END-ESSENTIAL SPANNING SURFACES FOR LINKS IN THICKENED SURFACES

THOMAS KINDRED

Abstract. Let D be an alternating link diagram on a surface Σ, such that D cuts Σ into disks and any disk $X \subset \Sigma$ intersecting D generically in two points contains no crossings. We prove that each checkerboard surface from D is incompressible and ∂-incompressible and contains no essential closed curve that is ∂-parallel in $\Sigma \times I$. Our chief motivation comes from technical aspects of a companion paper [Ki22a], where we prove that Tait's flyping conjecture holds for alternating virtual links. We also give an application to Turaev surfaces.

1. Introduction

Let Σ be a closed orientable surface, not necessarily connected or of positive genus, and let $D \subset \Sigma$ be an alternating diagram of a link $L \subset \Sigma \times I$[1] such that D cuts Σ into disks; such D is said to be **fully alternating** [Aetal19]. Then the disks of $\Sigma \setminus\setminus D$[2] admit a checkerboard coloring, from which one can construct the *checkerboard surfaces* B and W of D. These are **spanning surfaces** for L: embedded compact surfaces in $\text{int}(\Sigma \times I)$ with no closed components and boundary L.

A crossing c of D is **removably nugatory** if there is a disk $X \subset \Sigma$ such that $X \cap D = \{c\}$; in that case, one can remove c from D via a flype and a Reidemeister 1 move. If D has a removable nugatory crossing, then at least one of B or W is ∂-compressible. Our main result is the following strong converse of this fact:

Theorem 1.1. *If $D \subset \Sigma$ is a fully alternating diagram without removable nugatory crossings, then both checkerboard surfaces from D are* **end-essential***, meaning that they are π_1-injective and ∂-incompressible, and no essential closed curve on either surface is ∂-parallel in $\Sigma \times I$.*[3]

In some cases, Theorem 1.1 follows from work of Ozawa [Oz06] and Howie [Ho15]. Our strategy, roughly, is to prove the same result with

[1] We denote $I = [-1, 1]$. In $\Sigma \times I$, we identify Σ with $\Sigma \times \{0\}$ and denote $\Sigma \times \{\pm 1\} = \Sigma_\pm$.

[2] Throughout, $X \setminus\setminus Y$ denotes X cut along Y, i.e. the metric closure of $X \setminus Y$.

[3] For a more precise characterization of end-essentiality, see Definition 1.6.

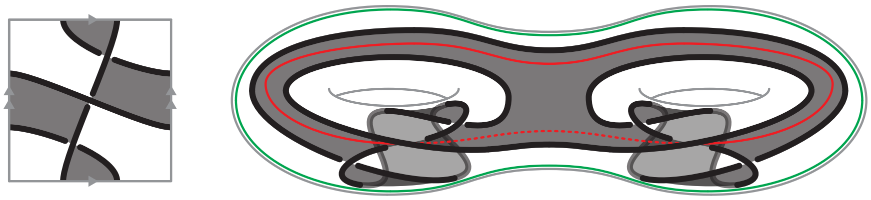

FIGURE 1. The surface F (left) is end-essential, as is its mirror image F', but not $F\natural F'$ (right).

an extra primeness assumption on D and then extend via connect sum.

If $\gamma \subset \Sigma$ is a separating curve such that the annulus $A = \gamma \times I$ intersects the link $L \subset \Sigma \times I$ transversally in two points, then (Σ, L) is a pairwise connect sum $(\Sigma_1, L_1)\#(\Sigma_2, L_2)$. If moreover F spans L and $|A \pitchfork F| = 1$,[4] then F is a **boundary connect sum** $F = F_1\natural F_2$, where each F_i spans L_i in $\Sigma_i \times I$. Interestingly:

Observation 1.2. *Even if $F_i \subset \Sigma_i \times I$ is end-essential for $i = 1, 2$, the surface $F_1\natural F_2 \subset (\Sigma_1\#\Sigma_2) \times I$ need not be end-essential.*

