J\subset\{\min I_r\},\ i_*\in I\}$. With Y_4,Y_5,Y_7,Y_8,Y_{10}, and Y_{11} here, we have $J\setminus\{\min I_*\}\neq\varnothing$, requiring us to split a unit interval into subintervals, in this case halves. Details on how this is done, including the definitions and purposes of the sets $V^-\subset V\subset I$, appear in §7.1, especially Table 10, and in Tables 11 and 12 in Appendix 1.

5.1.2 X_I when $I=\{0,1,2\}$

Still in dimension seven, consider

$$X_0\cap X_1\cap X_2 = \langle\alpha^2[0,2]\gamma^2[2,4]^2[2,5]\rangle\cup\langle\alpha^2[0,2]2\gamma^2[2,4]^2[2,5]\rangle.$$

Table 8 summarizes a handle decomposition $X_I = Y_1\cup\cdots\cup Y_{12}$. Again, the decomposition of X_I, $I=\{0,1,2\}$ is organized largely according to $\{(J,i_*) : J\subset\{\min I_r\},\ i_*\in I\}$. Here, we have a block I_r (in this case $I_r = I$) with $|I_r|\geq 3$, requiring us at times to split a unit interval into thirds, as seen here in Y_6-Y_8 and $Y_{14}-Y_{16}$. Details on this and the set U appear in §7.1, especially Table 10, and in Tables 11 and 12 in Appendix 1.

Another new complication arises here in Y_1-Y_4 and Y_9-Y_{12}, where $i_*+2\in I_*$, requiring us to split certain unit intervals into halves according to a different rule than

J	i_*	U	V	V^-	Y^*_z	h	z	glue to
$\varnothing$	0	$\varnothing$	{1,2}	{1}	$\alpha^-1\left\langle\beta^+2\gamma^3\right\rangle$	0	1	
				$\varnothing$	$\left\langle\alpha^+1\right\rangle\left\langle\beta^+2\gamma^3\right\rangle$	1	2	1
				{1,2}	$\alpha^-\left\langle 1\beta^-\right\rangle\left\langle 2\gamma^3\right\rangle$	1	3	1
				{2}	$\left\langle\alpha^+1\beta^-\right\rangle\left\langle 2\gamma^3\right\rangle$	2	4	2,3
	1	$\varnothing$	$\varnothing$	$\varnothing$	$\left\langle 0\alpha\right\rangle\left\langle\beta 2\gamma^3\right\rangle$	1	5	1,3
	2	{1}	$\varnothing$	$\varnothing$	$0\alpha_3^\circ\left\langle 1\beta\right\rangle\gamma^3$	1	6	5
					$\left\langle 0\alpha_3^-\right\rangle\left\langle 1\beta\right\rangle\gamma^3$	2	7	5,6
					$0\left\langle\alpha_3^+1\beta\right\rangle\gamma^3$	2	8	5,6
{0}	0	$\varnothing$	{1,2}	{1}	$\delta\alpha^-1\left\langle\beta^+2\gamma^2\right\rangle$	1	9	1,6,7
				$\varnothing$	$\delta\left\langle\alpha^+1\right\rangle\left\langle\beta^+2\gamma^2\right\rangle$	2	10	2,6,8
				{1,2}	$\delta\alpha^-\left\langle 1\beta^-\right\rangle\left\langle 2\gamma^2\right\rangle$	2	11	3,6,7
				{2}	$\delta\left\langle\alpha^+1\beta^-\right\rangle\left\langle 2\gamma^2\right\rangle$	**3**	12	4,6,8
	1	$\varnothing$	$\varnothing$	$\varnothing$	$\left\langle\delta 0\alpha\right\rangle\left\langle\beta 2\gamma^2\right\rangle$	2	13	5,9,11
	2	{1}	$\varnothing$	$\varnothing$	$\left\langle\delta 0\right\rangle\alpha_3^\circ\left\langle 1\beta\right\rangle\gamma^2$	2	14	6,13
					$\left\langle\delta 0\alpha_3^-\right\rangle\left\langle 1\beta\right\rangle\gamma^2$	**3**	15	7,13,14
					$\left\langle\delta 0\right\rangle\left\langle\alpha_3^+1\beta\right\rangle\gamma^2$	**3**	16	8,13,14

Table 8: X_I, $I=\{0,1,2\}$ from the quadrisection of T^7

in §5.1.1. Again, all the rules for splitting unit intervals into halves and thirds are detailed in §7.1, especially Table 10, and in Tables 11 and 12 in Appendix 1.

5.2 X_I, $I=\{0,1,2,3,5\}$ from T^{13}

There is one more complication, which arises, first in dimension 11, whenever X_I, $I=I_1\sqcup\cdots\sqcup I_m$, has some $I_r\not\ni i_*$ with $|I_r|\geq 3$. In fact, though, the *difficulty* of this complication only becomes apparent in dimension 13. From the septisection of T^{13}, consider X_I, $I=\{0,1,2,3,5\}$, which is given by

$$\left\langle\alpha 1\beta 2\gamma 3\delta^2[3,5]5\varepsilon^2[5,7]\right\rangle\cup\left\langle 0\alpha\beta 2\gamma 3\delta^2[3,5]5\varepsilon^2[5,7]\right\rangle\cup\left\langle 0\alpha 1\beta\gamma 3\delta^2[3,5]5\varepsilon^2[5,7]\right\rangle$$
$$\cup\left\langle 0\alpha 1\beta 2\gamma\delta^2[3,5]5\varepsilon^2[5,7]\right\rangle\cup\left\langle 0\alpha 1\beta 2\gamma 3\delta^2[3,5]\varepsilon^2[5,7]\right\rangle.$$

In this example, the new complication arises when $i_*=5$ and $0\notin J$, i.e in the part of X_I given by

$$\left\langle 0\alpha 1\beta 2\gamma 3\delta^2[3,5]\varepsilon^3\right\rangle,$$

part of which appears in the first several Y_z in the handle decomposition of this X_I. See Table 9. The tricky part here is how to order the pieces Y_z. See §7.1.4, especially

(23).

V^-	Y^*_z	h	z	glue to
$\varnothing$	$0\,\langle\alpha^+1\rangle\,\langle\beta^+2\rangle\,\langle\gamma^+3\delta^3\rangle\,\zeta^3$	0	1	
$\{1\}$	$\langle 0\alpha^-\rangle\,1\,\langle\beta^+2\rangle\,\langle\gamma^+3\delta^3\rangle\,\zeta^3$	1	2	1
$\{1,2\}$	$\langle 0\alpha^-\rangle\,\langle 1\beta^-\rangle\,2\,\langle\gamma^+3\delta^3\rangle\,\zeta^3$	1	3	2
$\{2\}$	$0\,\langle\alpha^+1\beta^-\rangle\,2\,\langle\gamma^+3\delta^3\rangle\,\zeta^3$	2	4	1,3
$\{2,3\}$	$0\,\langle\alpha^+1\beta^-\rangle\,\langle 2\gamma^-\rangle\,\langle 3\delta^3\rangle\,\zeta^3$	1	5	4
$\{1,2,3\}$	$\langle 0\alpha^-\rangle\,\langle 1\beta^-\rangle\,\langle 2\gamma^-\rangle\,\langle 3\delta^3\rangle\,\zeta^3$	2	6	3,5
$\{1,3\}$	$\langle 0\alpha^-\rangle\,1\,\langle\beta^+2\gamma^-\rangle\,\langle 3\delta^3\rangle\,\zeta^3$	2	7	2,6
$\{3\}$	$0\,\langle\alpha^+1\rangle\,\langle\beta^+2\gamma^-\rangle\,\langle 3\delta^3\rangle\,\zeta^3$	3	8	1,5,7

Table 9: From the septisection of T^{13}: the start of the handle decomposition of X_I when $I=\{0,1,2,3,5\}$. Here, $J=\varnothing$, $i_*=5$, $U=\varnothing$, and $V=\{1,2,3\}$.

Also see Table 18 in Appendix 1, which summarizes the start of the handle decomposition of X_I, $I=\{0,1,2,3,4,6\}$, from T^{15}

6 Combinatorics

This section proves several combinatorial facts about the decompositions of T^n. In particular, §6.2 proves that $T^n=\bigcup_{i\in\mathbb{Z}_k}X_i$, and §6.4 establishes a closed expression (2) for arbitrary X_I. Also, §6.3 establishes two combinatorial corollaries, which may be of independent interest but otherwise are not needed in this paper.

6.1 Notation

Because each X_i from our construction (1) is symmetric under the permutation action of S_n on the indices in T^n, it will often suffice, when considering an arbitrary point $\vec{x}=(x_1,\dots,x_n)\in(\mathbb{R}/k\mathbb{Z})^n=T^n$, to assume that $\vec{x}$ is **monotonic** in the sense that $x_1\le x_2\le\cdots\le x_n\le k+x_1$.

Denoting the main diagonal of T^n by Δ, note that each monotonic point $\vec{x}=(x_1,\dots,x_n)\in T^n\setminus\Delta$ corresponds to a unique point $(x_1,\dots,x_n)\in\mathbb{R}^n$ with $0\le x_1\le x_2\le\cdots\le x_n\le x_1+k<2k$. For such $\vec{x}$, extend the point $(x_1,\dots,x_n)\in\mathbb{R}^n$ to a point $\vec{x}_\infty=(x_r)_{r\in\mathbb{Z}}\in\mathbb{R}^{\mathbb{Z}}$ by defining for each $r\in\mathbb{Z}_k$ and $m\in\mathbb{Z}$:

$$x_{r+mn}=x_r+mk.$$

We will mainly be interested in $0 \le x_1 \le \cdots \le x_{2n}$, where

$$x_{2n} = x_n + k \le x_1 + 2k < 3k.$$

With this setup for any monotonic $\vec{x} \in T^n \setminus \Delta$, define the following **cutoff indices** $a_r(\vec{x}), b_r(\vec{x}) \in \mathbb{Z}$ for each $r \in \mathbb{Z}$:

$$a_r(\vec{x}) = \min\{a : x_{a+1} \ge r\} \text{ and}$$

$$b_r(\vec{x}) = \min\{b : x_{b+1} > r\}.$$

Note that, in all cases, we have $a_0(\vec{x}) \le 0$, with equality if and only if $x_n \ne k \equiv 0 \in \mathbb{R}/k\mathbb{Z}$. The main point is:

Observation 6.1 *Let $\vec{x} \in T^n \setminus \Delta$ be monotonic. Then $\vec{x} \in [0,1]^2 \cdots [0,k-1]^2[0,k]$ if and only if $b_s(\vec{x}) \ge 2s$ for every $s = 0, \ldots, k-1$.*

Note that $b_0(\vec{x}) \ge 0$ in all cases. In order to apply the principle of Observation 6.1 more broadly, denote for each $r \in \mathbb{Z}$:

$$\vec{x}_r = (x_{1+a_r(\vec{x})}, x_{2+a_r(\vec{x})}, \ldots, x_{n+a_r(\vec{x})})$$

The point regarding monotonic points off the main diagonal is:

Observation 6.2 *If $\vec{x} \in T^n \setminus \Delta$ is monotonic and $r \in \mathbb{Z}$, then*

$$r \le x_{1+a_r(\vec{x})} \le \cdots \le x_{n+a_r(\vec{x})} < r+k,$$

and the following conditions are equivalent:

- $\vec{x}_r \in [r, r+1]^2 \cdots [r, r+k-1]^2[r, r+k]$;
- $b_{r+s}(\vec{x}_r) \ge 2s$ *for every* $s = r+1, \ldots, r+k$;
- $b_{r+s}(\vec{x}) \ge a_r(\vec{x}) + 2s$ *for every* $s = r+1, \ldots, r+k$

Observations 6.1 and 6.2 apply more generally using:

Observation 6.3 *If $\vec{x} \in X_r \subset T^n \setminus \Delta$, then there is a permutation $\sigma \in S_n$ such that $\vec{x}_\sigma \in [r, r+1]^2 \cdots [r, r+k-1]^2[r, r+k]$ is monotonic.*

Note also that either class of cutoff indices provides two-sided bounds for the other class:

Observation 6.4 *If $\vec{x} \in T^n$ is nonzero and monotonic and $r \in \mathbb{Z}$, then*

$$\cdots \le a_r(\vec{x}) \le b_r(\vec{x}) \le a_{r+1}(\vec{x}) \le b_{r+1}(\vec{x}) \le \cdots$$

with $a_r(\vec{x}) = b_r(\vec{x})$ if and only if $x_{a_r(\vec{x})+1} \notin \mathbb{Z}_k$, and $b_r(\vec{x}) = a_{r+1}(\vec{x})$ if and only if $x_{b_r(\vec{x})+1} \geq r+1$.

Note that $x_{b_r(\vec{x})+1}$ is the first coordinate in $\vec{x}$ that exceeds r. Here is another convenient property:

Observation 6.5 *Any nonzero monotonic $\vec{x} \in T^n$, $r \in \mathbb{Z}_{\geq}0$ satisfy*

$$\begin{aligned} a_{r+k}(\vec{x}) &= n + a_r(\vec{x}), \\ b_{r+k}(\vec{x}) &= n + b_r(\vec{x}). \end{aligned} \tag{13}$$

Noting that $X_r \cap \Delta = \{(x,\ldots,x) : \ x \in [r, r+1]\}$, we can express each X_r in terms of cutoff indices as follows.

Proposition 6.6 *Let $\vec{x} \in T^n \setminus \Delta$ be monotonic, and let $r \in \mathbb{Z}_k$. Then $\vec{x} \in X_r$ if and only if $\vec{x}_r \in [r, r+1]^2 \cdots [r, r+k-1]^2[r, r+k]$. In particular,*

$$X_r \setminus \Delta = \langle \text{monotonic } \vec{x} : \ b_{r+s}(\vec{x}) \geq a_r(\vec{x}) + 2s \text{ for } s = 0,\ldots,k-1 \rangle. \tag{14}$$

Proof Write $\vec{x}_r = (x_1,\ldots,x_n)$. Note that $r \leq x_1 \leq \cdots \leq x_n < r+k$. To show that $\vec{x}_r \in [r, r+1]^2 \cdots [r, r+k-1]^2[r, r+k]$ if and only if $\vec{x}_r \in X_r$, we will prove both containments. One is trivial. For the other, suppose that $\vec{x}_r \notin [r, r+1]^2 \cdots [r, r+k-1]^2[r, r+k]$. Then Observation 6.2 implies that $b_{r+s}(\vec{x}_r) < 2s$ for some $s = 0,\ldots,k-1$, so

$$r+s < x_{2s},\ldots,x_n < r+k.$$

Thus, at least $n+1-2s$ of the coordinates of $\vec{x}$ lie in the open interval $(r+s, r+k)$. Yet, $2s$ of the n factors of $[r, r+1]^2 \cdots [r, r+k-1]^2[r, r+k]$ are disjoint from that open interval. Contradiction. Observation 6.3 now implies that $\vec{x} \in X_r \setminus \Delta$ if and only if $\vec{x}$ is an element of the RHS of (14). □

6.2 The X_r have disjoint interiors and cover T^n.

Proposition 6.7 *With the setup from Theorem 7.10, X_r and X_s have disjoint interiors whenever $0 \leq r < s \leq k-1$.*

This will follow from Lemma 6.13, but the following proof is much easier than that of the lemma; we include it for expository reasons.

Proof By the symmetry of the construction, we may assume that $r = 0$. Assume for contradiction that the interiors of X_r and X_s intersect. Then $X_r \cap X_s$ has positive measure, so there is a monotonic point $\vec{x} = (x_1, \dots, x_n) \in X_0 \cap X_j$ such that for every $i = 1, \dots, n$ we have $x_i \notin \mathbb{Z}_k$.

This implies that $a_i(\vec{x}) = b_i(\vec{x})$ for each $i = 1, \dots, n$, by Observation 6.4. In particular, since $\vec{x} \in X_0$, we have $a_0 = b_0 = 0$, and $a_s = b_s \geq 2s$ by Proposition 6.6. But then, since $\vec{x} \in X_s$ and $a_s \geq 2s$, Observation 6.5 and Proposition 6.6 give the following contradiction:

$$\begin{aligned} n &= n + b_0 = b_k = b_{s+(k-s)} \\ n &\geq a_s + 2(k-s) \\ n &\geq 2k. \square \end{aligned}$$

Lemma 6.8 *We have* $X_0 \cup \dots \cup X_{k-1} = T^n$.

Proof Let $\vec{x} \in T^n$. We will prove that $\vec{x} \in X_s$ for some s. If $\vec{x} = (x, \dots, x) \in \Delta$, then $\vec{x} \in X_{\lfloor x \rfloor}$. Assume instead that $\vec{x} \in T^n \setminus \Delta$. Also assume without loss of generality that $\vec{x}$ is monotonic with $a_0(\vec{x}) = 0$. Throughout this proof, denote each $a_s(\vec{x})$ by a_s and each $b_s(\vec{x})$ by b_s.

Let $s_0 = 0$, so that $a_{s_0} = a_0 = 0$. If $b_s \geq 2s = 2s - a_{s_0}$ for all $s = 1, \dots, k-1$, then $\vec{x} \in X_0 = X_{s_0}$. Otherwise, choose the smallest s_1 such that $b_{s_1} < 2s_1$. Thus, $b_s \geq 2s$ whenever $s < s_1$, so by Observation 6.4:

$$2s_1 - 2 \leq b_{s_1-1} \leq a_{s_1} \leq b_{s_1} \leq 2s_1 - 1.$$

Continue in this way: for each s_t, choose the minimum $s_{t+1} = s_t + 1, \dots, k-1$ such that $b_{s_{t+1}} < a_{s_t} + 2(s_{t+1} - s_t)$, if such s_{t+1} exists. Eventually this process terminates with some s_u, so that:

- $b_s \geq a_{s_t} + 2(s - s_t)$ whenever $s_t \leq s \leq s_{t+1}$ for $t = 0, \dots, u-1$,
- $b_s \geq a_{s_t} + 2(s - s_u)$ whenever $s_u \leq s \leq k-1$, and
- $b_{s_{t+1}} < a_{s_t} + 2(s_{t+1} - s_t)$ for each $t = 0, \dots, u-1$.

