Essence of a spanning surface

Thomas Kindred

Abstract. Murasugi sum, also called (generalized) plumbing, is a way of gluing two spanning surfaces along a disk to obtain another spanning surface. Gabai proved that plumbing π_1-essential Seifert surfaces always gives a π_1-essential surface, and Ozawa extended this result to unoriented spanning surfaces. We show that the analogous statement about geometrically essential surfaces is untrue.

We then extend the notion of π_1-essentiality to a new numerical invariant which we call the *essence* of a spanning surface $F \subset S^3$, which measures how F is from being compressible. We extend Ozawa's theorem by showing that plumbing respects this new invariant. We further extend this result in to spanning surfaces in thickened surfaces.

Contents

1. Introduction

Murasugi sum, also called (generalized) plumbing, is a way of gluing two spanning surfaces F_1 and F_2 along a disk U to obtain another spanning surface $F = F_1 * F_2$. (There is one extra condition; see Definition 2.9.) Gabai proved that plumbing respects several geometric properties of Seifert surfaces, including incompressibility [Ga83, Ga85], and Ozawa extended Gabai's result by proving that plumbing respects π_1-essentiality of *1- and* 2-sided spanning surfaces

Date: August 23, 2023.

[Oz11]. Section §2 states these results precisely and surveys other ways that plumbing has been applied.[1]

The main results of this paper concern possible extensions of Ozawa's theorem. First, in §3, we show that Ozawa's theorem does not extend from π_1-essential surfaces to geometrically essential ones.

Theorem 3.1. *A Murasugi sum of geometrically essential surfaces need not be geometrically essential.*[2]

Next, in §4, we introduce the *essence* $\text{ess}(F)$ of a spanning surface F, roughly a notion of *representativity* adapted to spanning surfaces. It measures how far a surface is from being compressible and generalizes π_1-essentiality, in the sense that F is π_1-essential if and only if $\text{ess}(F) \geq 2$. We extend Ozawa's theorem as follows:

Theorem 5.5. *If $F = F_1 * F_2$ is a Murasugi sum of π_1-essential spanning surfaces F_i, then $ess(F) \geq \max_{i=1,2} ess(F_i)$.*

2. Background

Definition 2.1. A **spanning surface** F for a link $L \subset S^3$ is a compact surface, orientable or nonorientable, with no closed components which is properly embedded in the link exterior $E = S^3 \setminus \mathring{\nu} L$, such that ∂F intersects each meridian on $\partial \nu L$ transversally in one point.

Alternatively, by attaching an annulus to F in each component of νL, one can view F as an embedded surface in S^3 with $\partial F = L$.

We will use both notions, each of which has advantages.[3]

Notation 2.2. Throughout, F, F', and F_i will denote spanning surfaces in S^3 with respective boundaries L, L', and L_i.

Given a diagram D of L, one can construct two spanning surfaces B and W by coloring the regions of $S^2 \setminus D$ black and white in checkerboard fashion. These **checkerboard surfaces** B and W intersect in *vertical arcs* which project to the crossings of D. Figure 1 shows the construction and the spatial graph $B \cap W$ comprised of L and the vertical arcs at the crossings.

More generally, given a *state* x of D (constructed by smoothing each crossing in one of two ways, $)(\xleftarrow{A} \times \xrightarrow{B} \asymp$), one can construct a spanning surface F_x for L, called a **state surface**, by attaching a

[1] E.g. to Alexander polynomials and crossing numbers of alternating knots; fiberedness, open book decompositions, and contact structures; uniqueness of minimal genus Seifert surfaces; the HOMFLY-PT polynomial; the Conway polynomial and periodicity; knot Floer homology; Khovanov homology; quasipositivity; and the slice-ribbon conjecture.

[2] Figure 13 shows an example of this phenomenon.

[3] For example, in the paragraph after Notation 2.2, the second sentence adopts the former perspective, while the third sentence adopts the latter.

FIGURE 1. Constructing checkerboard surfaces

disk to each state circle and attaching a half-twisted band at each crossing. [4][5]

Given a state x of a link diagram, the (abstract) **state graph** G_x is obtained by collapsing each state circle to a point, while keeping the A and B labels on the edges. The state x is **adequate** if G_x has no loops[6], and x is **homogeneous** if all edges in each cut component[7] of G_x have the same type, A or B. If both conditions hold, x is **homogeneously adequate**; in this case, F_x is π_1-essential. See Theorem 2.12.

Viewing a spanning surface F as a surface with boundary in S^3, one may cut S^3 along F to obtain a compact 3-manifold $S^3\backslash\backslash F$ with boundary; there is a natural map $\phi_F : S^3\backslash\backslash F \to S^3$ which restricts to a homeomorphism on $S^3\backslash\text{int}(F)$ and to a 2:1 covering map on $\text{int}(F)$. In particular, we may identify $L = \partial F$ with $\phi_F^{-1}(L) \subset \partial(S^3 \setminus\setminus F)$. When F is orientable, $S^3 \setminus\setminus F$ is a *sutured manifold* with sutures L, but when F is nonorientable, L does not separate $\partial(S^3 \setminus\setminus F)$, so $S^3 \setminus\setminus F$ is not quite a sutured manifold.