Indeed, consider the example shown in Figure 1. The surface $F \subset T^2 \times I$ shown left is end-essential (this will follow from Theorem 1.4). So too is its mirror image $F' \subset T^2 \times I$. Yet, as shown right in the figure, $F\natural F'$ is not end-essential in $(T^2\#T^2)\times I$: the red curve on $F\natural F'$ is parallel to the green curve on Σ_+. We note that this behavior is related to the following phenomenon in the classical setting:

Observation 1.3. *Even if $F_i \subset S^3$ is incompressible for $i = 1, 2$, the surface $F_1\natural F_2 \subset S^3$ need not be incompressible.*

Indeed, if M and M' are möbius bands, each spanning an unknot in $S^2 \times I$, then both M and M' are π_1-injective (but ∂-compressible; in fact, they are the *only* connected spanning surfaces in S^3 which are π_1-injective and ∂-compressible), and yet $M\natural M'$ is not π_1-injective.

The key to dealing with the complication presented by Observation 1.2 is to distinguish between pairwise connect sums $(\Sigma, D) = (\Sigma, D_1)\#(\Sigma, D_2)$ in general and those for which one of $\Sigma_i = S^2$. Indeed, this distinction is at the heart of the companion paper [Ki22b].

Following Howie–Purcell [HP20], we say that a (nontrivial) diagram (D, Σ) is **weakly prime** if, for any pairwise connect sum decomposition $(\Sigma, D) = (\Sigma, D_1)\#(S^2, D_2)$, either $D_2 = \bigcirc$ is the trivial diagram of the unknot, or $(\Sigma, D_1) = (S^2, \bigcirc)$. Note that no

[4] Here and throughout, $|X|$ denotes the number of components of X.

weakly prime, fully alternating diagram has any removable nugatory crossings.[5]

Theorem 1.1 will follow from the following two results:

Theorem 1.4. *If $D \subset \Sigma$ is a weakly prime, fully alternating link diagram, then both checkerboard surfaces from D are* **end-essential**.

Proposition 1.5. *Suppose $(\Sigma, L) = (\Sigma_1, L_1)\#(\Sigma_2, L_2)$. If $F = F_1 \natural F_2$ spans L, where each F_i is an essential spanning surface for L_i, then F is essential. If moreover $\Sigma_2 = S^2$ and F_1 is end-essential, then F is also end-essential.*

Before presenting the proofs, we give a more precise definition of end-essentiality. Note that these properties all concern curves and arcs in F which *may self-intersect*. There are alternative notions of incompressibility and ∂-incompressibility which do not allow such self-intersections; those notions are sometimes referred to as *geometric* (since they indicate whether or not certain types of surgery moves are possible on F) and these as *algebraic* (due to the equivalence between incompressibility and π_1-injectivity).

Definition 1.6. Let F be a spanning surface in $\Sigma \times I$. Denote $M_F = (\Sigma \times I) \setminus\setminus F$, and use the natural map $h_F : M_F \to \Sigma \times I$ to denote $h_F^{-1}(L) = \widetilde{L}$, $h_F^{-1}(\Sigma_\pm) = \widetilde{\Sigma_\pm}$, and $h_F^{-1}(F) = \widetilde{F}$, so that $h_F : \widetilde{L} \to L$ and $h_F : \widetilde{\Sigma_\pm} \to \Sigma_\pm$ are homeomorphisms and $h_F : \widetilde{F} \setminus \widetilde{L} \to \text{int}(F)$ is a 2:1 covering map. Then we say that F is:

(a) **incompressible** if any circle[6] $\gamma \subset \widetilde{F} \setminus \widetilde{L}$ that bounds a disk $X \subset M_F$ also bounds a disk in $\widetilde{F} \setminus \widetilde{L}$.[7][8][9]
(b) **end-incompressible** if any circle $\gamma \subset \widetilde{F} \setminus \widetilde{L}$ that is parallel in M_F to $\widetilde{\Sigma_\pm}$ bounds a disk in $\widetilde{F} \setminus \widetilde{L}$.
(c) **$\boldsymbol{\partial}$-incompressible** if, for any circle $\gamma \subset \widetilde{F}$ with $|\gamma \cap \widetilde{L}| = 1$ that bounds a disk in M_F, the arc $\gamma \setminus\setminus \widetilde{L}$ is parallel in $\widetilde{F} \setminus\setminus \widetilde{L}$ to $\widetilde{L}$.
(d) **essential** if F satisfies (a) and (c).
(e) **end-essential** if F satisfies (b) and (c).[10]

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[5] In [Ki22b], we call D **prime** if, for any pairwise connect sum decomposition $(\Sigma, D) = (\Sigma_1, D_1)\#(\Sigma_2, D_2)$, one of $(\Sigma_i, D_i) = (S^2, \bigcirc)$.