Hence, for each $t = 0, \dots, u-1$, Observation 6.4 gives:

$$a_{s_t} + 2(s_{t+1} - 1 - s_t) \leq b_{s_{t+1}-1} \leq a_{s_{t+1}} \leq b_{s_{t+1}} \leq a_{s_t} + 2(s_{t+1} - s_t) - 1.$$

Subtracting a_{s_t} from the first, middle, and last expressions gives:

$$2(s_{t+1} - s_t) - 2 \leq a_{s_{t+1}} - a_{s_t} \leq 2(s_{t+1} - s_t) - 1.$$

Therefore, for any $t = 0, \dots, u-1$:

$$\begin{aligned} a_{s_u} - a_{s_t} &= \sum_{r=t}^{u-1} (a_{s_{r+1}} - a_{s_r}) \\ &\leq \sum_{r=t}^{u-1} \big(2(s_{r+1} - s_r) - 1\big) \\ &= 2(s_u - s_t) - (u - t) \\ a_{s_u} - a_{s_t} &\leq 2(s_u - s_t) - 1. \end{aligned}$$

Rearranging gives

$$a_{s_u} - 2s_u \leq a_{s_t} - 2s_t - 1. \tag{15}$$

We claim that $\vec{x} \in X_{s_u}$. This is true if (and only if) $b_s \geq a_{s_u} + 2(s - s_u)$ for each $s = s_u, \dots, s_u + k - 1$. Fix some $s = k, \dots, s_u + k - 1$. Then $s_t \leq s - k \leq s_{t+1} - 1$ for some $t = 0, \dots, u-1$. By construction, we have $b_{s-k} \geq a_{s_t} + 2(s - k - s_t)$. Together with (13) and (15), this gives:

$$\begin{aligned} b_s &= b_{s-k} + n \\ &\geq a_{s_t} + 2(s - k - s_t) + 2k - 1 \\ &= (a_{s_t} - 2s_t - 1) + 2s \\ &\geq (a_{s_u} - 2s_u) + 2s \\ &= a_{s_u} + 2(s - s_u). \square \end{aligned}$$

6.3 Combinatorial corollaries

This subsection establishes two combinatorial corollaries, which may be of independent interest but otherwise are not needed in this paper.

We have proven that the pieces X_r of the multisection of T^n have disjoint interiors and cover T^n. Also, each $X_r = X_0 + (r, \dots, r)$, so all X_r have the same number of unit cubes. Since there are k^n unit cubes in $T^n = (\mathbb{R}/k\mathbb{Z})^n$, each X_r contains k^{n-1} unit cubes. By counting these unit cubes a different way, we obtain the following.[12]

[12]Note that by definition, if $a, b \in \mathbb{Z}$ with $b < 0$, then $\binom{a}{b} = 0$.

Corollary 6.9 *For any $n = 2k-1$, we have:*

$$k^{n-1} = \sum_{i_0=2}^{n} \binom{n}{i_0} \sum_{i_2=4-i_0}^{n-i_0} \binom{n-i_0}{i_1} \sum_{i_3=6-i_0-i_1}^{n-i_0-i_1} \binom{n-i_0-i_1}{i_2} \cdots \tag{16}$$
$$\cdots \sum_{i_{k-2}=2k-2-\sum_{j=0}^{k-3} i_j}^{n-\sum_{j=0}^{k-3} i_j} \binom{n-\sum_{j=0}^{k-3} i_j}{i_{k-1}}.$$

Note that (16) is also the number of spanning trees of the complete bipartite graph $K_{j,j}$ where $j = k$ [OEIS].

Proof X_0 consists of k^{n-1} subcubes, each of the form $\prod_{r=1}^{n}[w_r, w_r+1]$ for some $w_1, \dots, w_n \in \mathbb{Z}_k$. For each $s = 0, \dots, k-2$, there are at least $2s+2$ indices among $r = 1, \dots, n$ with $w_r \in \{0, \dots, s\}$, and conversely any subcube of that form with this property will be in X_0. (This characterization follows from the expression (1) for X_0.) In (16), each $i_s = \#\{r : w_r = s\}$, so $i_0 \geq 2$, $i_0 + i_1 \geq 4$, and so on. □

As noted above, each subcube of T^n has the form $\prod_{r=1}^{n}[w_r, w_r+1]$ for some $w_1, \dots, w_n \in \mathbb{Z}_k$. Say that two subcubes $\prod_{r=1}^{n}[w_r, w_r+1]$ and $\prod_{r=1}^{n}[w'_r, w'_r+1]$ have the same *combinatorial type* if $(w'_1, \dots, w'_n)$ is a permutation of $(w_1, \dots, w_n)$. Counting combinatorial *cube types* in three different ways yields:

Corollary 6.10 *For any $n = 2k-1$, we have:*

$$k \sum_{i_0=2}^{n} \sum_{i_1=\max\{0,4-i_0\}}^{n-i_0} \sum_{i_2=\max\{0,6-i_0-i_1\}}^{n-i_0-i_1} \cdots \sum_{i_{k-2}=\max\left\{0,2k-2-\sum_{j=0}^{k-3} i_j\right\}}^{n-\sum_{j=0}^{k-3} i_j} 1 \tag{17}$$
$$= \sum_{i_0=0}^{n} \sum_{i_1=0}^{n-i_0} \sum_{i_2=0}^{n-i_0-i_1} \cdots \sum_{i_{k-2}=0}^{n-\sum_{j=0}^{k-3} i_j} 1$$
$$= \binom{3k-2}{k-1}.$$

Proof The first expression is k times the number of *cube types* in X_0, counted using the same principle and notation as in Corollary 6.9. The second counts the number of cube types in T^n, each of which we may write in the form $\prod_{r=0}^{k-1}[r, r+1]^{i_r}$ and is thus characterized by a tuple $(i_0, \dots, i_{k-1})$ with $\sum_{r=0}^{k-1} i_r = n$. The third counts the number of cube types in T^n by denoting $a_0 = 0, a_k = 3k-1$ and associating to each $A = \{a_1, \ldots, a_{k-1}\} \subset \{1, \ldots, 3k-2\}$ satisfying $a_1 < \cdots < a_{k-1}$ with the cube type

$$\prod_{i=1}^{k} \prod_{j=a_{i-1}+1}^{a_i - 1} [i-1, i]. \qquad \square$$

See [OEIS] for other interpretations of (17).

6.4 Verification of the formula $X_I =$ (2)

Next, we will use the cutoff indices $a_r(\vec{x}), b_r(\vec{x})$ to verify (2). To prepare this, we define subsets $C_{I,s} \subset T^n$ as follows. Let $I \subset \mathbb{Z}_k$ following Convention 3.9, with $s \in \mathbb{Z}_\ell$, and denote $i_s = i_*$. Then define:[13]

$$\begin{aligned} C_{I,s} = & \left(\prod_{t=0}^{s-1} \{i_t\} \times [i_t, i_t+1]^2 \times \cdots \times [i_t, i_{t+1}-1]^2 \times [i_t, i_{t+1}] \right) \\ & \times [i_*, i_*+1]^2 \times \cdots \times [i_*, i_{s+1}-1]^2 \times [i_*, i_{s+1}] \\ & \times \left(\prod_{t=s+1}^{\ell-1} \{i_t\} \times [i_t, i_t+1]^2 \times \cdots \times [i_t, i_{t+1}-1]^2 \times [i_t, i_{t+1}] \right). \end{aligned} \tag{18}$$

Note the "missing" $\{i_*\}$ at the start of the second line; this corresponds to the $\widehat{i_*}$ in (2). Observe that the expression on the RHS of (2) equals

$$\bigcup_{s \in \mathbb{Z}_\ell} \langle C_{I,s} \rangle .$$

Proposition 6.11 *Let $I \subset \mathbb{Z}_k$ follow Convention 3.9, $s \in \mathbb{Z}_\ell$, and $C_{I,s}$ as in (18). Suppose $\vec{x} \in T^n \setminus \Delta$ is monotonic. Then $\vec{x} \in C_{I,s}$ if and only if all of the following conditions hold:*

- $b_t(\vec{x}) \geq 2t+1$ *for* $0 \leq t < i_*$,
- $b_t(\vec{x}) \geq 2t$ *for* $i_* \leq t \leq k-1$,
- $a_t(\vec{x}) \leq 2t$ *for* $t = i_0, \ldots, i_*$, *and*
- $a_t(\vec{x}) \leq 2t-1$ *for* $t = i_{s+1}, \ldots, i_{\ell-1}$.

[13] Note that the first line in (18) contributes no factors to $C_{I,s}$ if $s = 0$, and likewise for the third line if $s = \ell - 1$. In particular, if $I = \{0\}$, then $s = 0$ and $C_{I,s} = [0,1]^2 \times \cdots \times [0, k-1]^2 [0, k]$, so $\langle C_{I,s} \rangle = X_0$.

Proof This follows immediately from the definitions, upon consideration of each entry in $\vec{x}$. $\square$

Also note the following generalization of Observation 6.3:

Observation 6.12 *Let $I \subset \mathbb{Z}_k$ follow Convention 3.9, $s \in \mathbb{Z}_\ell$, and $C_{I,s}$ as in (18). Suppose $\vec{x} \in \langle C_{I,s} \rangle$. Then there is a permutation $\sigma \in S_n$ such that $\vec{x}_\sigma \in C_{I,s}$ is monotonic.*

Lemma 6.13 *Given nonempty $I \subset \mathbb{Z}_k$ (following Convention 3.9),*

$$X_I = \bigcup_{s \in \mathbb{Z}_\ell} \langle C_{I,s} \rangle . \tag{19}$$

In particular,

$$\bigcap_{i_* \in \mathbb{Z}_k} X_i = \bigcup_{i_* \in I} \left\langle (i_1, \ldots, \widehat{i_*}, \ldots, i_\ell) \prod_{i \in \mathbb{Z}_k} [i, i+1] \right\rangle . \tag{3}$$

Note that the formula (19) is equivalent to (2).

Proof We argue by induction on ℓ. When $\ell = 1$, $X_I = X_0 = \langle C_{I,0} \rangle = (2)$.

Assume now that $\ell > 1$. First, we will show that

$$X_I \subset \bigcup_{s \in \mathbb{Z}_\ell} \langle C_{I,s} \rangle . \tag{20}$$

Let $\vec{x} \in X_I$, and define $I' = I \setminus \{i_{\ell-1}\}$. Note that I' is simple and $X_I = X_{I'} \cap X_{i_{\ell-1}}$. Since $\vec{x} \in X_{I'}$, the induction hypothesis implies that $\vec{x} \in \left\langle C_{I',s_0} \right\rangle$ for some $s_0 \in \mathbb{Z}_{\ell-1}$. By Observation 6.12, there exists $\sigma \in S_n$ such that $\vec{x}_\sigma$ is monotonic and $\vec{x}_\sigma \in C_{I',s_0}$. Proposition 6.11 implies that:

- $b_t(\vec{x}_\sigma) \geq 2t + 1$ for $0 \leq t \leq i_{s_0} - 1$,
- $b_t(\vec{x}_\sigma) \geq 2t$ for $i_{s_0} \leq t \leq k - 1$,
- $a_t(\vec{x}_\sigma) \leq 2t$ for $t = i_0, \ldots, i_{s_0}$, and
- $a_t(\vec{x}_\sigma) \leq 2t - 1$ for $t = i_{s_0+1}, \ldots, i_{\ell-2}$.

If also $a_{i_{\ell-1}}(\vec{x}_\sigma) \leq 2i_{\ell-1} - 1$, then Proposition 6.11 implies that $\vec{x}_\sigma \in C_{I,s_0}$. In that case, we are done proving the forward containment. Assume instead that $a_{i_{\ell-1}}(\vec{x}_\sigma) \geq 2i_{\ell-1}$. We now split into two cases:

Case 1: Assume that $a_{i_{\ell-1}}(\vec{x}_\sigma) = 2i_{\ell-1}$. We claim that $\vec{x}_\sigma \in C_{I,\ell-1}$. By Proposition 6.11, since $\vec{x}_\sigma$ is monotonic, it will suffice to show:

(a) $b_t(\vec{x}_\sigma) \geq 2t+1$ for $0 \leq t \leq i_{\ell-1}-1$,

(b) $b_t(\vec{x}_\sigma) \geq 2t$ for $i_{\ell-1} \leq t \leq k-1$, and

(c) $a_t(\vec{x}_\sigma) \leq 2t$ for $t = i_0, \dots, i_{\ell-1}$.

Observation 6.5, Proposition 6.6, and the facts that $\vec{x}_\sigma \in X_{i_{\ell-1}}$ and $a_{i_{\ell-1}}(\vec{x}_\sigma) = 2i_{\ell-1}$ imply for each $t = 0, \dots, i_{\ell-1}-1$ that:

$$\begin{aligned}
b_t(\vec{x}_\sigma) &= b_{t+k}(\vec{x}_\sigma) - n \\
&\geq 2(t+k) + a_{i_{\ell-1}}(\vec{x}_\sigma) - 2i_{\ell-1} - n \\
&\geq 2t+1.
\end{aligned}$$

This verifies (a). Taking $t = i_{\ell-1}, \dots, k-1$, similar reasoning confirms (b):

$$b_t(\vec{x}_\sigma) \geq a_{i_{\ell-1}}(\vec{x}_\sigma) + 2(t - i_{\ell-1}) \geq 2t.$$

Finally, we have $a_t(\vec{x}_\sigma) \leq 2t$ for each $t = i_0, \dots, i_{\ell-1}$. For $t = i_0, \dots, i_{\ell-2}$, this is because $\vec{x}_\sigma \in X_{I'}$; for $t = i_{\ell-1}$, it is our assumption in Case 1. Thus, in Case 1, (a), (b), and (c) hold, and so $\vec{x}_\sigma \in C_{I,\ell-1}$.

Case 2: Assume instead that $a_{i_{\ell-1}}(\vec{x}_\sigma) \geq 2i_{\ell-1}+1$. Denoting $\vec{x}_\sigma = (x_1, \dots, x_n)$, we claim in this case that $x_1 = x_2 = 0 \equiv k$ and that $\vec{y} = (x_2, \dots, x_n, x_1) \in C_{I,\ell-1}$. By similar reasoning to Case 1, we have:

$$\begin{aligned}
b_0(\vec{x}_\sigma) &= b_k(\vec{x}_\sigma) - n \\
&\geq a_{i_{\ell-1}}(\vec{x}_\sigma) + 2(k - i_{\ell-1}) - n \\
&\geq 2.
\end{aligned}$$

Thus, $x_1 = x_2 = 0 \equiv k$. Define $\vec{y}$ as above. Note that, since $\vec{x}_\sigma$ is monotonic, $\vec{y}$ is also monotonic. It remains to show that $\vec{y} \in C_{I,\ell-1}$. The arguments are almost identical to those in Case 1, except that we need to check that $a_{i_{\ell-1}}(\vec{y}) \leq 2i_{\ell-1}$. Using Observations 6.4 and 6.5 and the fact that $a_{i_{\ell-1}}(\vec{y}) = a_{i_{\ell-1}}(\vec{x}_\sigma) - 1$, we compute:

$$\begin{aligned}
a_{i_{\ell-1}}(\vec{y}) &= a_{i_{\ell-1}}(\vec{x}_\sigma) - 1 \\
&\leq b_{k-1}(\vec{x}_\sigma) - 2(k-1-i_{\ell-1}) - 1 \\
&\leq a_k(\vec{x}_\sigma) - 2k + 1 + 2i_{\ell-1} \\
&= a_0(\vec{x}_\sigma) + (n+1-2k) + 2i_{\ell-1} \\
&= a_0(\vec{x}_\sigma) + 2i_{\ell-1} \\
&\leq 2i_{\ell-1}.
\end{aligned}$$

This completes the proof of the forward containment (20). For the reverse containment, keep the same subset $I \subset \mathbb{Z}_k$ from the start of the induction step of the proof, fix some $s \in \mathbb{Z}_\ell$, let $\vec{x} \in C_{I,s}$ be monotonic, and let $t \in I = \{i_0, \dots, i_{\ell-1}\}$. We will show for each $r = 0, \dots, k-1$ that $b_{t+r}(\vec{x}) \geq a_t(\vec{x}) + 2r$. Proposition 6.6 will then imply that $\vec{x} \in X_t$. Since t is arbitrary, this will imply that $\vec{x} \in X_I$, completing the proof. We will split into cases, but first note, since $\vec{x}$ is monotonic, that Proposition 6.11 implies:

- $b_t(\vec{x}) \geq 2t + 1$ for $0 \leq t \leq i_s - 1$,
- $b_t(\vec{x}) \geq 2t$ for $i_s \leq t \leq k - 1$,
- $a_t(\vec{x}) \leq 2t$ for $t = i_0, \dots, i_s$, and
- $a_t(\vec{x}) \leq 2t - 1$ for $t = i_{s+1}, \dots, i_{\ell-2}$.

<u>Case 1:</u> If $t + r \leq k - 1$, then $b_{t+r}(\vec{x}) \geq 2(t + r) \geq a_t(\vec{x}) + 2r$.