[6] We use "circle" as shorthand for "simple closed curve".

[7] The disk $h_F(X)$ is then called a **fake compressing disk** for F.

[8] Thus, F is **compressible** if some *essential* circle $\gamma \subset \widetilde{F} \setminus \widetilde{L}$ bounds a **compressing disk** $Y \subset M_F$.

[9] F is incompressible if and only if F is π_1-injective, meaning that inclusion $\text{int}(F) \hookrightarrow (\Sigma \times I) \setminus L$ induces an injection of fundamental groups (for all possible choices of basepoint).

[10] Note that any end-essential surface is essential. Observe moreover that the converse is true when (each component of) Σ is a 2-sphere.

Sometimes, when considering a spanning surface F for a link $L \subset \Sigma \times I$, it is convenient to cut out a regular neighborhood νL of the link and view F (which we identify with $F \setminus\setminus \nu L$) as a properly embedded surface in the link exterior $E = (\Sigma \times I) \setminus\setminus \nu L$. This change of perspective affect the aspects of end-essentiality as follows. Denote $E_F = E \setminus\setminus F$ and use the natural map $g_F : E_F \to E$ to denote $\widehat{\Sigma_\pm} = g_F^{-1}(\Sigma_\pm)$, $\widehat{F} = g_F^{-1}(F)$, and $\widehat{\partial \nu L} = g_F^{-1}(\partial \nu L)$. Then F is

(a) **incompressible** if no essential circle $\gamma \subset \widehat{F}$ bounds a disk in E_F.[11] also bounds a disk in $\widehat{F} \setminus \widehat{L}$.
(b) **end-incompressible** if no essential circle $\gamma \subset \widehat{F}$ is parallel in E_F to $\widehat{\Sigma_\pm}$.
(c) **∂-incompressible** if, for any pair of arcs $\alpha \subset \widehat{\partial \nu L}$ and $\beta \subset \widehat{F}$ that are parallel through a disk $X \subset E_F$, α and β are also parallel in $\widehat{\partial \nu L} \cup \widehat{F}$.[12][13]

2. Proofs

Proof of Theorem 1.4. Assume without loss of generality that Σ is connected. Denote the checkerboard surfaces of D by B and W, where W is the all-A state surface of D and B is the all-B state surface; equivalently, W is the negative-definite surface from D, and B is the positive-definite surface [**?**]. View B and W as properly embedded surfaces in the link exterior $(\Sigma \times I) \setminus\setminus \nu L$. Denote $B \cap W = v$, which consists of one vertical arc at each crossing.

Suppose first that B is compressible. Among all compressing disks for B, choose one, X, so as lexicographically to minimize $(|\partial X \pitchfork v|, |X \pitchfork W|)$. Then $X \cap W$ consists only of arcs (each with both endpoints on v), since any circle of $X \cap W$ would lie entirely in some disk of $W \setminus v$ and thus bound a disk in W; an innermost circle of $X \cap W$ would then bound a disk W' of $W \setminus\setminus X$, and surgering X along W' would give a (sphere and a) compressing disk X' for F with $\partial X' = \partial X$ and $|X' \pitchfork W| < |X \pitchfork W|$, contrary to assumption.

Further, $X \cap W$ is nonempty, since v cuts B into disks and each point of $\partial X \cap v$ is an endpoint of an arc of $X \cap W$. Therefore, there are arcs $\alpha \subset \partial X \cap B \setminus\setminus v$ and $\beta \subset X \cap W$ that cobound an outermost disk X_0 of $X \setminus\setminus W$. Because D is alternating, X_0 appears as shown left in Figure 2, contradicting the fact that D is weakly prime.

[11] If it did bound such a disk X, then $g_F(X)$ would be a **compressing disk** for F.

[12] In particular, α is parallel in $\widehat{\partial \nu L}$ to $\partial \widehat{F} = g_F^{-1}(\partial F)$, and β is parallel in $\widehat{F}$ to $\partial \widehat{F}$.

[13] If α and β were not parallel in $\widehat{\partial \nu L} \cup \widehat{F}$, then $g_F(X)$ would be a **∂-compressing disk** for F.

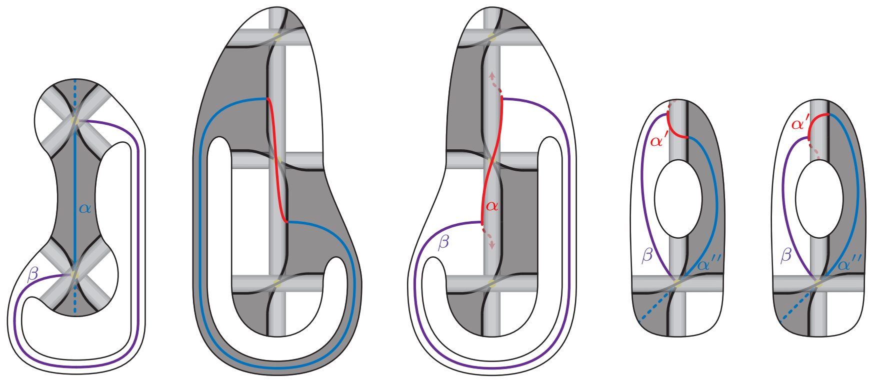

FIGURE 2. Contradictions from the proof of Theorem 1.4

Second, assume that B is end-compressible. Then without loss of generality there is an end-annulus N whose boundary consists of two closed curves, $\gamma \subset B$ and $\gamma_+ \subset \Sigma_+$, where γ is essential in B (possibly with self-intersections). Among all such end-annuli N, choose one which lexicographically minimizes $(|\gamma \pitchfork v|, |N \pitchfork W|)$. Then $N \cap W$ consists entirely of arcs with both endpoints on $\gamma \cap v$: there are no circles, since any circle of $N \cap W$, being disjoint from v, would bound a disk in W, implying that γ_+ bounds a disk in Σ_+, hence that B is compressible. Further, γ must intersect v, since it is essential in B. This leads to an outermost disk of $N \backslash\backslash W$, giving the same contradiction as above, using Figure 2, left.

Third, assume that B is ∂-compressible. Among all ∂-compressing disks for B, choose one, X, so as lexicographically to minimize $(|\partial X \pitchfork v|, |X \pitchfork W|)$, provided $\partial X \cap \partial v = \varnothing$. Then $X \cap W$ consists only of arcs with endpoints on either v or $\partial \nu L$, and X must intersect W (otherwise, X appears as shown center-left in Figure 2, contradicting the fact that D is weakly prime).

There are at least two outermost disks of $X \backslash\backslash W$ and just two points of $\partial X \cap \partial B$, neither of them in $X \cap W$ (since $\partial X \cap \partial v = \varnothing$) so there is an outermost disk X' of $X \backslash\backslash W$ that contains at most one of the two points of $\partial X \cap \partial B$. Its boundary consists of an arc β of $X \cap W$ and an arc $\alpha \subset \partial X$.