<u>Case 2:</u> If instead $t + r \geq k$ and $t + r \leq k + i_s - 1$, then

$$\begin{aligned} b_{t+r}(\vec{x}) &= n + b_{t+r-k}(\vec{x}) \geq n + 2(t + r - k) + 1 = 2t + 2r + (n + 1 - 2k) \\ b_{t+r}(\vec{x}) &\geq a_t(\vec{x}) + 2r \end{aligned}$$

<u>Case 3:</u> Similarly, if $t + r \geq k$ and $t \geq i_{s+1}$, then

$$\begin{aligned} b_{t+r}(\vec{x}) &= n + b_{t+r-k}(\vec{x}) \geq n + 2(t + r - k) = (2t - 1) + 2r + (n + 1 - 2k) \\ b_{t+r}(\vec{x}) &\geq a_t(\vec{x}) + 2r \end{aligned}$$

Are there other cases? If there were, they would satisfy $t + r \geq k + i_s$ and $t \leq i_s$, giving

$$\begin{aligned} k + i_s \leq t + r &\leq i_s + r \\ k &\leq r. \end{aligned}$$

Yet $r \leq k - 1$ by assumption. Therefore, in every case, $b_{t+r}(\vec{x}) \geq a_t(\vec{x}) + 2r$, and so $\vec{x} \in X_{i_t}$ for arbitrary $t \in I$. Thus, $\vec{x} \in X_I$. This completes the proof of the reverse containment, and thus of the equality in (2)=(19). □

7 General construction

This section confirms the remaining details of our main construction and completes the proof of our main result, Theorem 7.10. Namely, §7.1 describes how to decompose arbitrary X_I, and §7.2 shows that this decomposition does in fact give an appropriate handle structure for X_I.

Section 7 uses Notations 3.3, 3.6, 3.8, and Convention 3.9.

7.1 Handle decompositions: the general case

Throughout §7.1, fix arbitrary $I = \{i_s\}_{s\in\mathbb{Z}_\ell} = \bigsqcup_{r\in\mathbb{Z}_m} I_r \subsetneq \mathbb{Z}_k$, following Convention 3.9. Recall in particular that $T = \{t \in \mathbb{Z}_\ell : i_t - 1 \notin I\} = \{t_r\}_{r\in\mathbb{Z}_m}$, so that $\{\min I_r\}_{r\in\mathbb{Z}_m} = \{i_t\}_{t\in T}$.

7.1.1 Overview

In §7.1, we will decompose X_I into handles in several steps as follows. First, we will decompose X_I into pieces X_{I,J,i_*} determined by all pairs (J, i_*) where $J \subset \{\min I_r\}$ and $i_* \in I$. Second, for fixed (J, i_*), we will define disjoint subsets $U, V \subset I$ for the purpose of dividing each interval $[i-1, i]$, $i \in I$, into thirds if $i \in U$, into halves if $i \in V$, or neither if $i \notin U, V$. Third, still fixing (J, i_*), after dividing certain intervals into halves and thirds as just described, we will decompose each piece X_{I,J,i_*} into pieces $X_{I,J,i_*,V^-,U^\circ,U^-}$; these pieces are determined by all triples (V^-, U°, U^-) where $V^- \subset V$, $U^\circ \subset U$, and $U^- \subset U \setminus U^\circ$. For each of the first three steps, we will describe what to do within each block I_r; then we will take a product across all blocks and extend by permutations of the indices.

Fourth, we will order the possibilities of the tuple $(J, i_*, V^-, U^\circ, U^-)$, thus determining an order on the pieces $X_{I,J,i_*,V^-,U^\circ,U^-}$. The order will be lexicographical, and will thus require defining orders on $\{J \subset \{\min I_r\}\}$, $\{i_* \in I\}$, $\{V^- \subset V\}$, $\{U^\circ \subset U\}$, and $\{U^- \subset U \setminus U^\circ\}$. Of these five orders, only the third will be somewhat complicated. Once we define this order, we will use it to relabel the various pieces $X_{I,J,i_*,V^-,U^\circ,U^-}$ as Y_z, with $z = 1, 2, 3, \ldots$. Fifth and finally, we will decompose each Y_z into handles, one of which we denote Y_z^* (each handle H from Y_z is related to Y_z^* by $H = \{\vec{x}_\tau : \vec{x} \in Y_z^*\}$ for some fixed permutation $\tau \in S_n$).

7.1.2 Decomposing X_I according to (J, i^*)

Fix arbitrary $J \subset \{\min I_r\}$ and $i_* \in I$ for all of §7.1.2. Momentarily fixing arbitrary $r \in \mathbb{Z}_m$, denote

$$a = \min I_r, \; b = \max I_r, \; c = \min I_{r+1}, \text{ and } \widehat{C}_r = \prod_{j=b+1}^{c-1} [b, j]^2, \tag{21}$$

and define

$$
(22)\qquad C_r = \begin{Bmatrix} [a-1,a] & i_* = a \in J \\ [a-1,a]\times\{a\} & i_* \neq a \in J \\ \{a\} & i_* \neq a \notin J \\ \text{(no factor)} & i_* = a \notin J \end{Bmatrix} \times \prod_{i=a+1}^{b} \begin{Bmatrix} [i-1,i]\times\{i\} & i \neq i_* \\ [i-1,i] & i = i_* \end{Bmatrix} \times \begin{Bmatrix} \widehat{C}_r \times [b,c-1] & c \notin J \\ \widehat{C}_r & c \in J \end{Bmatrix}.
$$

Now the piece of X_I corresponding to the pair (J, i_*) is given by

$$
X_{I,J,i_*} = \left\langle \prod_{r\in\mathbb{Z}_m} C_r \right\rangle .
$$

7.1.3 The index subsets $U, V \subset I$

Fix arbitrary $J \subset \{\min I_r\}$ and $i_* \in I$ for all of §7.1.3. For each $r \in \mathbb{Z}_m$, define subsets $U_r, V_r \subset I_r$ following Table 10 (or equivalently according to Tables 11 and 12 in Appendix 1, which present U_r and V_r more explicitly). Note that $\min I_r \notin (U_r \cup V_r)$ unless $I_r \neq I_*$ and $\min I_r = \max I_r \in J$. See Table 7 for an example of this exceptional case: X_I, $I = \{0,2\}$, from T^7.

	$i_* \notin I_r$, $a \notin J$	$i_* \notin I_r$, $a \in J$	$i_* \in I_r$, $i_* \leq b-2$	$i_* \in I_r$, $i_* \geq b-1$
U_r	$\varnothing$	$I_r \setminus \{a,b\}$	$I_r \setminus \{a, i_*, i_*+1, b\}$	$I_r \setminus \{a, i_*, b\}$
V_r	$I_r \setminus \{a\}$	$\{b\}$	$\{i_*+1, b\}$	$\varnothing$
$I_r \setminus (U_r \cup V_r)$	$\{a\}$	$\{a\} \setminus \{b\}$	$\{a, i_*\}$	$\{a, i_*, b\}$

Table 10: The index subsets $U_r, V_r \subset I_r$ when $I_r = \{a, \ldots, b\}$.

Define

$$
U = \bigcup_{r\in\mathbb{Z}_m} U_r \text{ and } V = \bigcup_{r\in\mathbb{Z}_m} V_r.
$$

Next, decompose each X_{I,J,i_*} into pieces $X_{I,J,i_*,V^-,U^\circ,U^-}$ as follows. Denote

$$
\begin{aligned}
2^V &= \{V^- \subset V\}, \\
2^U &= \{U^\circ \subset U\},
\end{aligned}
$$

and given $U^\circ \subset U$, denote

$$2^{U\setminus U^\circ} = \{U^- \subset U \setminus U^\circ\}.$$

Given $V^- \subset V$, denote $V^+ = V \setminus V^-$, and given $U^\circ \subset U$ and $U^- \subset U \setminus U^\circ$, denote $U^+ = U \setminus (U^\circ \cup U^-)$. Then $V = V^- \sqcup V^+$ and $U = U^- \sqcup U^\circ \sqcup U^+$. Momentarily fixing $r \in \mathbb{Z}_m$, denote a, b, c, and $\widehat{C}_r$ as in (21), and for each $i \in I_r$ define

$$\rho_i = \begin{cases} [i-1, i-\frac{2}{3}] & i \in U^- \\ [i-\frac{2}{3}, i-\frac{1}{3}] & i \in U^\circ \\ [i-\frac{1}{3}, i] & i \in U^+ \\ [i-1, i-\frac{1}{2}] & i \in V^- \\ [i-\frac{1}{2}, i] & i \in V^+ \\ [\max I_{r-1}, i-1] & i = a \notin J \cup V \\ [i-1, i] & \text{else.} \end{cases}$$

Note that $\rho_i \subset [i-1, i]$ for each $i = a+1, \dots, b$, that $\rho_a \subset [a-1, a]$ if $a \in J$, and that $\rho_c = [b, c-1]$ if $c \notin J$. Still fixing $r \in \mathbb{Z}_m$, define

$$X_{I,J,i_*,V^-,U^\circ,U^-,r} = \left\{\begin{matrix} \rho_a & i_* = a \in J \\ \rho_a \times \{a\} & i_* \neq a \in J \\ \{a\} & i_* \neq a \notin J \\ \text{(no factor)} & i_* = a \notin J \end{matrix}\right\}$$
$$\times \prod_{i=a+1}^{b} \left\{\begin{matrix} \rho_i \times \{i\} & i \neq i_* \\ \rho_i & i = i_* \end{matrix}\right\} \times \left\{\begin{matrix} \widehat{C}_r \times \rho_c & c \notin J \\ \widehat{C}_r & c \in J \end{matrix}\right\}.$$

The piece of X_I corresponding to the tuple $(J, i_*, V^-, U^\circ, U^-$ is:

$$X_{I,J,i_*,V^-,U^\circ,U^-} = \left\langle \prod_{r \in \mathbb{Z}_m} X_{I,J,i_*,V^-,U^\circ,U^-,r} \right\rangle$$

Note that $X_{I,J,i_*} = \bigcup_{V^-,U^\circ,U^-} X_{I,J,i_*,V^-,U^\circ,U^-}$.

7.1.4 Ordering the pieces $X_{I,J,i_*,V^-,U^\circ,U^-}$

Next, we define orders $\prec$ on $\{J \subset \{\min I_r\}\}$, I, 2^V, 2^U, and $2^{U\setminus U^\circ}$ and use these to order the pieces $X_{I,J,i_*,V^-,U^\circ,U^-}$ lexicographically and then relabel them as $Y_1, Y_2, Y_3, \dots$. Order $\{J \subset \{\min I_r\}\}$ and 2^U partially by inclusion, so that $J' \prec J$ if $J' \subsetneq J$ and

$U'^{\circ} \prec U^{\circ}$ if $U'^{\circ} \subsetneq U^{\circ}$; extend these partial orders arbitrarily to total orders. Define an arbitrary total order $\prec$ on $2^{U \setminus U^{\circ}}$. Partially order I such that $i < i'$ if $i \in I_r$, $i' \in I_s$, and $i - \min I_r < i_s - \min I_s$; extend arbitrarily to a total order on I.

It remains to order 2^V. This will be slightly more complicated. To do this, we first define a total order $\prec_r$ on 2^{V_r} for each $r \in \mathbb{Z}_m$. First consider the case $I_r \ni i_*$, i.e. $I_r = I_*$. If $i_* \geq \max I_* - 1$, we have $V_r = \varnothing$, so there is nothing to do. Otherwise, we have $i_* \leq \max I_* - 2$ and $V_r = \{i_* + 1, \max I_*\}$; in this case, order 2^{V_r} as follows:

$$\{i_* + 1\} \prec_r \varnothing \prec_r \{i_* + 1, \max I_*\} \prec_r \{\max I_*\}.$$

Now consider the case $I_r \not\ni i_*$. Define $\prec_r$ on 2^{V_r} recursively by $V_r^- \prec_r V_r'^-$ if:

- $\max V_r^- < \max V_r'^-$, or
- $\max V_r^- = \max V_r'^-$ and $V_r'^- \setminus \{\max V_r'^-\} \prec_r V_r^- \setminus \{\max V_r^-\}$.

Note the reversal of order on the line above. If we assume without loss of generality that $V_r = \{0, \dots, b\}$, we can write the order explicitly:

$$\begin{aligned} &\varnothing \prec_r \{0\} \prec_r \{0,1\} \prec_r \{1\} \prec_r \{1,2\} \prec_r \{0,1,2\} \prec_r \{0,2\} \prec_r \{2\} \\ &\quad \prec_r \{2,3\} \prec_r \{0,2,3\} \prec_r \{0,1,2,3\} \prec_r \{1,2,3\} \prec_r \{1,3\} \prec_r \cdots \\ &\cdots \prec_r \{0,1,2,b\} \prec_r \{1,2,b\} \prec_r \{1,b\} \prec_r \{0,1,b\} \prec_r \{0,b\} \prec_r \{b\}. \end{aligned} \tag{23}$$

See Tables 9 and 18, and the part of Table 16 where $i_* = 4$.

Use the orderings $\prec_r$ on 2^{V_r} to define a partial order on 2^V by declaring $V^- \prec V'^-$ if

- $V^- \cap I_r \prec_r V'^- \cap I_r$ for some r, and
- there is no r for which $V'^- \cap I_r \prec_r V^- \cap I_r$.

Extend $\prec$ arbitrarily to a total order on 2^V. This determines a total order on

$$\left\{(J, i_*, V^-, U^{\circ}, U^-)\right\}_{J \subset T,\ i_* \in I,\ V^- \subset V,\ U^{\circ} \subset U,\ U^- \subset U \setminus U^{\circ}}, \tag{24}$$

and thus on the pieces $X_{I,J,i_*,V^-,U^{\circ},U^-}$. Relabel these pieces as Y_z, $z = 1, \dots, \#(24)$, according to this order.

7.1.5 Decomposing each Y_z into handles

Each Y_z is now given by an expression of the form

$$\left\langle \prod_{r=1}^{n} \chi_r \right\rangle, \tag{25}$$

where each χ_r is either a closed interval or a singleton. Fixing arbitrary z, use the expression (25) to define the coarsest equivalence relation $\sim$ on $\{1,\dots,n\}$ that obeys the following property: whenever $\chi_r \subset \chi_s$, we have $r \sim s$. Denote the set of equivalence classes under $\sim$ by $P = \{R_1,\dots,R_p\}$, and for each $r = 1,\dots,p$, denote $\left\langle \prod_{s\in R_r} \chi_s \right\rangle = \xi_r$. Define

$$Y_z^* = \prod_{r=1}^{p} \xi_r. \tag{26}$$

In §7.2, we will see that each Y_z^* is a handle, and that attaching Y_z to $\bigcup_{s=1}^{z-1} Y_s$ amounts to attaching a collection of handles, each of which is related to Y_z^* as follows. Let

$$G = \{\sigma \in S_n : \vec{x}_\sigma \in Y_z^* \text{ whenever } \vec{x} \in Y_z^*\} = S_{n_{|R_1|}} \times \cdots \times S_{n_{|R_p|}}$$

consist of the permutations on the indices of T^n which fix Y_z^* setwise. Then there is a one-to-one correspondence between the left cosets of G and the handles comprising Y_z:

$$\tau G \longleftrightarrow \{\vec{x}_\tau : \vec{x} \in Y_z^*\}.$$

Example 7.1 Consider $X_I \subset T^9$ where $I = \{0,1,2,3\}$, which is detailed in Tables 14 and 15. Note that $T = \{0\}$. In particular, consider the first and twelfth rows of Table 14 (after the headings), where $J = \varnothing$, $i_* = 0$, $U = \{2\}$, and $V = \{1,3\}$. The first row of Table 14 corresponds to

$$Y_1 = X_{I,J,s,V^-,U^\circ,U^-} = \left\langle \alpha^- 1 \beta_3^\circ 2 \gamma^+ 3 \delta^3 \right\rangle, \tag{27}$$

where $V^- = \{1\}$, $U^\circ = \{2\}$, and $U^- = \varnothing$ with

$$\chi_1 = \alpha^- = \left[0, \frac{1}{2}\right], \ \chi_2 = \{1\}, \ \chi_3 = \beta_3^\circ = \left[\frac{4}{3}, \frac{5}{3}\right], \ \chi_4 = \{2\},$$
$$\chi_5 = \gamma^+ = \left[\frac{5}{2}, 3\right], \ \chi_6 = \{3\}, \text{ and } \chi_7 = \chi_8 = \chi_9 = \delta = [3,4].$$

The ensuing partition of $\{1,\dots,9\}$ gives

$$P = \{\{1\},\{2\},\{3\},\{4\},\{5,6,7,8,9\}\},$$

and so

$$\begin{aligned} Y_1^* &= \chi_1 \times \chi_2 \times \chi_3 \times \chi_4 \times \langle \chi_5 \times \chi_6 \times \chi_7 \times \chi_8 \times \chi_9 \rangle \\ &= \alpha^- 1 \beta_3^\circ 2 \left\langle \gamma^+ 3 \delta^3 \right\rangle, \end{aligned}$$

where

$$\xi_1 = \alpha^-, \ \xi_2 = \{1\}, \ \xi_3 = \beta_3^\circ, \xi_4 = \{2\}, \text{ and } \xi_5 = \left\langle \gamma^+ 3 \delta^3 \right\rangle.$$

The twelfth row of Table 14 corresponds to

$$Y_{12} = X_{I,J,s,V^-,U^\circ,U^-} = \left\langle \alpha^+ 1 \beta_3^+ 2\gamma^- 3\delta^3 \right\rangle,$$

where $V^- = \{3\}$, $U^\circ = \varnothing = U^-$. The ensuing partition of $\{1,\dots,9\}$ gives

$$P = \{\{1,2\},\{3,4,5\},\{6,7,8,9\}\},$$

and so

$$\begin{aligned} Y_{12}^* &= \langle \chi_1 \times \chi_2 \rangle \times \langle \chi_3 \times \chi_4 \times \chi_5 \rangle \times \langle \chi_6 \times \chi_7 \times \chi_8 \times \chi_9 \rangle \\ &= \underbrace{\left\langle \alpha^+ 1 \right\rangle}_{\xi_1} \underbrace{\left\langle \beta_3^+ 2\gamma^- \right\rangle}_{\xi_2} \underbrace{\left\langle 3\delta^3 \right\rangle}_{\xi_3}. \end{aligned}$$

7.2 Properties of handle decompositions

7.2.1 Combinatorics

Proposition 7.2 *Let $i \in I_s \cap V^-$ for some $s \in \mathbb{Z}_m$, where $i_* \notin I_s$. Denote $b = \max I_s$, $c = \max(I_s \cap V^-)$. Let $V'^- = V^- \setminus \{i\}$. Then $V'^- \prec V^-$ if and only if $|V^- \cap \{i+1,\dots,b\}|$ is even.*

Proof We argue by induction on $c - i$. When $c - i = 0$, we have $c = i > \max(I_s \cap V^- \setminus \{i\})$ and $I_r \cap V^- = I_r \cap V^- \setminus \{i\}$ for all $r \neq s$, so $V'^- \prec V^-$.