There are three possibilities for α, each giving a contradiction. If both endpoints of α lie on v, then $\alpha \subset B$, giving the same contradiction (left in Figure 2) for a third time. If both endpoints of α lie on $\partial \nu L$, then (again using the fact that D is alternating) $\alpha \subset \partial \nu L$ appears as shown center in Figure 2, again contradicting the fact that D is weakly prime. If one endpoint of α lies on v and the other lies on $\partial \nu L$, then α consists of an arc $\alpha' \subset \partial \nu L$ and an arc $\alpha'' \subset B$,

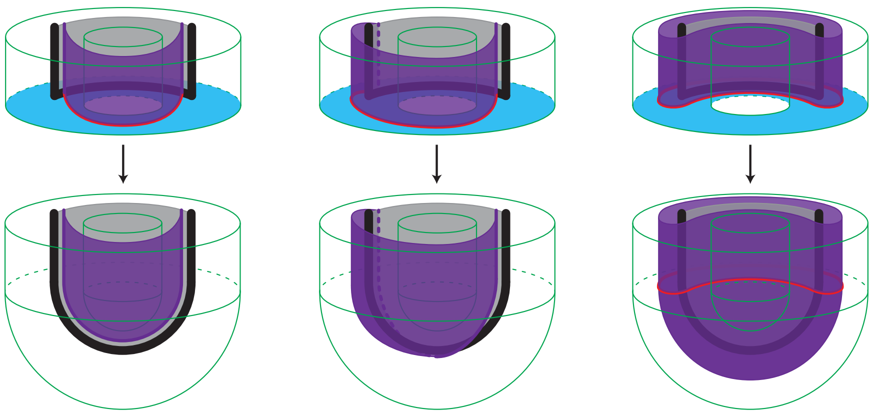

FIGURE 3. An outermost disk within a compressing disk for $F_1 \natural F_2$ extends through a 2-handle to give a possibly fake compressing disk or ∂-compressing disk for F_1 or F_2.

and X' must appear as shown center-right or right in Figure 2, but this contradicts the minimality of $|\partial X \cap v|$, since D is weakly prime.

The same arguments prove that W, too, is end-essential. □

Proof of Proposition 1.5. Assume that $(\Sigma, L) = (\Sigma_1, L_1)\#_\gamma(\Sigma_2, L_2)$ and $F = F_1 \natural_\gamma F_2$, where both F_1 and F_2 are essential; denote the annulus $A = \gamma \times I$.

Suppose first that F is compressible. Choose a compressing disk X for F which minimizes $|X \pitchfork A|$. The incompressibility of F_1, F_2 imply that $X \cap A \neq \varnothing$, and the minimality of $|X \cap A|$ implies that $X \cap A$ contains no circles. Hence, there is an outermost disk X' of $X \setminus\setminus A$; its boundary consists of an arc α of $X \cap A$ and an arc $\beta \subset \partial X \subset F$.

Assume without loss of generality that $\beta \subset F_1$. Viewing each $\Sigma_i \times I$ as a component of $(\Sigma \times I) \setminus\setminus A$ with a 2-handle attached, X' extends as shown in Figure 3 through this 2-handle in $\Sigma_1 \times I$ to give a disk $X'' \subset \Sigma_1 \times I$ that is either a compressing disk, possibly fake, or a ∂-compressing disk for F_1.[14] Since F_1 is essential, the only possibility is that X'' is a fake compressing disk for F_i, implying that $\beta \subset F_1$ is parallel through F_1 to A, but this contradicts the minimality of $|X \cap A|$.

If F is ∂-compressible, then we may choose a ∂-compressing disk X for F which minimizes $|X \pitchfork A|$, provided $\partial X \cap \partial \nu L \cap A = \varnothing$. This last condition, with the same arguments used above, ensures that there is an outermost disk X' exactly as above, the key point being

[14] Color guide for Figures 3 and 4: X' purple, A light blue, α red, $\Sigma_\pm$ green.

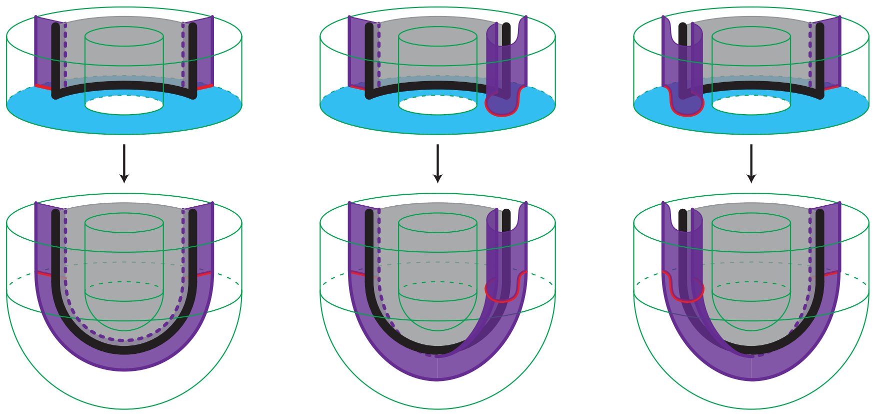

FIGURE 4. Extending a disk X' in an end-annulus for $F_1 \natural F_2$ through a 2-handle to get an annulus Y

that $\beta \subset F$, i.e. $\beta \cap \partial \nu L = \varnothing$. This gives the same contradiction as above.

We have shown that F is essential. Now assume that $\Sigma_2 = S^2$ and F_1 is end-essential.

Suppose that F is end-compressible. Then without loss of generality there is an end-annulus N, whose boundary consists of two closed curves, $\omega \subset F$ and $\omega_+ \subset \Sigma_+$, where ω is essential in F (possibly with self-intersections). Among all such annuli N, choose one which minimizes $|N \pitchfork A|$. Then $N \cap A$ is nonempty, since F_1 and F_2 are end-essential. Further, $N \cap A$ consists only of arcs, no circles, since any circle in $A \setminus F$ bounds a disk in $\Sigma_2 \times I = S^2 \times I$. Moreover, no arc of $N \cap A$ is parallel in $A \setminus\setminus N$ to ∂A or to $\text{int}(F)$; otherwise, there would be an outermost disk of $A \setminus\setminus N$ along which to surger N, contradicting minimality.

Hence, all arcs of $N \cap A$ have one endpoint on F and one on ∂A, and so each component of $N \setminus\setminus A$ is a disk whose boundary consists of arcs of $N \cap A$ together with an arc of $\omega \setminus\setminus A$ and an arc of $\omega_+ \setminus\setminus A$. There is at least one disk X' of $N \setminus\setminus A$ on the same side of A as F_2. Denote $\partial X' \cap F = \beta$ and $\alpha = \partial X' \cap A$. Extend X' through a 2-handle as in Figure 4 to obtain an annulus $Y \subset \Sigma_2 \times I = S^2 \times I$. Then attach a disk to Y along the component of ∂Y that lies in $\Sigma_2^+ = S^2$, and push the resulting disk Z into the interior of $\Sigma_2 \times I$. Since F_2 is essential, Z cannot be a compressing disk or a ∂-compressing disk; instead, Z must be a *fake* compressing disk, implying that β is parallel in F_2 to A, but this contradicts the minimality of $|N \cap A|$. □

Proof of Theorem 1.1. Let $D \subset \Sigma$ be a fully alternating diagram without removable nugatory crossings. If D is weakly prime, then its checkerboard surfaces are end-essential, by Theorem 1.4. Otherwise, decompose $(\Sigma, D) = (\Sigma, D_0) = (\Sigma, D_1)\#(S^2, D_1')$ such that D_1' is a prime diagram on S^2. Continue in this way, decomposing $(\Sigma, D_i) = (\Sigma, D_{i+1})\#(S^2, D_i')$, where D_i' is prime on S^2 until some (Σ, D_n) is weakly prime. Then D_n is fully alternating on Σ, so its checkerboard surfaces are both end-essential by Theorem 1.4. Likewise, $D_1', \dots, D_{n-1}'$ are all fully alternating on S^2, so all of their checkerboard surfaces are essential. Therefore, by Proposition 1.5, both checkerboard surfaces from D are end-essential. □

2.1. **End-essentiality and stabilization.**

Proposition 2.1. *If an end-incompressible surface F spans a split link $L \subset \Sigma \times I$, then the boundary of each connected component of F lies in a single split component of L.*