Now assume that $c - i = t > 0$, and assume that the claim is true whenever $\max(I_s \cap V^-) - i < t$. Let $W^- = V^- \setminus \{c\}$ and $W'^- = V'^- \setminus \{c\}$. Then $|V^- \cap \{i+1,\dots,b\}|$ and $|W^- \cap \{i+1,\dots,b\}|$ have opposite parities. Also, by construction, $V^- \prec V'^-$ if and only if $W'^- \prec W^-$. The result now follows by induction. □

Notation 7.3 Denote the symmetric difference of sets R and S by

$$R \ominus S = (R \setminus S) \cup (S \setminus R).$$

Proposition 7.4 *Let $A \subset V$ such that $V^- \prec V^- \ominus \{a\}$ for each $a \in A$. Then $V^- \prec V^- \setminus A$.*

Proof Suppose first that $A \subset I_s$ for some $s \in \mathbb{Z}_m$. Denote $A = \{a_1,\dots,a_q\}$ with $\min I_s \le a_1 \le \cdots \le a_q \le \max I_s = b$. Assume that $i_* \notin I_s$ and $|I_s| \ge 3$ (the other cases are trivial). Proposition 7.2 implies, for each $a \in A$, that $|V^- \cap \{a+1,\dots,b\}|$ is odd if and only if $a \in V^-$. For each $r = 1,\dots,q$, denote the symmetric difference

$V_r^- = V^- \ominus \{a_1, \dots, a_r\}$. Then, $|V_a^- \cap \{a+1, \dots, b\}| = |V^- \cap \{a+1, \dots, b\}|$ for each $a = 0, \dots, q-1$. Since this quantity is odd if and only if $a \in V^-$, Proposition 7.2 implies:

$$V^- \prec V_1^- \prec \cdots \prec V_q^- = V^- \setminus A.$$

For the general case, apply this argument repeatedly for each $s \in \mathbb{Z}_m$. □

7.2.2 Topology

Observation 7.5 *In X_I, if Y_z comes from $(J, i_*, V^-, U^\circ, U^-)$ and Y_w comes from $(J, i_*, V^-, U^\circ, U'^-)$, then $Y_z \cap Y_w = \varnothing$ unless $U^- = U'^-$. That is, if $U^- \neq U'^-$, then*

$$X_{I,J,i_*,V^-,U^\circ,U^-} \cap X_{I,J,i_*,V^-,U^\circ,U'^-} = \varnothing.$$

Lemma 7.6 *Each factor ξ_r in the expression (26) for Y_z^* has one of the forms described in Lemma 4.2, and thus is PL homeomorphic to $D^{d(r)}$ for some $d(r) \geq 0$.*

Moreover, $\sum_{r=1}^p d(r) = n + 1 - |I|$, so $Y_z^* \cong D^{n+1-|I|}$.

Proof Regarding the first claim, we examine the equivalence relation $\sim$ that led to (26). Suppose $\chi_r \subset \chi_{r'}$. Then, by construction, either χ_r is a singleton (in $I \setminus \{i_*\}$) and $\chi_{r'}$ is an interval with this singleton as an endpoint, or else $\chi_r \supset [\max I_s, \max I_s + 1]$ for some $s \in \mathbb{Z}_\ell$. Moreover, by construction, if $\chi_{r'}$ contains a point of $I \setminus \{i_*\}$, then it contains only one such point and it contains no interval of the form $[\max I_s, \max I_s + 1]$, and no $\chi_{r'}$ contains more than one interval of the form $[\max I_s, \max I_s + 1]$. The first claim now follows. (For an explicit accounting of the types of factors $\xi_r = \left\langle \prod_{s \in R_r} \chi_s \right\rangle$, see Tables 19, 20, and 21.)

Regarding the second claim, note for each $r = 1, \dots, p$, that $d(r)$ equals the number of intervals among $\{\chi_s\}_{s \in R_r}$, which equals the order of R_r minus the number of singletons among $\{\chi_s\}_{s \in R_r}$. Since $\sum_{r=1}^p |R_r| = |P| = n$ and $\{\chi_s : s = 1, \dots n\}$ contains a total of $|I| - 1$ singletons, it follows that $\sum_{r=1}^p d(r) = n + 1 - |I|$. Thus, $Y_z^* \cong D^{n+1-|I|}$. □

We wish to show, in arbitrary X_I, that attaching any Y_z to $\bigcup_{w<z} Y_w$ amounts to attaching a collection of $(n+1-|I|)$-dimensional $h(z)$-handles for some $h(z)$. Indeed, Lemma 7.6 confirms that each Y_z^* from X_I is a compact $(n+1-|I|)$-ball, so it remains to consider how everything is glued together. Our goal is to show that

$$Y_z^* \cap \bigcup_{w<z} Y_w \cong S^{h(z)-1} \times D^{n+1-|I|-h(z)} \tag{28}$$

and

$$Y_z^* \cap (Y_z \setminus \setminus Y_z^*) \subset Y_z^* \cap \bigcup_{w<z} Y_w. \tag{29}$$

The former will imply that attaching Y_z^* to $\bigcup_{w<z} Y_w$ amounts to attaching an $(n+1-|I|)$-dimensional $h(z)$-handle, and the latter will further imply that if we attach all the copies of Y_z^* one at a time to $\bigcup_{w<z} Y_w$, then attaching each copy amounts to attaching another $(n+1-|I|)$-dimensional $h(z)$-handle.

Recall that each Y_z^* has the form $\prod_{r=1}^{p} \xi_r(z)$. Hence,

$$\partial Y_z^* = \bigcup_{a=1}^{p} \left(\prod_{r=1}^{a-1} \xi_r(z) \times \partial \xi_a(z) \times \prod_{r=a+1}^{p} \xi_r(z) \right).$$

We will show, given arbitrary Y_z^* in X_I, that there is a subset $S(z) \subset \{1, \ldots, p\}$ such that

$$Y_z^* \cap \bigcup_{w<z} Y_w = \bigcup_{a \in S(z)} \left(\prod_{r=1}^{a-1} \xi_r(z) \times \xi_a(z) \times \prod_{r=a+1}^{p} \xi_r(z) \right). \tag{30}$$

Then, denoting $h(z) = \sum_{r \in S(z)} \dim(\xi_r)$, we will obtain (28):

$$\begin{aligned} Y_z^* \cap \bigcup_{w<z} Y_w &= \bigcup_{a \in S(z)} \left(\prod_{r=1}^{a-1} \xi_r(z) \times \xi_a(z) \times \prod_{r=a+1}^{p} \xi_r(z) \right) \\ &\cong \left(\partial \prod_{r \in S(z)} \xi_r(z) \right) \times \prod_{r \notin S(z)} \xi_r(z) \\ &\cong \partial D^{h(z)} \times D^{n+1-|I|-h(z)} \\ &= S^{h(z)-1} \times D^{n+1-|I|-h(z)}, \end{aligned}$$

Our next step is to describe the subset $S(z) \subset \{1, \ldots, p\}$. To do so, we characterize each $\xi_r(z)$ as type (A) or type (B); then $S(z)$ will consist of those $r = 1, \ldots, p$ for which $\xi_r(z)$ has type (A). After that, Lemmas 7.7 and 7.8 will establish (30) by double containment, implying (28), and Lemma 7.9 will establish (29).

Consider an arbitrary $Y_z^* = \prod_{r=1}^{p} \xi_r(z)$ from an arbitrary X_I. In the following way, classify each factor $\xi_r(z)$ into one of two classes, (A) or (B). Say that $\xi_r(z)$ is in class (B) if

- $\xi_r(z) = \left[i - \frac{2}{3}, i - \frac{1}{3}\right]$ for some $i \in I$;

- $\xi_r(z) = \left[i_*, i_* + \frac{1}{2}\right]$;
- $[\max I_s, j]$ is a factor in the expression for $\xi_r(z)$ for some s, j; or
- Some $\{i\}$ is a factor in the expression for $\xi_r(z)$ and:
  - $i \in V^+$ and $i+1 \in U^\circ \cup U^+ \cup V^+$, or $i \in U^- \cup U^\circ \cup V^-$ and $i+1 \in V^-$; and
  - $|V^- \cap \{i+1, \ldots, \max I_s\}|$ is even, where $i \in I_s$.

All other types of $\xi_r(z)$ are of class (A). Tables 19, 20, and 21 in Appendix 1 list the possibilities explicitly.

Lemma 7.7 *Suppose $Y_z^* = \prod_{r=1}^p \xi_r(z)$ comes from $(J, i_*, V^-, U^\circ, U^-)$. If, for some $a = 1, \ldots, p$, $\xi_a(z)$ is of class (A) and*

$$\vec{x} = (x_1, \ldots, x_n) \in \prod_{r=1}^{a-1} \xi_r(z) \times \partial\xi_a(z) \times \prod_{r=a+1}^{p} \xi_r(z),$$

then $\vec{x} \in Y_w$ for some $w < z$.

Proof Suppose first that some $\{i\}$ appears in the expression for $\xi_a(z)$, with $i \in I_s$; $i \in V^+$ and $i+1 \in U^\circ \cup U^+ \cup V^+$, or $i \in U^- \cup U^\circ \cup V^-$ and $i+1 \in V^-$; and $|V^- \cap \{i+1, \ldots, \max I_s\}|$ is odd. Then $\vec{x}$ is in the Y_w coming from $(J, i_*, V'^-, U^\circ, U^-)$ where V'^- is either $V^- \cup \{i\}$ or $V^- \setminus \{i+1\}$. In either case, Proposition 7.2 implies that $V'^- \prec V$ and thus $w < z$.

Next, suppose that $\xi_a(z)$ has no singleton factors. There are two possibilities. If $\xi_a(z) = [i_* - 1, i_*]$ with $i_* \in J$, then $\vec{x}$ is in some Y_w coming from $J \setminus \{i_*\} \prec J$. Otherwise, $\xi_a(z) = \left[i-1, i-\frac{1}{2}\right]$ for some $i \in J \cap V^-$; in this case, $i+1 \notin I$, and so $\vec{x}$ is in some Y_w coming either from $J \setminus \{i\} \prec J$ or the from same J and i_* and $V'^- = V^- \setminus \{i\}$, where Proposition 7.2 implies that $V'^- \prec V^-$ because $i+1 \notin I$.

The remaining cases follow by similar reasoning. The interested reader may find Table 21 useful for this. □

Lemma 7.8 *Let $Y_z^* = \prod_{r=1}^p \xi_r(z)$ come from some $(J, i_*, V^-, U^\circ, U^-)$. If*

$$\vec{x} = (x_1, \ldots, x_n) \in Y_z^* \cap \bigcup_{w<z} Y_w,$$

then

$$\vec{x} \in \prod_{r=1}^{a-1} \xi_r(z) \times \partial\xi_a(z) \times \prod_{r=a+1}^{p} \xi_r(z)$$

for some $a = 1, \ldots, p$, such that $\xi_a(z)$ is of class (A).

Proof Let $\vec{x} = (x_1, \ldots, x_n) \in Y_z^* \cap Y_{w'}$ for some $w' < z$. Choose the smallest $w < z$ such that $\vec{x} \in Y_w$, and assume that Y_w comes from some $(J', i'_*, V'^-, U'^\circ, U'^-)$ with $V'^- \subset V'$ and $U'^\circ \subset U'$, whereas Y_z comes from some $(J, i_*, V^-, U^\circ, U^-)$ with $V^- \subset V$ and $U^\circ \subset U$. Denote

$$S = \left\{ a = 1, \ldots, p : \ \vec{x} \in \prod_{r=0}^{a-1} \xi_r(z) \times \partial\xi_a(z) \times \prod_{r=a+1}^{p} \xi_r(z) \right\}.$$

Assume for contradiction that $\xi_a(z)$ is of class (B) for every $a \in S$. If $S = \varnothing$, then no coordinate of $\vec{x}$ equals i_*, so $i'_* = i_*$. Also, in that case, no coordinate of $\vec{x}$ equals $\min I_s - 1$ for any $s \in \mathbb{Z}_m$, and so J and J' completely determine the number of coordinates that $\vec{x}$ has in each open interval $(\min I_s - 1, \min I_{s+1} - 1)$. It follows that either $J' = J$ or $J' = T \setminus J$. If $J' = T \setminus J$, then considering the coordinates of $\vec{x}$ in $[\min I_*, \max I_*]$ yields a contradiction. If $J' = J$, then the fact that $S = \varnothing$ implies that $V^- = V'^-$, $U^\circ = U'^\circ$, and $U^- = U'^-$, contradicting the fact that $w < z$.

Therefore, $S \neq \varnothing$. If no coordinate of $\vec{x}$ equals i_*, then $i'_* = i_*$, so again either $J' = J$ or $J' = T \setminus J$. The latter case gives the same contradiction as before. Therefore $J' = J$, and so $V' = V$.

For each $i \in V^- \ominus V'^-$, $\vec{x}$ has a coordinate $x_t = i - \frac{1}{2}$ (using the fact that $i'_* = i_*$ and $J' = J$). The corresponding $\xi_r(z)$ has $r \in S$, and so by assumption $\xi_r(z)$ is of class (B). Therefore, $V^- \prec V^- \ominus \{i\}$ for each $i \in V^- \ominus V'^-$. Proposition 7.4 implies that $V^- \prec V'^-$ unless $V^- = V'^-$. Since $w < z$, we must have $V^- = V'^-$.

Each $i \in U'^\circ$ must also be in U°, or else the corresponding coordinate of $\vec{x}$ would equal $i - \frac{1}{3}$ or $i - \frac{2}{3}$, and the corresponding $\xi_a(z)$ would be of class (A) with $a \in S$, contrary to assumption. Thus, $U^\circ \subset U'^\circ$. Similarly, each $i \in U^\circ$ must also be in U'°, or else the $Y_{w'}$ coming from $J, i_*, V, U'^\circ \cup \{i\}, U^- \setminus \{i\}$ would still contain $\vec{x}$ but with $w' < w$, contrary to assumption. Thus, $U'^\circ = U^\circ$.

Finally, we must have $U'^- = U^-$, by Observation 7.5. This implies, contrary to assumption, that $Y_w = Y_z$. □

Lemma 7.9 *Let $Y_z^* = \prod_{r=1}^{p} \xi_r(z)$ come from some $(J, i_*, V^-, U^\circ, U^-)$. If*

$$\vec{x} = (x_1, \ldots, x_n) \in Y_z^* \cap (Y_z \setminus \setminus Y_z^*),$$

then

$$\vec{x} \in \prod_{r=1}^{a-1} \xi_r(z) \times \partial \xi_a(z) \times \prod_{r=a+1}^{p} \xi_r(z)$$

for some $a = 1, \ldots, p$*, such that* $\xi_a(z)$ *is of class* (A)*.*

Proof This follows from a case analysis, for which the interested reader may find Tables 19–21 useful. It comes down to this. Consider two pieces $\xi_a(z)$ and $\xi_b(z)$ of Y_z^* for which the infimum $\min \xi_b(z)$ of all coordinates in $(0,k)$ among all points in $\xi_b(z)$ equals the supremum $\max \xi_a(z)$ of all coordinates in $(0,k)$ among all points in $\xi_a(z)$. Denote $\max \xi_a(z) = \min \xi_b(z) = c$. Then $c \in \mathbb{Z}_k$. If c equals $i-1$ for some $i \in T$, then $i \in J$ and $\xi_b(z)$ is of class (A). Otherwise, $c = i_*$ and $\xi_a(z)$ is of class (A). □

7.3 Proof of the main result

The results of §§6, 7.2 provide all the details we need to prove:

Theorem 7.10 *For* $n = 2k-1 \in \mathbb{Z}_+$*, the* n*-torus admits a multisection* $T^n = \bigcup_{r \in \mathbb{Z}_k} X_r$ *defined by*

$$(1) \qquad \begin{aligned} X_0 &= \left\{ \vec{x}_\sigma : \ \vec{x} \in [0,1]^2 \cdots [0,k-1]^2 [0,k] / \sim, \ \sigma \in S_n \right\}, \\ X_i &= \{ \vec{x} + (i, \ldots, i) : \ \vec{x} \in X_0 \}. \end{aligned}$$

Proof Lemma 6.8 implies that $X = \bigcup_{i \in \mathbb{Z}_k} X_i$, so it remains only to prove for each nonempty proper subset $I \subset \mathbb{Z}_k$, that $X_I = \bigcap_{i \in I} X_i$ is an $(n+1-|I|)$-dimensional submanifold of X with a spine of dimension $|I|$.

Fix some such I. Assume WLOG that I is simple. Then $X_I =$ (2), by Lemma 6.13. Decompose $X_I = \bigcup_z Y_z$ as described in §7.1. Lemmas 7.6 and 7.7 imply that Y_1^* is an $(n+1-|I|)$-dimensional 0-handle with no pieces $\xi_r(1)$ of class (A); Lemma 7.9 and the symmetry of the construction imply further that Y_1 is a union of $(n+1-|I|)$-dimensional 0-handles.

For each z, denote $S(z) = \{ r : \ \xi_r(z) \text{ is of class (A)} \}$. Lemmas 7.6, 7.7, and 7.8 imply that attaching Y_z^* to $\bigcup_{w<z} Y_z$ amounts to attaching an $(n+1-|I|)$-dimensional h-handle, where $h(z)$ is the sum of the dimensions of those $\xi_r(z)$ of class (A):

$$h(z) = \sum_{r \in S(z)} \dim(\xi_r(z)) \leq |I|.$$

Lemma 7.9 and the symmetry of the construction imply further that attaching all of Y_z to $\bigcup_{w<z} Y_w$ amounts to attaching several such handles. Thus, X_I is an $(n+1-|I|)$-dimensional $|I|$-handlebody in T^n.

It remains to check that $X_{\mathbb{Z}_k} = \bigcap_{i\in\mathbb{Z}_k} X_i$ is a closed k-manifold. We know from Lemma 6.13 that $X_{\mathbb{Z}_k}$ is given by (3).

Since $X_{\mathbb{Z}_k\setminus\{k-1\}}$ is $(k+1)$-manifold, it suffices to check that $X_{\mathbb{Z}_k}$ equals $\partial X_{\mathbb{Z}_k\setminus\{k-1\}}$, which is the union of those k-faces of the Y_z from the handle decomposition of $X_{\mathbb{Z}_k\setminus\{k-1\}}$ that are not glued to any other Y_w. Case analysis confirms that this union equals the expression from (3). (The reader may find Tables 19-21 useful). □

Alternatively, one can construct a handle decomposition of $X_{\mathbb{Z}_k}$ as follows. Cut each unit interval $[i, i+1]$ into thirds and, for each $i_* \in \mathbb{Z}_k$, further cut $\left[i_* - \frac{1}{3}, i_*\right]$ and $\left[i_*, i_* + \frac{1}{3}\right]$ into halves. Then, for each $i_* \in \mathbb{Z}_k$, $U^\circ \subset \mathbb{Z}_k$, $U^- \subset \mathbb{Z}_k \setminus U^\circ$, and $U^* \subset (\{i_*+1\} \cap U^-) \cup (\{i_*\} \setminus (U^\circ \cup U^-)$, define

$$\rho_i = \begin{cases} [i-\frac{2}{3}, i-\frac{1}{3}] & i \in U^\circ \\ [i-1, i-\frac{2}{3}] & i_*+1 \neq i \in U^- \\ [i-\frac{1}{3}, i] & i_* \neq i \in \mathbb{Z}_k \setminus (U^\circ \cup U^-) \\ [i_*, i_*+\frac{1}{6}] & i_*+1 = i \in U^* \\ [i_*+\frac{1}{6}, i_*+\frac{1}{3}] & i_*+1 = i \in U^- \setminus U^* \\ [i_*-\frac{1}{6}, i_*] & i_* = i \in U^* \\ [i_*-\frac{1}{3}, i_*-\frac{1}{6}] & i_* = i \in U^+ \setminus U^*, \end{cases}$$

$$X_{\mathbb{Z}_k, i_*, U^\circ, U^-, U^*} = \prod_{i\in\mathbb{Z}_k} \begin{Bmatrix} \rho_i \times \{i\} & i \neq i_* \\ \rho_i & = i_* \end{Bmatrix}.$$

Order the pieces $X_{\mathbb{Z}_k, i_*, U^\circ, U^-, U^*}$ as Y_z, $z = 1, 2, 3, \ldots$, lexicographically according to the following orders on the possibilities for (i_*, U°, U^-, U^*). Order $\{i_* \in I\}$ and $U^- \subset U^\circ$ arbitrarily. Partially order $\{U^\circ \subset \mathbb{Z}_k\}$ by inclusion, with $U^\circ \prec U'^\circ$ if $U^\circ \subset U'^\circ$, and extend arbitrarily to a total order. Order the possibilities for U^* the same way. Then

$$\bigcup_{i=1,\ldots,k} Y_z = \bigcup_{i_*\in\mathbb{Z}_k} X_{\mathbb{Z}_k, i_*, \mathbb{Z}_k, \varnothing, \varnothing}$$

is a union of 0-handles, and to attach each $Y_z = X_{\mathbb{Z}_k, i_*, U^\circ, U^-, U^*}$ to $\bigcup_{w<z} Y_w$ is to attach a collection of $h(z)$-handles for $h(z) = k - |U^\circ| - |U^*|$.

We leave the following question open:

Question 6 Are the multisections in Theorem 7.10 smoothable?

That is, for odd n, does T^n (under its standard smooth structure) admit a smooth multisection such that, when one passes to the unique PL structure on T^n, there is a PL homeomorphism $f : T^n \to T^n$ sending each piece of this smooth multisection to a piece of the multisection from Theorem 7.10?

8 Cubulated manifolds of odd dimension

This section extends Theorem 7.10 to certain cubulated manifolds. Consider a covering space $p : M \to T^n$, where $n = 2k-1$. Multisect $T^n = \bigcup_{i\in\mathbb{Z}_k} X_i$ as in Theorem 7.10. Then, by Corollary 17 of [RuTi20], $M = \bigcup_{i\in\mathbb{Z}_k} p^{-1}(X_i)$ determines a PL multisection of M. In general, one expects such multisections to be less efficient than those from Theorem 7.10. Also, there seems to be no reason to expect that one can extend the main construction to cubulated odd-dimensional manifolds in general. There is, however, an intermediate case to which our construction does extend.

First, we propose a modest generalization of the usual notion of a cubulation. The generalization is similar to Hatcher's Δ-complexes vis a vis simplicial complexes [Ha02]. A *cube* is a homeomorphic copy of I^n for some $n \geq 0$, with the usual cell structure; its *faces* are defined in the traditional way.

Consider an arbitrary edge of I^n, joining $\vec{a} = (a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n)$ and $\vec{b} = (a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n)$. Orient this edge so that it runs from $\vec{a}$ to $\vec{b}$. Do the same with every edge of the n-cube. Call these the *standard orientations* on the edges of the n-cube. Call a face of I^n *positive* if it contains $\vec{0}$; otherwise it is *negative*, containing $\vec{1} = (1, \dots, 1)$.

Definition 8.1 A **□-complex** K is a quotient space of a collection of disjoint cubes obtained by identifying certain faces of theirs via PL homeomorphisms.[14] If all of these face identifications glue a positive face of one cube to a negative face of another (not necessarily distinct) cube and respect the standard orientations on all edges, then K is a **directed** □-complex.

Note that, by definition, a □-complex comes equipped with a cell structure.

[14]Unlike the traditional notion of cubulation, we do not require that these identifications are between faces of *distinct* cubes.

Definition 8.2 A **generalized cubulation** of a manifold M is a PL homeomorphism to a ⧉-complex. A **directed cubulation** of M is a PL homeomorphism to a ⧉-complex.

In other words, a generalized cubulation of an n-manifold M imposes a cell structure on M in which every n-cell “looks like" an n-cube, and in a directed cubulation, the n-cells are glued in a particularly nice way.

Example 8.3 The usual cell structure on T^n determines a generalized cubulation, and in fact a directed cubulation, but not a cubulation in the traditional sense.

Let $f : M \to K$ be a directed cubulation of an n-manifold, $n = 2k - 1$, let $g : I^n = [0,k]^n \to T^n = (\mathbb{R}/k\mathbb{Z})^n = [0,k]^n/\sim$ be the quotient map, and multisect $T^n = \bigcup_{i \in \mathbb{Z}_k} X_i$ as in Theorem 7.10. Multisect M as follows. For each n-cell C in K, let $h_C : I^n \to C$ be the identification from K. For each $i \in \mathbb{Z}_k$, define

$$X_i' = \bigcup_{n\text{-cubes } C \text{ in } K} f^{-1}(h_C(g^{-1}(X_i))).$$

Proposition 8.4 *With the setup above, $M = \bigcup_{i \in \mathbb{Z}_k} X_i'$ determines a multisection of M.*

Proof First consider the case where $p : M \to T^n$ is a covering space. Let $I \subset \mathbb{Z}_k$ be arbitrary. Construct a handle structure on $X_I \subset T^n$, as in §7.1. By construction, each handle is a subset of some open cube $(a, a+k)^n \subset T^n$. Hence, the handle structure on $X_I \subset T^n$ pulls back to a handle structure on $X_I' \subset M$. The general case follows for the same reason, due to the fact that the multisection of T^n is fixed by the permutation action on the indices. □

Remark In any multisection $M = \bigcup_{i \in \mathbb{Z}_k} X_i'$ from Proposition 8.4, all X_i' have genus $n\#(n\text{-cubes in } K)$. In particular, if $p : M \to T^n$ is an $r : 1$ covering space, then M has a multisection $M = \bigcup_{i \in \mathbb{Z}_k} X_i'$ in which each X_i' has genus nr.

Example 8.5 Consider the quotient space M obtained from I^3 by identifying the front and right faces, the left and top faces, and the bottom and back faces, all in the way that respects the standard orientations on the edges of I^n. See Figure 10, left. The natural cell structure on M consists of one vertex, three edges, three faces, and one 3-cell. It is easy to check that the link of the vertex is a 2-sphere, and so M is a 3-manifold. Geometrically, M is geometrically flat, since there is a 27:1 covering space $T^3 \to M$ (see Figure 10, right). But M is not T^3, since $H_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}_3$. Proposition 8.4 gives a genus 3 Heegaard splitting of M. Does M have an efficient (genus 2) splitting? We leave this as a puzzle for the reader.

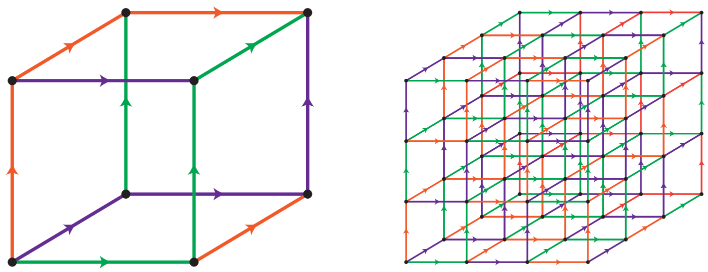

Figure 10: Face identifications (left) for the 3-manifold M from Example 8.5, and a 27:1 covering space (right) $T^3 \to M$.

Example 8.6 Generalizing Example 8.5, let $n = 2k - 1$, and let $\sigma \in S_n$ be an even permutation. Denote the faces of I^n by $F_i^{\pm}$, where $F_i^+ = \{(x_1, \dots, x_n) : x_i = 1\}$ and $F_i^- = \{(x_1, \dots, x_n) : x_i = 0\}$. Identify each F_i^+ with $F_{\sigma(i)}^-$ by identifying each point $(x_1, \dots, 1, \dots, x_n) \in F_i^+$ (where the 1 is in the i^{th} spot) with $(x_{\sigma^{-1}(1)}, \dots, 0, \dots, x_{\sigma^{-1}(n)}) \in F_{\sigma(i)}^-$ (where the 0 is in the $\sigma^{-1}(i)^{\text{th}}$ spot).

Question 7 For what n and $\sigma \in S_n$ does the construction in Example 8.6 produce a manifold M? When it is a manifold, is M always distinct from T^n? Is the multisection of M from Proposition 8.4 ever efficient?

Appendix 1: Additional tables detailing handle decompositions

Tables 11 and 12 explicitly detail $U_r, V_r \subset I_r$ for arbitrary I_r (following Notation 3.8). For simplicity, these tables have $I_r = I_0 = \{0, \dots, w\}$, listing U_0, V_0; this is not necessarily consistent with Convention 3.9. To adapt $U_0, V_0 \subset I_0$ to the general case $U_r, V_r \subset I_r$, add $\min I_r$ in each coordinate.

I_0	$0 \notin J$		$0 \in J$	
	U_0	V_0	U_0	V_0
$\{0\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\{0\}$
$\{0,1\}$	$\varnothing$	$\{1\}$	$\varnothing$	$\{1\}$
$\{0,1,2\}$	$\varnothing$	$\{1,2\}$	$\{1\}$	$\{2\}$
$\{0,1,2,3\}$	$\varnothing$	$\{1,2,3\}$	$\{1,2\}$	$\{3\}$
$\{0,1,2,3,4\}$	$\varnothing$	$\{1,2,3,4\}$	$\{1,2,3\}$	$\{4\}$
$\{0,\ldots,w\}$	$\varnothing$	$\{1,\ldots,w\}$	$\{1,\ldots,w-1\}$	$\{w\}$

Table 11: The index subsets $U_0, V_0 \subset I_0$ when $i_* \notin I_0$.

I_0	U_0	V_0
$\{0\}$	$\varnothing$	$\varnothing$
$\{0,1\}$	$\varnothing$	$\varnothing$
$\{0,1,2\}$	$\left\{\begin{matrix} \{1\} & i_* = 2 \\ \varnothing & i_* \neq 2 \end{matrix}\right\}$	$\left\{\begin{matrix} \{1,2\} & i_* = 0 \\ \varnothing & i_* \neq 0 \end{matrix}\right\}$
$\{0,1,2,3\}$	$\left\{\begin{matrix} \{2\} & i_* = 0 \\ \varnothing & i_* = 1 \\ \{1\} & i_* = 2 \\ \{1,2\} & i_* = 3 \end{matrix}\right\}$	$\left\{\begin{matrix} \{i_*+1,3\} & i_* \leq 1 \\ \varnothing & i_* \geq 2 \end{matrix}\right\}$
$\{0,\ldots,w\}$	$I_0 \setminus \{0, i_*, i_*+1, w\}$	$\left\{\begin{matrix} \{i_*+1,w\} & i_* \leq w-2 \\ \varnothing & i_* \geq w-1 \end{matrix}\right\}$

Table 12: The index subsets $U_0, V_0 \subset I_0$ when $i_* \in I_0$.

Table 13 details the handle decomposition of X_I from T^9 with $I = \{0,1,3\} = I_1 \sqcup I_2$, $I_1 = \{0,1\}$, $I_2 = \{3\}$. The interesting feature of this example is how the two blocks of indices I_1, I_2 interact.

J	i_*	U	V	V^-	Y_z^*	h	z	glue to
$\varnothing$	0	$\varnothing$	$\varnothing$		$\langle\alpha 1\beta^3\rangle\langle 3\delta^3\rangle$	0	1	
	1	$\varnothing$	$\varnothing$	$\varnothing$	$\langle 0\alpha\rangle\beta^3\langle 3\delta^3\rangle$	1	2	1
	3	$\varnothing$	$\{1\}$	$\varnothing$	$0\langle\alpha^+1\beta^3\rangle\delta^3$	0	3	
				$\{1\}$	$\langle 0\alpha^-\rangle\langle 1\beta^3\rangle\delta^3$	1	4	3
$\{0\}$	0	$\varnothing$	$\varnothing$	$\varnothing$	$\langle\alpha 1\beta^3\rangle\langle 3\delta^2\rangle\varepsilon$	1	5	1,3,4
	1	$\varnothing$	$\varnothing$	$\varnothing$	$\langle\varepsilon 0\alpha\rangle\beta^3\langle 3\delta^2\rangle$	2	6	2,5
	3	$\varnothing$	$\{1\}$	$\varnothing$	$\langle\varepsilon 0\rangle\langle\alpha^+1\beta^3\rangle\delta^2$	1	7	3
				$\{1\}$	$\langle\varepsilon 0\alpha^-\rangle\langle 1\beta^3\rangle\delta^2$	2	8	4,7
$\{3\}$	0	$\varnothing$	$\{3\}$	$\varnothing$	$\langle\alpha 1\beta^2\rangle\langle\gamma^+3\delta^3\rangle$	0	9	
				$\{3\}$	$\langle\alpha 1\beta^2\rangle\gamma^-\langle 3\delta^3\rangle$	1	10	1,9
	1	$\varnothing$	$\{3\}$	$\varnothing$	$\langle 0\alpha\rangle\beta^2\langle\gamma^+3\delta^3\rangle$	1	11	9
				$\{3\}$	$\langle 0\alpha\rangle\beta^2\gamma^-\langle 3\delta^3\rangle$	2	12	2,10,11
	3	$\varnothing$	$\{1\}$	$\varnothing$	$0\langle\alpha^+1\beta^2\rangle\gamma\delta^3$	1	13	2,3
				$\{1\}$	$\langle 0\alpha^-\rangle\langle 1\beta^2\rangle\gamma\delta^3$	2	14	2,4,13
$\{0,3\}$	0	$\varnothing$	$\{3\}$	$\varnothing$	$\langle\alpha 1\beta^2\rangle\langle\gamma^+3\delta^2\rangle\varepsilon$	1	15	9,13,14
				$\{3\}$	$\langle\alpha 1\beta^2\rangle\gamma^-\langle 3\delta^2\rangle\varepsilon$	2	16	10,13,14,15
	1	$\varnothing$	$\{3\}$	$\varnothing$	$\langle\varepsilon 0\alpha\rangle\beta^2\langle\gamma^+3\delta^2\rangle$	2	17	11,15
				$\{3\}$	$\langle\varepsilon 0\alpha\rangle\beta^2\gamma^-\langle 3\delta^2\rangle$	**3**	18	6,12,16,17
	3	$\varnothing$	$\{1\}$	$\varnothing$	$\langle\varepsilon 0\rangle\langle\alpha^+1\beta^2\rangle\gamma\delta^2$	2	19	6,7,13
				$\{1\}$	$\langle\varepsilon 0\alpha^-\rangle\langle 1\beta^2\rangle\gamma\delta^2$	**3**	20	6,8,14,19