Proof. Let $A = \bigsqcup_t A_t \subset \Sigma \times I$ be a disjoint union of properly embedded annuli, each with one boundary component on each of $\Sigma_\pm$, such that for each component Y of $(\Sigma \times I) \setminus\setminus A$, $Y \cap L$ is a nonempty nonsplit link. Isotope F so that it intersects A transversally and minimally. This forces $F \cap A \neq \varnothing$: otherwise, $A \setminus\setminus F$ contains either a disk or an annulus whose boundary is a circle of $F \cap A$ and a circle of ∂A, and either possibility contradicts the assumptions that $F \pitchfork A$ is minimal and F is end-incompressible. The result follows. □

3. Applications to Turaev surfaces

Our chief motivating applications appear in the companion paper [Ki22a], where one of the main results states that any two weakly prime, fully alternating diagrams of a given link $L \subset \Sigma \times I$ are related by a sequence of flype moves. The idea of the proof is to show that the checkerboard surfaces for any two such diagrams are related by a sequence of re-plumbing moves (this is where we use the results of this paper), and that these re-plumbing moves on the checkerboard surfaces correspond to flype moves on the diagrams. The result is false without the requirement of weak primeness, and yet, it holds for (links and) diagrams which are weakly prime but not (pairwise) prime. This is noteworthy because it implies that there are infinitely many distinct ways to take the connect sum of any two non-classical virtual knots. See [Ki22a] for details.

We conclude this paper with a different sort of application. Given any diagram $D \subset S^2$ of a classical link $L \subset S^3$, construct the all-A state surface F_A and the all-B state surface F_B from D such that near each crossing each state surface has a standard crossing band, and away from these crossing bands F_A and F_B lie entirely on opposite sides of S^2. Then the interiors of F_A and F_B intersect only in vertical arcs, one at each crossing. Then a regular neighborhood $F_A \cup F_B$ is a thickened surface; its core surface Σ_D, called the **Turaev surface** of D [Tu87], is a Heegaard surface for S^3 (provided D is connected) on which D forms a fully alternating diagram [DFKLS08]. We note the following consequence of Theorem 1.1:

Corollary 3.1. *For any diagram $D \subset S^2$ without nugatory crossings, denote the all-A and all-B state surfaces from D by F_A and F_B, so that $\nu(F_A \cup F_B) = \Sigma_D \times I$ is a regular neighborhood of the Turaev surface Σ_D from D. Then F_A and F_B are end-essential in $\Sigma_D \times I$. Hence, even if F_A, say, is compressible in S^3, any compressing disk must intersect $\partial(\Sigma_D \times I)$ generically in at least two circles.*

In particular, suppose D is a nontrivial diagram of the unknot without nugatory crossings. Then F_A and F_B are compressible in $S^3 \supset \Sigma_D \times I$, but given any compressing disk X for either surface, $X \pitchfork \partial(\Sigma_D \times I)$ is a disconnected multicurve ω on $\partial(\Sigma_D \times I)$; it is reasonable to require further that (each component of) ω be essential, and no two components of ω be parallel, in $\partial(\Sigma_D \times I)$. What are the possibilities for this multicurve ω? In particular, let us denote

$$\Omega_A(D) = \left\{ \omega = X \pitchfork \partial(\Sigma_D \times I) \;\middle|\; \begin{array}{c} X \text{ is a compressing disk for } F_A, \\ \omega \text{ cuts off no disk or annulus} \\ \text{from } \partial(\Sigma_D \times I) \end{array} \right\},$$

and

$$n_A(D) = \min_{\omega \in \Omega_A(D)} |\omega|,$$

and likewise denote $\Omega_B(D)$ and n_B. With this setup, we ask:

Question 3.2. How can the sets $\Omega_A(D)$ and $\Omega_B(D)$ relate to each other (depending on D)?

Question 3.3. How do the quantities $|\Omega_A(D)|$, $n_A(D)$ and $\beta_1(F_A)$ relate to each other?

Question 3.4. How do $|\Omega_A(D)|$, $n_A(D)$, $|\Omega_B(D)|$, and $n_B(D)$ relate to the number of crossings in D and the genus of the Turaev surface?

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Department of Mathematics & Statistics, Wake Forest University, Winston-Salem North Carolina, 27109

E-mail address: thomas.kindred@wfu.edu

URL: www.thomaskindred.com