Table 13: A genus 9 quintisection of T^9: X_I when $I = \{0,1,3\}$

Tables 14-15 detail the handle decomposition of X_I, $I = \{0,1,2,3\}$, from the quintisection of T^9. Note that, since $I = I_1$ consists of a single block in this example, we always have $I_1 = I_*$.

i_*	U	V	V^-	Y_z^*	h	z	glue to
0	$\{2\}$	$\{1,3\}$	$\varnothing$	$\alpha^-1\beta_3^\circ 2\langle\gamma^+3\delta^3\rangle$	0	1	
				$\alpha^-\langle 1\beta_3^-\rangle 2\langle\gamma^+3\delta^3\rangle$	1	2	1
				$\alpha^-1\langle\beta_3^+2\rangle\langle\gamma^+3\delta^3\rangle$	1	3	1
			$\{1\}$	$\langle\alpha^+1\rangle\beta_3^\circ 2\langle\gamma^+3\delta^3\rangle$	1	4	1
				$\langle\alpha^+1\beta_3^-\rangle 2\langle\gamma^+3\delta^3\rangle$	2	5	2,4
				$\langle\alpha^+1\rangle\langle\beta_3^+2\rangle\langle\gamma^+3\delta^3\rangle$	2	6	3,4
			$\{3\}$	$\alpha^-1\beta_3^\circ\langle 2\gamma^-\rangle\langle 3\delta^3\rangle$	1	7	1
				$\alpha^-\langle 1\beta_3^-\rangle\langle 2\gamma^-\rangle\langle 3\delta^3\rangle$	2	8	2,7
				$\alpha^-1\langle\beta_3^+2\gamma^-\rangle\langle 3\delta^3\rangle$	2	9	3,7
			$\{1,3\}$	$\langle\alpha^+1\rangle\beta_3^\circ\langle 2\gamma^-\rangle\langle 3\delta^3\rangle$	2	10	4,7
				$\langle\alpha^+1\beta_3^-\rangle\langle 2\gamma^-\rangle\langle 3\delta^3\rangle$	3	11	5,8,10
				$\langle\alpha^+1\rangle\langle\beta_3^+2\gamma^-\rangle\langle 3\delta^3\rangle$	3	12	6,9,10
1	$\varnothing$	$\{2,3\}$	$\varnothing$	$\langle 0\alpha\rangle\beta^-2\langle\gamma^+3\delta^3\rangle$	1	13	1,2
			$\{2\}$	$\langle 0\alpha\rangle\langle\beta^+2\rangle\langle\gamma^+3\delta^3\rangle$	2	14	1,3,13
			$\{3\}$	$\langle 0\alpha\rangle\beta^-\langle 2\gamma^-\rangle\langle 3\delta^3\rangle$	2	15	7,8,13
			$\{2,3\}$	$\langle 0\alpha\rangle\langle\beta^+2\gamma^-\rangle\langle 3\delta^3\rangle$	3	16	7,9,14,15
2	$\{1\}$	$\varnothing$	$\varnothing$	$0\alpha_3^\circ\langle 1\beta\rangle\langle\gamma 3\delta^3\rangle$	1	17	13
				$\langle 0\alpha_3^-\rangle\langle 1\beta\rangle\langle\gamma 3\delta^3\rangle$	2	18	13,17
				$0\langle\alpha_3^+1\beta\rangle\langle\gamma 3\delta^3\rangle$	2	19	13,17
3	$\{1,2\}$	$\varnothing$	$\varnothing$	$0\alpha_3^\circ 1\beta_3^\circ\langle 2\gamma\rangle\delta^3$	1	20	17
				$\langle 0\alpha_3^-\rangle 1\beta_3^\circ\langle 2\gamma\rangle\delta^3$	2	21	18,20
				$0\langle\alpha_3^+1\rangle\beta_3^\circ\langle 2\gamma\rangle\delta^3$	2	22	19,20
				$0\alpha_3^\circ\langle 1\beta_3^-\rangle\langle 2\gamma\rangle\delta^3$	2	23	17,20
				$\langle 0\alpha_3^-\rangle\langle 1\beta_3^-\rangle\langle 2\gamma\rangle\delta^3$	3	24	18,20,23
				$0\langle\alpha_3^+1\beta_3^-\rangle\langle 2\gamma\rangle\delta^3$	3	25	19,22,23
				$0\alpha_3^\circ 1\langle\beta_3^+2\gamma\rangle\delta^3$	2	26	17,20
				$\langle 0\alpha_3^-\rangle 1\langle\beta_3^+2\gamma\rangle\delta^3$	3	27	18,21,26
				$0\langle\alpha_3^+1\rangle\langle\beta_3^+2\gamma\rangle\delta^3$	3	28	19,22,26

Table 14: X_I, $I=\{0,1,2,3\}$, from T^9. Part 1: $J=\varnothing$.

i_*	U	V	V^-	Y_z^*	h	z	glue to
0	{2}	{1,3}	$\varnothing$	$\alpha^- 1\beta_3^\circ 2\left\langle\gamma^+3\delta^2\right\rangle\varepsilon$	1	29	1,19,20
				$\alpha^-\left\langle 1\beta_3^-\right\rangle 2\left\langle\gamma^+3\delta^2\right\rangle\varepsilon$	2	30	2,22,23,29
				$\alpha^- 1\beta_3^+ 2\left\langle\gamma^+3\delta^2\right\rangle\varepsilon$	2	31	3,25,26,29
			{1}	$\langle\alpha^+1\rangle\,\beta_3^\circ 2\left\langle\gamma^+3\delta^2\right\rangle\varepsilon$	2	32	4,19,21
				$\left\langle\alpha^+1\beta_3^-\right\rangle 2\left\langle\gamma^+3\delta^2\right\rangle\varepsilon$	3	33	5,22,24,30,32
				$\langle\alpha^+1\rangle\,\beta_3^+ 2\left\langle\gamma^+3\delta^2\right\rangle\varepsilon$	3	34	6,25,27,31,32
			{3}	$\alpha^- 1\beta_3^\circ\,\langle 2\gamma^-\rangle\left\langle 3\delta^2\right\rangle\varepsilon$	2	35	7,19,20,29
				$\alpha^-\left\langle 1\beta_3^-\right\rangle\langle 2\gamma^-\rangle\left\langle 3\delta^2\right\rangle\varepsilon$	3	36	8,22,23,30,35
				$\alpha^- 1\left\langle\beta_3^+2\gamma^-\right\rangle\left\langle 3\delta^2\right\rangle\varepsilon$	3	37	9,25,26,31,35
			{1,3}	$\langle\alpha^+1\rangle\,\beta_3^\circ\,\langle 2\gamma^-\rangle\left\langle 3\delta^2\right\rangle\varepsilon$	3	38	10,19,21,32,35
				$\left\langle\alpha^+1\beta_3^-\right\rangle\langle 2\gamma^-\rangle\left\langle 3\delta^2\right\rangle\varepsilon$	**4**	39	11,22,24,33,36,38
				$\langle\alpha^+1\rangle\left\langle\beta_3^+2\gamma^-\right\rangle\left\langle 3\delta^2\right\rangle\varepsilon$	**4**	40	12,25,27,34,37,38
1	$\varnothing$	{2,3}	$\varnothing$	$\langle\varepsilon 0\alpha\rangle\,\beta^- 2\left\langle\gamma^+3\delta^2\right\rangle$	2	41	13,29,30
			{2}	$\langle\varepsilon 0\alpha\rangle\,\langle\beta^+2\rangle\left\langle\gamma^+3\delta^2\right\rangle$	3	42	14,29,31,41
			{3}	$\langle\varepsilon 0\alpha\rangle\,\beta^-\,\langle 2\gamma^-\rangle\left\langle 3\delta^2\right\rangle$	3	43	15,35,36,41
			{2,3}	$\langle\varepsilon 0\alpha\rangle\,\langle\beta^+2\gamma^-\rangle\left\langle 3\delta^2\right\rangle$	**4**	44	16,35,37,42,43
2	{1}	$\varnothing$	$\varnothing$	$\langle\varepsilon 0\rangle\,\alpha_3^\circ\,\langle 1\beta\rangle\left\langle\gamma 3\delta^2\right\rangle$	2	45	17,41,43
				$\left\langle\varepsilon 0\alpha_3^-\right\rangle\langle 1\beta\rangle\left\langle\gamma 3\delta^2\right\rangle$	3	46	18,41,43,45
				$\langle\varepsilon 0\rangle\left\langle\alpha_3^+1\beta\right\rangle\left\langle\gamma 3\delta^2\right\rangle$	3	47	19,41,43,45
3	{1,2}	$\varnothing$	$\varnothing$	$\langle\varepsilon 0\rangle\,\alpha_3^\circ 1\beta_3^\circ\,\langle 2\gamma\rangle\,\delta^2$	2	48	20,45
				$\left\langle\varepsilon 0\alpha_3^-\right\rangle 1\beta_3^\circ\,\langle 2\gamma\rangle\,\delta^2$	3	49	21,46,48
				$\langle\varepsilon 0\rangle\left\langle\alpha_3^+1\right\rangle\beta_3^\circ\,\langle 2\gamma\rangle\,\delta^2$	3	50	22,47,48
				$\langle\varepsilon 0\rangle\,\alpha_3^\circ\left\langle 1\beta_3^-\right\rangle\langle 2\gamma\rangle\,\delta^2$	3	51	23,45,48
				$\left\langle\varepsilon 0\alpha_3^-\right\rangle\left\langle 1\beta_3^-\right\rangle\langle 2\gamma\rangle\,\delta^2$	**4**	52	24,46,49,51
				$\langle\varepsilon 0\rangle\left\langle\alpha_3^+1\beta_3^-\right\rangle\langle 2\gamma\rangle\,\delta^2$	**4**	53	25,47,50,51
				$\langle\varepsilon 0\rangle\,\alpha_3^\circ 1\left\langle\beta_3^+2\gamma\right\rangle\delta^2$	3	54	26,45,48
				$\left\langle\varepsilon 0\alpha_3^-\right\rangle 1\left\langle\beta_3^+2\gamma\right\rangle\delta^2$	**4**	55	27,46,49,54
				$\langle\varepsilon 0\rangle\left\langle\alpha_3^+1\right\rangle\left\langle\beta_3^+2\gamma\right\rangle\delta^2$	**4**	56	28,47,50,54

Table 15: X_I, $I=\{0,1,2,3\}$, from T^9. Part 2: $J=\{0\}$.

Tables 16 and 17 detail handle decompositions of X_I, $I=\{0,1,2,4\}$ from the sexasection of T^{11}. The parts of these tables with $i_*=4$ and $0\notin J$ feature a complication that does not appear in dimensions $n\leq 9$. Also see Tables 9 and 18 for more complicated examples of this pattern.

J	i_*	U	V	V^-	Y_z^*	h	z	glue to
$\varnothing$	4	$\varnothing$	{1,2}	$\varnothing$	$0\,\langle\alpha^+1\rangle\,\langle\beta^+2\gamma^3\rangle\,\varepsilon^3$	0	1	
				{1}	$\langle 0\alpha^-\rangle\,1\,\langle\beta^+2\gamma^3\rangle\,\varepsilon^3$	1	2	1
				{1,2}	$\langle 0\alpha^-\rangle\,\langle 1\beta^-\rangle\,\langle 2\gamma^3\rangle\,\varepsilon^3$	1	3	2
				{2}	$0\,\langle\alpha^+1\beta^-\rangle\,\langle 2\gamma^3\rangle\,\varepsilon^3$	2	4	1,3
	0	$\varnothing$	{1,2}	$\varnothing$	$\alpha^-1\,\langle\beta^+2\gamma^3\rangle\,\langle 4\varepsilon^3\rangle$	0	5	
				{1}	$\langle\alpha^+1\rangle\,\langle\beta^+2\gamma^3\rangle\,\langle 4\varepsilon^3\rangle$	1	6	5
				{2}	$\alpha^-\,\langle 1\beta^-\rangle\,\langle 2\gamma^3\rangle\,\langle 4\varepsilon^3\rangle$	1	7	5
				{1,2}	$\langle\alpha^+1\beta^-\rangle\,\langle 2\gamma^3\rangle\,\langle 4\varepsilon^3\rangle$	2	8	5,6
	1	$\varnothing$	$\varnothing$	$\varnothing$	$\langle 0\alpha\rangle\,\langle\beta 2\gamma^3\rangle\,\langle 4\varepsilon^3\rangle$	1	9	5,7
	2	{1}	$\varnothing$	$\varnothing$	$0\alpha_3^\circ\,\langle 1\beta\rangle\,\gamma^3\,\langle 4\varepsilon^3\rangle$	1	10	9
					$\langle 0\alpha_3^-\rangle\,\langle 1\beta\rangle\,\gamma^3\,\langle 4\varepsilon^3\rangle$	2	11	9,10
					$0\,\langle\alpha_3^+1\beta\rangle\,\gamma^3\,\langle 4\varepsilon^3\rangle$	2	12	9,10
{4}	4	$\varnothing$	{1,2}	$\varnothing$	$0\,\langle\alpha^+1\rangle\,\langle\beta^+2\gamma^2\rangle\,\delta\varepsilon^3$	1	13	1,10,12
				{1}	$\langle 0\alpha^-\rangle\,1\,\langle\beta^+2\gamma^2\rangle\,\delta\varepsilon^3$	2	14	2,10,11,13
				{1,2}	$\langle 0\alpha^-\rangle\,\langle 1\beta^-\rangle\,\langle 2\gamma^2\rangle\,\delta\varepsilon^3$	2	15	3,10,11,14
				{2}	$0\,\langle\alpha^+1\beta^-\rangle\,\langle 2\gamma^2\rangle\,\delta\varepsilon^3$	3	16	4,10,12,13,15
	0	$\varnothing$	{1,2}	$\varnothing$	$\alpha^-1\,\langle\beta^+2\gamma^2\rangle\,\langle\delta^+4\varepsilon^3\rangle$	0	17	
					$\alpha^-1\,\langle\beta^+2\gamma^2\rangle\,\delta^-\,\langle 4\varepsilon^3\rangle$	1	18	5,17
				{1}	$\langle\alpha^+1\rangle\,\langle\beta^+2\gamma^2\rangle\,\langle\delta^+4\varepsilon^3\rangle$	1	19	17
					$\langle\alpha^+1\rangle\,\langle\beta^+2\gamma^2\rangle\,\delta^-\,\langle 4\varepsilon^3\rangle$	2	20	6,18,19
				{2}	$\alpha^-\,\langle 1\beta^-\rangle\,\langle 2\gamma^2\rangle\,\langle\delta^+4\varepsilon^3\rangle$	1	21	19
					$\alpha^-\,\langle 1\beta^-\rangle\,\langle 2\gamma^2\rangle\,\delta^-\,\langle 4\varepsilon^3\rangle$	2	22	7,20,21
				{1,2}	$\langle\alpha^+1\beta^-\rangle\,\langle 2\gamma^2\rangle\,\langle\delta^+4\varepsilon^3\rangle$	2	23	19,21
					$\langle\alpha^+1\beta^-\rangle\,\langle 2\gamma^2\rangle\,\delta^-\,\langle 4\varepsilon^3\rangle$	3	24	8,20,22,23
	1	$\varnothing$	{4}	$\varnothing$	$\langle 0\alpha\rangle\,\langle\beta 2\gamma^2\rangle\,\langle\delta^+4\varepsilon^3\rangle$	1	25	17,21
				{4}	$\langle 0\alpha\rangle\,\langle\beta 2\gamma^2\rangle\,\delta^-\,\langle 4\varepsilon^3\rangle$	2	26	9,18,22,25
	2	{1}	{4}	$\varnothing$	$0\alpha_3^\circ\,\langle 1\beta\rangle\,\gamma^2\,\langle\delta^+4\varepsilon^3\rangle$	1	27	25
					$\langle 0\alpha_3^-\rangle\,\langle 1\beta\rangle\,\gamma^2\,\langle\delta^+4\varepsilon^3\rangle$	2	28	25,27
					$0\,\langle\alpha_3^+1\beta\rangle\,\gamma^2\,\langle\delta^+4\varepsilon^3\rangle$	2	29	25,27
				{4}	$0\alpha_3^\circ\,\langle 1\beta\rangle\,\gamma^2\delta^-\,\langle 4\varepsilon^3\rangle$	2	30	10,26,27
					$\langle 0\alpha_3^-\rangle\,\langle 1\beta\rangle\,\gamma^2\delta^-\,\langle 4\varepsilon^3\rangle$	3	31	11,26,28,30
					$0\,\langle\alpha_3^+1\beta\rangle\,\gamma^2\delta^-\,\langle 4\varepsilon^3\rangle$	3	32	12,26,29,30

Table 16: Part 1 of X_I, $I=\{0,1,2,4\}$, from T^{11}.

J	i_*	U	V	V^-	Y_z^*	h	z	glue to
$\{0\}$	4	$\{1\}$	$\{2\}$	$\varnothing$	$\langle\zeta 0\rangle\,\alpha_3^\circ 1\,\langle\beta^+ 2\gamma^3\rangle\,\varepsilon^2$	1	33	1,2
					$\langle\zeta 0\rangle\,\langle\alpha_3^+ 1\rangle\,\langle\beta^+ 2\gamma^3\rangle\,\varepsilon^2$	2	34	1,33
					$\langle\zeta 0\alpha_3^-\rangle\,1\,\langle\beta^+ 2\gamma^3\rangle\,\varepsilon^2$	2	35	2,33
				$\{2\}$	$\langle\zeta 0\rangle\,\alpha_3^\circ\,\langle 1\beta^-\rangle\,\langle 2\gamma^3\rangle\,\varepsilon^2$	2	36	3,4,33
					$\langle\zeta 0\rangle\,\langle\alpha_3^+ 1\beta^-\rangle\,\langle 2\gamma^3\rangle\,\varepsilon^2$	3	37	3,34,36
					$\langle\zeta 0\alpha_3^-\rangle\,\langle 1\beta^-\rangle\,\langle 2\gamma^3\rangle\,\varepsilon^2$	3	38	4,35,36
	0	$\varnothing$	$\{1,2\}$	$\varnothing$	$\zeta\alpha^- 1\,\langle\beta^+ 2\gamma^3\rangle\,\langle 4\varepsilon^2\rangle$	1	39	2,5
				$\{1\}$	$\zeta\,\langle\alpha^+ 1\rangle\,\langle\beta^+ 2\gamma^3\rangle\,\langle 4\varepsilon^2\rangle$	2	40	1,6,39
				$\{2\}$	$\zeta\alpha^-\,\langle 1\beta^-\rangle\,\langle 2\gamma^3\rangle\,\langle 4\varepsilon^2\rangle$	2	41	3,7,39
				$\{1,2\}$	$\zeta\,\langle\alpha^+ 1\beta^-\rangle\,\langle 2\gamma^3\rangle\,\langle 4\varepsilon^2\rangle$	3	42	4,8,40
	1	$\varnothing$	$\varnothing$	$\varnothing$	$\langle\zeta 0\alpha\rangle\,\langle\beta 2\gamma^3\rangle\,\langle 4\varepsilon^2\rangle$	2	43	9,39,41
	2	$\{1\}$	$\varnothing$	$\varnothing$	$\langle\zeta 0\rangle\,\alpha_3^\circ\,\langle 1\beta\rangle\,\gamma^3\,\langle 4\varepsilon^2\rangle$	2	44	10,43
					$\langle\zeta 0\alpha_3^-\rangle\,\langle 1\beta\rangle\,\gamma^3\,\langle 4\varepsilon^2\rangle$	3	45	11,43,44
					$\langle\zeta 0\rangle\,\langle\alpha_3^+ 1\beta\rangle\,\gamma^3\,\langle 4\varepsilon^2\rangle$	3	46	12,43,44
$\{0,4\}$	4	$\{1\}$	$\{2\}$	$\varnothing$	$\langle\zeta 0\rangle\,\alpha_3^\circ 1\,\langle\beta^+ 2\gamma^2\rangle\,\delta\varepsilon^2$	2	47	13,14,33,44
					$\langle\zeta 0\rangle\,\langle\alpha_3^+ 1\rangle\,\langle\beta^+ 2\gamma^2\rangle\,\delta\varepsilon^2$	3	48	13,34,45,47
					$\langle\zeta 0\alpha_3^-\rangle\,1\,\langle\beta^+ 2\gamma^2\rangle\,\delta\varepsilon^2$	3	49	14,35,46,47
				$\{2\}$	$\langle\zeta 0\rangle\,\alpha_3^\circ\,\langle 1\beta^-\rangle\,\langle 2\gamma^2\rangle\,\delta\varepsilon^2$	3	50	15,16,36,44,47
					$\langle\zeta 0\rangle\,\langle\alpha_3^+ 1\beta^-\rangle\,\langle 2\gamma^2\rangle\,\delta\varepsilon^2$	4	51	16,37,45,48,50
					$\langle\zeta 0\alpha_3^-\rangle\,\langle 1\beta^-\rangle\,\langle 2\gamma^2\rangle\,\delta\varepsilon^2$	4	52	15,38,46,49,50
	0	$\varnothing$	$\{1,2\}$	$\varnothing$	$\zeta\alpha^- 1\,\langle\beta^+ 2\gamma^2\rangle\,\langle\delta^+ 4\varepsilon^2\rangle$	1	53	14,17
					$\zeta\alpha^- 1\,\langle\beta^+ 2\gamma^2\rangle\,\delta^-\,\langle 4\varepsilon^2\rangle$	2	54	14,18,39,53
				$\{1\}$	$\zeta\,\langle\alpha^+ 1\rangle\,\langle\beta^+ 2\gamma^2\rangle\,\langle\delta^+ 4\varepsilon^2\rangle$	2	55	13,19,53
					$\zeta\,\langle\alpha^+ 1\rangle\,\langle\beta^+ 2\gamma^2\rangle\,\delta^-\,\langle 4\varepsilon^2\rangle$	3	56	13,20,40,54,55
				$\{2\}$	$\zeta\alpha^-\,\langle 1\beta^-\rangle\,\langle 2\gamma^2\rangle\,\langle\delta^+ 4\varepsilon^2\rangle$	2	57	15,21,53
					$\zeta\alpha^-\,\langle 1\beta^-\rangle\,\langle 2\gamma^2\rangle\,\delta^-\,\langle 4\varepsilon^2\rangle$	3	58	15,22,41,54,57
				$\{1,2\}$	$\zeta\,\langle\alpha^+ 1\beta^-\rangle\,\langle 2\gamma^2\rangle\,\langle\delta^+ 4\varepsilon^2\rangle$	3	59	16,23,55,57
					$\zeta\,\langle\alpha^+ 1\beta^-\rangle\,\langle 2\gamma^2\rangle\,\delta^-\,\langle 4\varepsilon^2\rangle$	4	60	16,24,42,56,58,59
	1	$\varnothing$	$\{4\}$	$\varnothing$	$\langle\zeta 0\alpha\rangle\,\langle\beta 2\gamma^2\rangle\,\langle\delta^+ 4\varepsilon^2\rangle$	2	61	25,53,57
				$\{4\}$	$\langle\zeta 0\alpha\rangle\,\langle\beta 2\gamma^2\rangle\,\delta^-\,\langle 4\varepsilon^2\rangle$	3	62	26,43,54,58,61
	2	$\{1\}$	$\{4\}$	$\varnothing$	$\langle\zeta 0\rangle\,\alpha_3^\circ\,\langle 1\beta\rangle\,\gamma^2\,\langle\delta^+ 4\varepsilon^2\rangle$	2	63	27,61
					$\langle\zeta 0\alpha_3^-\rangle\,\langle 1\beta\rangle\,\gamma^2\,\langle\delta^+ 4\varepsilon^2\rangle$	3	64	28,61,63
					$\langle\zeta 0\rangle\,\langle\alpha_3^+ 1\beta\rangle\,\gamma^2\,\langle\delta^+ 4\varepsilon^2\rangle$	3	65	29,61,63
				$\{4\}$	$\langle\zeta 0\rangle\,\alpha_3^\circ\,\langle 1\beta\rangle\,\gamma^2\delta^-\,\langle 4\varepsilon^2\rangle$	3	66	30,44,62,63
					$\langle\zeta 0\alpha_3^-\rangle\,\langle 1\beta\rangle\,\gamma^2\delta^-\,\langle 4\varepsilon^2\rangle$	4	67	31,44,62,64,66
					$\langle\zeta 0\rangle\,\langle\alpha_3^+ 1\beta\rangle\,\gamma^2\delta^-\,\langle 4\varepsilon^2\rangle$	4	68	32,45,62,65,66

Table 17: Part 2 of X_I, $I=\{0,1,2,4\}$, from T^{11}.

Table 18 details the start of the handle decomposition of X_I from T^{15} with $I = \{0,1,2,3,4,6\}$, focusing on the first few pieces Y_z. Those pieces have $J = \varnothing$, $i_* = 6$, $U = \varnothing$, $V = \{1,2,3,4\}$. The interesting feature of this example is the ordering of these pieces. Compare to (23) and Tables 9, 16, 17.

V^-	Y_z^*	h	z	glue to
$\varnothing$	$0\,\langle\alpha^+1\rangle\,\langle\beta^+2\rangle\,\langle\gamma^+3\rangle\,\langle\delta^+4\varepsilon^3\rangle\,\eta^3$	0	1	
$\{1\}$	$\langle 0\alpha^-\rangle\,1\,\langle\beta^+2\rangle\,\langle\gamma^+3\rangle\,\langle\delta^+4\varepsilon^3\rangle\,\eta^3$	1	2	1
$\{1,2\}$	$\langle 0\alpha^-\rangle\,\langle 1\beta^-\rangle\,2\,\langle\gamma^+3\rangle\,\langle\delta^+4\varepsilon^3\rangle\,\eta^3$	1	3	2
$\{2\}$	$0\,\langle\alpha^+1\beta^-\rangle\,2\,\langle\gamma^+3\rangle\,\langle\delta^+4\varepsilon^3\rangle\,\eta^3$	2	4	1,3
$\{2,3\}$	$0\,\langle\alpha^+1\beta^-\rangle\,\langle 2\gamma^-\rangle\,3\,\langle\delta^+4\varepsilon^3\rangle\,\eta^3$	1	5	4
$\{1,2,3\}$	$\langle 0\alpha^-\rangle\,\langle 1\beta^-\rangle\,\langle 2\gamma^-\rangle\,3\,\langle\delta^+4\varepsilon^3\rangle\,\eta^3$	2	6	3,5
$\{1,3\}$	$\langle 0\alpha^-\rangle\,1\,\langle\beta^+2\gamma^-\rangle\,3\,\langle\delta^+4\varepsilon^3\rangle\,\eta^3$	2	7	2,6
$\{3\}$	$0\,\langle\alpha^+1\rangle\,\langle\beta^+2\gamma^-\rangle\,3\,\langle\delta^+4\varepsilon^3\rangle\,\eta^3$	3	8	1,5,7
$\{3,4\}$	$0\,\langle\alpha^+1\rangle\,\langle\beta^+2\gamma^-\rangle\,\langle 3\delta^-\rangle\,\langle 4\varepsilon^3\rangle\,\eta^3$	1	9	8
$\{1,3,4\}$	$\langle 0\alpha^-\rangle\,1\,\langle\beta^+2\gamma^-\rangle\,\langle 3\delta^-\rangle\,\langle 4\varepsilon^3\rangle\,\eta^3$	2	10	7,9
$\{1,2,3,4\}$	$\langle 0\alpha^-\rangle\,\langle 1\beta^-\rangle\,\langle 2\gamma^-\rangle\,\langle 3\delta^-\rangle\,\langle 4\varepsilon^3\rangle\,\eta^3$	2	11	6,10
$\{2,3,4\}$	$0\,\langle\alpha^+1\beta^-\rangle\,\langle 2\gamma^-\rangle\,\langle 3\delta^-\rangle\,\langle 4\varepsilon^3\rangle\,\eta^3$	3	12	5,9,11
$\{2,4\}$	$0\,\langle\alpha^+1\beta^-\rangle\,2\,\langle\gamma^+3\delta^-\rangle\,\langle 4\varepsilon^3\rangle\,\eta^3$	2	13	4,12
$\{1,2,4\}$	$\langle 0\alpha^-\rangle\,\langle 1\beta^-\rangle\,2\,\langle\gamma^+3\delta^-\rangle\,\langle 4\varepsilon^3\rangle\,\eta^3$	3	14	3,11,13
$\{1,4\}$	$\langle 0\alpha^-\rangle\,1\,\langle\beta^+2\rangle\,\langle\gamma^+3\delta^-\rangle\,\langle 4\varepsilon^3\rangle\,\eta^3$	3	15	2,10,14
$\{4\}$	$0\,\langle\alpha^+1\rangle\,\langle\beta^+2\rangle\,\langle\gamma^+3\delta^-\rangle\,\langle 4\varepsilon^3\rangle\,\eta^3$	4	16	1,9,13,15

Table 18: Start of the handle decomposition from T^{15} with $I = \{0,1,2,3,4,6\}$, $J = \varnothing$, $i_* = 6$, $U = \varnothing$, $V = \{1,2,3,4\}$.

Tables 19, 20, and 21 list the possible forms for $\xi_r(z)$. Table 19 lists those *with no* singleton factor. Table 20 lists those *with* a singleton factor $\{i\}$, where $i \in V^+$ and $i+1 \in U^\circ \cup U^+ \cup V^+$, or $i \in U^- \cup U^\circ \cup V^-$ and $i+1 \in V^-$; the class of this case depends on the parity of $\#(V^- \cap \{i+1,\dots,\max I_s\})$, where $i \in I_s$. Table 21 lists the remaining possibilities for $\xi_r(z)$.

class	$\xi_r(z)$	conditions
(A)	$[i_* - 1, i_*]$	$i_* \in J$
(A)	$\left[i-1, i-\frac{1}{2}\right]$	$i \in J \cap V^- \implies i \neq i_*, i+1 \notin I$
(B)	$\left[i_*, i_* + \frac{1}{2}\right]$	$a \leq i_* \leq b-2,\ i_*+1 \in V^-$
(B)	$\left[i-\frac{2}{3}, i-\frac{1}{3}\right]$	$i \in U^\circ$
(B)	$\prod_{j=i_*+1}^{c-1} [i_*, j]^2$	$i_* = b,\ c \in J$
(B)	$\prod_{j=i_*+1}^{c-2} [i_*, j]^2 [i_*, c-1]^3$	$i_* = b,\ c \notin J$

Table 19: The possible forms for $\xi_r(z)$ with no singleton factor, where $i_* \in I_s$, $a = \min I_s$, $b = \max I_s$, $c = \min I_{s+1}$.

class	$\xi_r(z)$	conditions on i	conditions on $i+1$	parity
(A)	$\left[i-\frac{1}{2}, i\right]\{i\}$	$i \in V^+$	$i+1 \in U^\circ \cup U^+ \cup V^+$	odd
(A)	$\{i\}\left[i, i+\frac{1}{2}\right]$	$i \in U^- \cup U^\circ \cup V^-$	$i+1 \in V^-$	odd
(A)	$\left[i-\frac{1}{2}, i\right]\{i\}\left[i, i+\frac{1}{2}\right]$	$i \in V^+$	$i+1 \in V^-$	odd
(B)	$\left[i-\frac{1}{2}, i\right]\{i\}$	$i \in V^+$	$i+1 \in U^\circ \cup U^+ \cup V^+$	even
(B)	$\{i\}\left[i, i+\frac{1}{2}\right]$	$i \in U^- \cup U^\circ \cup V^-$	$i+1 \in V^-$	even
(B)	$\left[i-\frac{1}{2}, i\right]\{i\}\left[i, i+\frac{1}{2}\right]$	$i \in V^+$	$i+1 \in V^-$	even

Table 20: The possible forms for each $\xi_r(z)$ containing a singleton factor $\{i\}$, where $i \in V^+$ and $i+1 \in U^\circ \cup U^+ \cup V^+$, or $i \in U^- \cup U^\circ \cup V^-$ and $i+1 \in V^-$; the class depends on the parity of $\#(V^- \cap \{i+1, \ldots, \max I_s\})$, where $i \in I_s$.

class	$\xi_r(z)$	conditions on i
(A)	$[i-1,i]\{i\}$	$i \in J,\, i+1 \in U^\circ \cup U^+ \cup V^+$
(A)	$[i-1,i]\{i\}[i,i+1]$	$i \in J,\, i_* = i+1$
(A)	$[i-1,i]\{i\}\left[i,i+\frac{1}{3}\right]$	$i \in J,\, i+1 \in U^-$
(A)	$[i-1,i]\{i\}\left[i,i+\frac{1}{2}\right]$	$i \in J,\, i+1 \in V^-$
(A)	$\left[i-\frac{1}{3},i\right]\{i\}$	$i \in U^+,\, i+1 \in U^\circ \cup U^+ \cup V^+$
(A)	$\{i\}\left[i,i+\frac{1}{3}\right]$	$i+1 \in U^-,\, i \in U^- \cup U^\circ \cup V^-$
(A)	$\left[i-\frac{1}{3},i\right]\{i\}\left[i,i+\frac{1}{3}\right]$	$i \in U^+,\, i+1 \in U^-$
(A)	$\left[i-\frac{1}{3},i\right]\{i\}\left[i,i+\frac{1}{2}\right],$	$i \in U^+,\, i+1 \in V^-$ $\implies i+1 = \max I_s \neq i_*$
(A)	$\left[i-\frac{1}{2},i\right]\{i\}\left[i,i+\frac{1}{3}\right],$	$i \in V^+,\, i+1 \in U^-$ $\implies i = i_*+1 \leq \max I_s - 1$
(A)	$\{i\}$	$i \in (T \setminus J) \cup U^- \cup U^\circ \cup V^-,$ $i+1 \in U^\circ \cup U^+ \cup V^+$
(B)	$[i-1,i]\{i\}\prod_{j=i+1}^{c-2}[i,j]^2[i,c-1]^q$	$i_* = \min I_s = i-1 = \max I_s - 1$
(B)	$\left[i-\frac{1}{2},i\right]\{i\}\prod_{j=i+1}^{c-2}[i,j]^2[i,c-1]^q$	$i = \max I_s \in V^+$
(B)	$\{i\}\prod_{j=i+1}^{c-2}[i,j]^2[i,c-1]^q$	$i = \max I_s \in V^-$

Table 21: The possible forms for each $\xi_r(z)$ not listed in Tables 19, 20. Each contains a singleton factor $\{i\}$, $i_* \neq i \in I_s$, $s \in \mathbb{Z}_m$. Denote $c = \min I_{s+1}$ with $q \in \{2,3\}$.

Appendix 2: Four other attempts to multisect T^n for n odd

From the handle decomposition

The n-torus has a natural handle decomposition, with $\binom{n}{h}$ h-handles for each $h = 0,\dots,n$, which one can construct as follows. View T^n as $(\mathbb{R}/2\mathbb{Z})^n$, and decompose it into the 2^n subcubes with vertices in $(\mathbb{Z}/2\mathbb{Z})^n$. Then, using notation 3.4, for each $h = 0,\dots,n$, the h-handles are the subcubes which are permutations of $\alpha^{n-h}\beta^h$.[15]

One might hope that $X_i = \langle \alpha^{n-i}\beta^i \rangle \cup \left\langle \alpha^{n+1-i}\beta^{i-1} \right\rangle$ determines a multisection.[16] Indeed, in dimension 3, this is the Heegaard splitting shown in Figure 1. Yet, the construction does not work beyond dimension 3, as one can see by noting, e.g., that $X_0 \cap X_{k-1} = \bigcup_{r=0}^{n-2} \left\langle \alpha\beta 0^r 1^{n-2-r} \right\rangle$ is always 2-dimensional.

[15]Note that this handle decomposition is optimal in the sense that it has the minimum possible number of handles of each index, since $H_h(T^n)$ has rank $\binom{n}{h}$.

[16]Note that n is odd throughout Appendix 2.

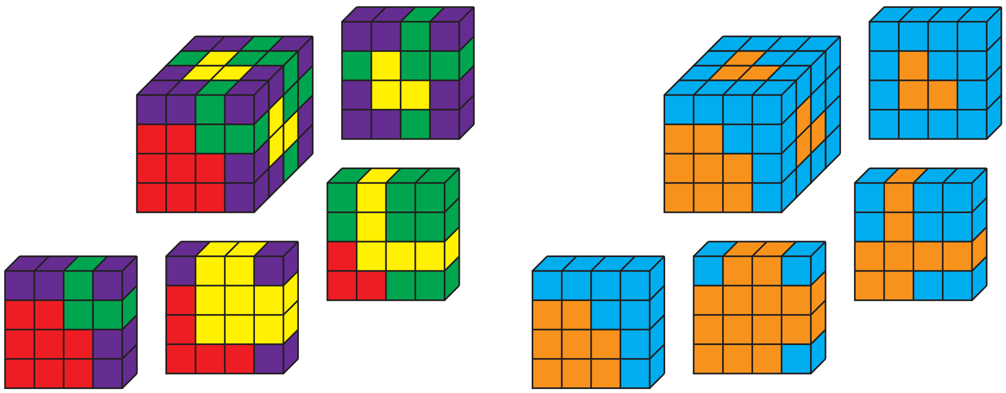

Figure 11: Another construction of the minimal genus Heegaard splitting of S^3

By gluing pairs of balls

Instead, one might attempt to generalize the following construction. See Figure 11. View T^n as $(\mathbb{R}/2k\mathbb{Z})^n = [0,2k]^n/\sim$. Partition the $(2k)^n$ unit cubes with vertices in the lattice $\left(\mathbb{Z}/2k\mathbb{Z}\right)^n$ so as to form $V_0,\dots,V_n$ subject to the following conditions:[17]

- If $\vec{x} \in V_0$, then $\vec{x} + (r,\dots,r) \in V_r$;
- The permutation action on the indices fixes each V_r;
- V_0 contains $[0,1]^n$, is star-shaped about $(0,\dots,0)$, and contains no points with any coordinate in $(n-1,n)$.

Then, for $i = 0,\dots,k = \frac{n+1}{2}$, let $X_i = V_{2i} \cup V_{2i+1}$. Figure 11 shows that this construction does in fact give a genus 3 Heegaard splitting of T^3.

In higher dimensions, this construction is promising for many of the same reasons as the construction behind Theorem 7.10. This construction has at least one additional advantage, namely that each V_i is a ball. This makes it easy to check that each X_i is indeed an n-dimensional handlebody of genus n. Unfortunately, the complexity of this construction grows much more rapidly than the construction behind Theorem 7.10, making it hard to check the other details, even in dimension 5. Indeed, see Figure 12.

Question 8 Does this construction also give a trisection of T^5? Does it give a multisection of T^n for arbitrary $n = 2k-1$?

[17] These conditions uniquely determine $V_0,\dots,V_n$.

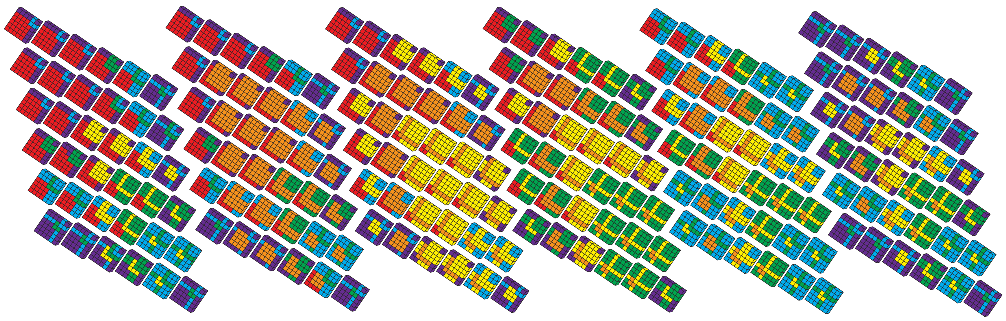

Figure 12: Decomposing $T^5 = [0,6]^5/\sim$ as $V_0 \cup \cdots \cup V_5$. Does $(V_0 \cup V_1, V_2 \cup V_3, V_4 \cup V_5)$ determine a trisection?

By summing coordinates

As shown in Figure 13, the genus 3 Heegaard splitting of $T^3 = [0,2]^3/\sim$ can be constructed as $T^3 = X_0 \cup X_1$ where each

$$X_i = \{(x_1,x_2,x_3) : 3i \le x_1 + x_2 + x_3 \le 3(i+1)\}/\sim .$$

The splitting surface consists of the hexagon $\{(x_1,x_2,x_3) : x_1 + x_2 + x_3 = 3\}/\sim$ together with three other hexagons. One is $\{(0,x_2,x_3) : 1 \le x_2 + x_3 \le 5\}/\sim$, and the others are obtained from this one by permuting coordinates. A co-core of one 1-handle in X_0 is the triangle $\{(0,x_2,x_3) : x_2 + x_3 \le 1\}/\sim$, and a co-core of a 1-handle in X_0 is the triangle $\{(0,x_2,x_3) : 5 \le x_2 + x_3\}/\sim$; the other 1-handles of X_0 and X_1 are related to these by permuting coordinates.

One might attempt to trisect $T^5 = [0,3]^5/\sim$ as $T^5 = X_0 \cup X_1 \cup X_2$ with

$$X_i = \{(x_1,\ldots,x_5) : 5i \le x_1 + \cdots + x_5 \le 5(i+1)\}/\sim .$$

Then each X_i is in fact a 4-dimensional 1-handlebody of genus 4: a co-core of a 1-handle of X_0 is the 4-simplex $\{(0,x_2,x_3,x_4,x_5) : x_2 + x_3 + x_4 + x_5 \le 2\}/\sim$, a co-core of a 1-handle of X_1 is $\{(0,x_2,x_3,x_4,x_5) : 5 \le x_2+x_3+x_4+x_5 \le 7\}/\sim$, and a co-core of a 1-handle of X_2 is $\{(0,x_2,x_3,x_4,x_5) : 12 \le x_2 + x_3 + x_4 + x_5\}/\sim$; the other 1-handles of X_0, X_1, and X_2 are related to these by permuting coordinates.

Yet, this is not a trisection, because

$$X_0 \cap X_2 = \{(x_1,x_2,x_3,0,0)_\sigma :\ 4 \le x_1 + x_2 + x_3 \le 5,\ \sigma \in S_5\}/\sim$$

is 3-dimensional, not 4-.

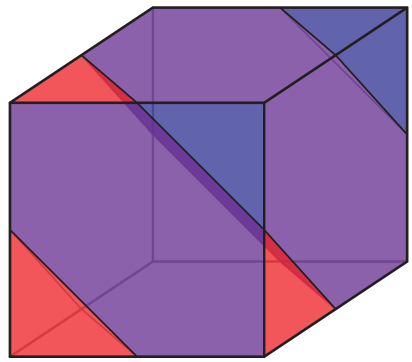

Figure 13: The efficient Heegaard splitting $T^3 = X_0 \cup X_1$ constructed by summing coordinates. Four purple hexagons comprise the splitting surface. Red and blue triangles are co-cores of the 1-handles in X_0 and X_1, respectively.

To fix this problem, one could choose $0 = a_0 < a_1 < a_2 < a_3 = 15$ differently and define each

$$X_i = \{(x_1, \dots, x_5) : a_i \leq x_1 + \cdots + x_5 \leq a_{i+1}\}.$$

Then $X_0 \cap X_2$ will be 4-dimensional if and only if $a_2 - a_1 < 3$. This creates a new problem: if $a_2 - a_1 < 3$, then X_1 is contractible, hence a 5-ball. It now follows from Proposition 2.6 that no choice of a_1 and a_2 produces a trisection of T^5. The same difficulty prevails in all other dimensions $n > 3$ (including even dimensions).

Using the symmetric space T^n/S_n

Given a triangulation K of an n-manifold X, Rubinstein–Tillmann multisect X by mapping each n-simplex of K to the standard $(k-1)$-simplex

$$\Delta_{k-1} = [\vec{v}_0, \dots, \vec{v}_{k-1}] = \left\{ \sum_{j \in \mathbb{Z}_k} a_j \vec{v}_j : \ 0 \leq a_j, \ \sum_{j \in \mathbb{Z}_k} a_j = 1 \right\}, \tag{31}$$

decomposing $\Delta_{k-1} = \bigcup_{i \in \mathbb{Z}_k} Z_i$ where each

$$Z_i = \{\vec{x} \in \Delta_{k-1} : \ |\vec{x} - \vec{v}_i| \leq |\vec{x} - \vec{v}_j| \ \forall j \in \mathbb{Z}_k\}, \tag{32}$$

(see Figure 14), and pulling back. Their maps from the n-simplices of K to Δ_{k-1} are simplest to construct in odd dimension $n = 2k - 1$. Namely:

- map the barycenter of each r-face to $\vec{v}_j \in \Delta_{k-1}$, $j = 2r, 2r + 1$; and
- extend linearly in the first barycentric subdivision of K.

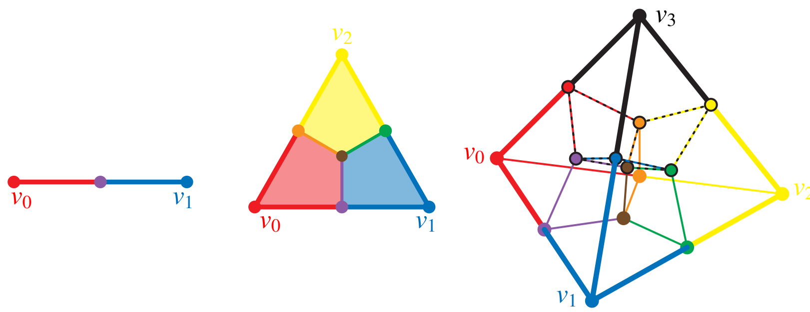

Figure 14: The decompositions $\Delta_{k-1} = \bigcup_{i\in\mathbb{Z}_k} Z_i$ of the 1-, 2-, and 3-simplices following Rubinstein–Tillmann.

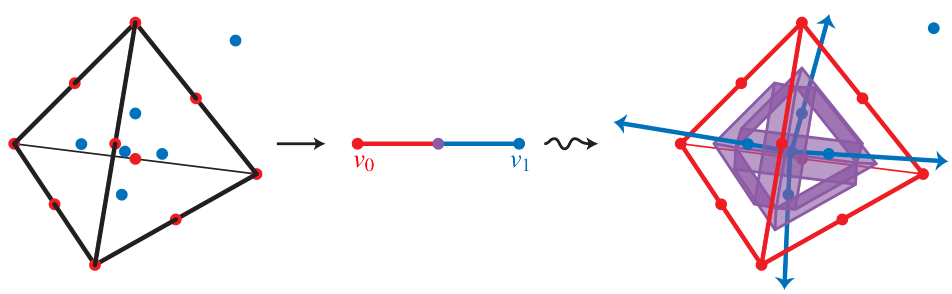

Figure 15: A genus 3 Heegaard splitting (right) of S^3, following Rubinstein–Tillmann's construction.

The even-dimensional case is similar, but with an extra move.

For example, the triangulation of S^3 with two 3-simplices gives a genus 3 Heegaard splitting, as shown in Figure 15.

Following Rubinstein-Tillmann, one might try to construct a, say PL, multisection of T^n using the symmetric space T^n/S_n, which is homeomorphic to a disk-bundle over the circle; this bundle is twisted when n is even and untwisted when n is odd.

One can also view the symmetric space T^n/S_n as an n-simplex $\Delta_n = [\vec{v}_0, \dots, \vec{v}_n]$ with certain faces identified. When $n = 2k-1$, one can also view Δ_n as an iterated join of intervals,

$$\Delta_n = [\vec{v}_0, \vec{v}_1] * \cdots * [\vec{v}_{2(k-1)}, \vec{v}_{2k-1}].$$

Hence, there is a map $\phi : \Delta_n \to \Delta_{k-1} = [\vec{v}_0, \dots, \vec{v}_n]$ given by

$$\phi : \vec{x} = \sum_{i=0}^{k-1} w_i(c_i\vec{v}_{2i} + (1-c_i)\vec{v}_{2i+1}) \mapsto \sum_{i=0}^{k-1} w_i\vec{v}_i.$$

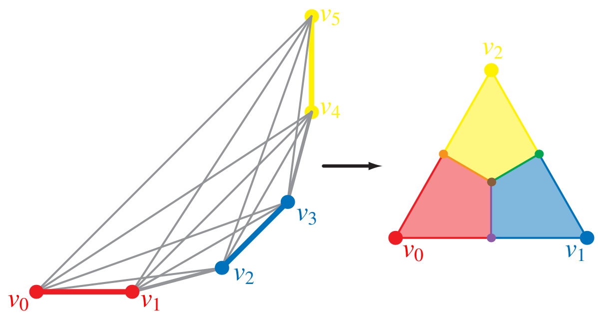

Figure 16: Try viewing T^n/S_n as $\Delta_n/\sim$ and Δ_n as an iterated join of k intervals. Then map $\Delta_n \to \Delta_{k-1}$, decompose Δ_{k-1}, and pull back. It fails, even for $n=5$, shown.

One can then decompose Δ_{k-1} symmetrically into k pieces using barycentric coordinates as in (32) and Figure 16. Following Rubinstein–Tillmann's construction of PL multisections from triangulations [RuTi20], one might attempt to construct a multisection of T^n by pulling back each X_i via ϕ, mapping forward by the quotient map $\Delta_n \to T^n/S_n$, and pulling back by the quotient map $T^n \to T^n/S_n$.

This construction works for T^3 and cuts any T^n into k 1-handlebodies of genus n. Unfortunately, the needed intersection properties fail, even for T^5, so the decomposition is not a multisection. Note that by writing

$$\Delta_n = [\vec{v}_0, \vec{v}_1] * \cdots * [\vec{v}_{2(k-1)}, \vec{v}_{2k-1}]$$

we made an asymmetric choice, and that the resulting decomposition is generally different than the one obtained by writing

$$\Delta_n = [\vec{v}_{\sigma(0)}, \vec{v}_{\sigma(1)}] * \cdots * [\vec{v}_{\sigma(2k-2)}, \vec{v}_{\sigma(2k-1)}]$$

for arbitrary $\sigma \in S_n$ and then following the same procedure.

References

[Fr86] C. Frohman, *Minimal surfaces and Heegaard splittings of the three-torus*, Pacific J. Math. 124 (1986), no. 1, 119-130.

[GaKi16] D. Gay, R. Kirby, *Trisecting 4-manifolds*, Geom. Topol. 20 (2016), no. 6, 3097-3132.

[Ha02] A. Hatcher, *Algebraic topology*, Cambridge Univ. Press (2002), xii+544 pp.

[Ko21] D. Koenig, *Trisections of 3-manifold bundles over* S^1, Algebr. Geom. Topol. 21 (2021), no. 6, 2677-2702.

[LCMe18] P. Lambert-Cole, J. Meier, *Bridge trisections in rational surfaces*, arXiv:1810.10450.

[LCMi21] P. Lambert-Cole, M. Miller, *Trisections of 5-manifolds*, MATRIX annals 4, Springer (2021), 117-134.

[MeScZu16] J. Meier, T. Schirmer, A. Zupan, *Classification of trisections and the generalized property R conjecture* Proc. Amer. Math. Soc. 144 (2016), no. 11, 4983-4997.

[Mo67] H.R. Morton, *Symmetric products of the circle*, Proc. Cambridge Philos. Soc. 63 (1967), 349-352.

[OEIS] *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org.

[RoSa82] C. Rourke, B. Sanderson, *Introduction to piecewise-linear topology*, Springer-Verlag (1982), viii+123 pp.

[RuTi20] J.H. Rubinstein, S. Tillmann, *Multisections of piecewise linear manifolds*, Indiana Univ. Math. J. 69 (2020), no. 6, 2209-2239.

[RuTi18] J.H. Rubinstein, S. Tillmann, *Generalized trisections in all dimensions*, Proc. Natl. Acad. Sci. USA 115 (2018), no. 43, 10908-10913.

[Wi20] M. Williams, Trisections of flat surface bundles over surfaces, Ph.D. Thesis, University of Nebraska-Lincoln (2020), 81 pp.

Department of Mathematics and Statistics, Wake Forest University, Winston Salem, NC 27109, USA

kindret@wfu.edu

www.thomaskindred.com