Notation 2.3. Throughout, denote $S_F = S^3\backslash\backslash F$ and $\phi_F : S^3\backslash\backslash F \to S^3$ the natural map described above. Also denote $\widetilde{L} = {\phi_F}^{-1}(L) \subset \partial S_F$ and $\widetilde{F} = \phi_F^{-1}(\text{int}(F)) = \partial S_F \setminus \widetilde{L}$.

2.1. Geometrically and algebraically essential surfaces.

[4]The isotopy class of F_x may depend on the *layering* of the disks relative to the projection sphere; to avoid such ambiguity, we assume, unless stated otherwise, that all state circles are capped with disks *on the same side* of the projection sphere S^2. For an interesting example of a state surface with different layering, see Figure 4.

[5]Every state surface is a checkerboard surface of some diagram.

[6]That is, the endpoints of each crossing arc lie on distinct state circles.

[7]If G_x has no cut vertices (ones whose deletion disconnects G_x), then G_x has a single cut component; otherwise, cut G_x at a cut vertex. Cut each resulting component at a cut vertex, if one exists. Continue until no component has a cut vertex. The resulting components are the *cut components* of G_x.

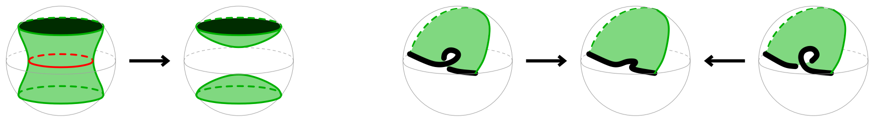


FIGURE 2. Geometric compression and ∂-compression of a spanning surface

Definition 2.4. A spanning surface F is **geometrically essential** if F cannot be compressed or ∂-compressed to a spanning surface. (See Figure 2.) Equivalently, F is geometrically essential if *both*:

- Every simple closed curve in $\text{int}(F)$ bounding a disk in S^3 bounds a disk in F, and
- Every properly embedded arc in F which is parallel through (an embedded disk in) S^3 to ∂F is also ∂-parallel in F.

If F satisfies the first condition, it is called **geometrically incompressible**, whether or not it satisfies the second.

Definition 2.5. A spanning surface F is $\boldsymbol{\pi_1}$**-essential** if it satisfies the following *equivalent conditions*:

- S_F has incompressible, ∂-incompressible boundary;
- Inclusion $\text{int}(F) \hookrightarrow S^3 \setminus L$ induces an injection of fundamental groups, and F is not a möbius band spanning the unknot.

Remark 2.6. If F is π_1-essential, then F is geometrically essential.

Remark 2.7. A 2-sided spanning surface is π_1-essential if and only if it is geometrically incompressible.

Figure 3 shows a surface F_1 which is geometrically incompressible, because if F_1 admitted a geometric compression, then the resulting surface would be a disk with the same nonzero boundary slope as F_1. There is, however, a compressing disk $\widetilde{X}$ in S_{F_1} for $\widetilde{F_1}$; $X = p_{F_1}(\widetilde{X})$ is an immersed disk in S^3, whose interior is embedded, but whose boundary self-intersects. With the exception of the möbius bands spanning the unknot, any π_1-inessential surface admits such a disk X; call such X an **algebraic compressing disk**.

Modify F_1 as shown in Figure 4 by *plumbing on* six annuli, each with two full positive twists, to get a surface F_2. (A careful definition of plumbing follows in §2.2.) Interestingly:

Proposition 2.8. *The surface F_2 shown in Figure 4 is geometrically essential but π_1-inessential.*

The proof of Proposition 2.8 is not too hard. Still, the argument will be clearer with the extra technical setup of §2.3, which generalizes the notion of outermost disks. The proof appears in §**??**.

FIGURE 3. Left: A geometrically compressible surface. Right: A geometrically incompressible surface F_1 which admits an algebraic compressing disk X.

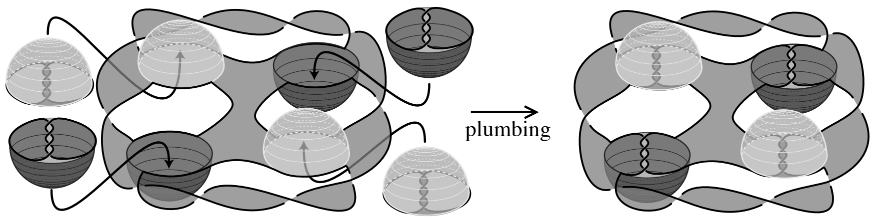


FIGURE 4. Constructing a surface F_2 that is geometrically essential but π_1-inessential

2.2. Plumbing. In §2.2, view F as a compact surface in S^3 with $\partial F = L$, rather than as a properly embedded surface in the link exterior.

Definition 2.9. Let $W \subset S^3$ be an embedded disk with $W \cap F = \partial W$ such that

- ∂W bounds a disk $U \subset F$.
- Denoting $S^3 \setminus \backslash(U \cup W) = B_1 \sqcup B_2$, neither $F_i = F \cap Y_i$ is a disk.

Then W is a **plumbing cap** for F, and U is its **shadow**. (If W satisfies the first condition but not the second, W is a *fake plumbing cap*.)

Say that F is obtained by (generalized) **plumbing** F_1 and F_2 along U, denoted $F_1 * F_2 = F$. This operation is also called **Murasugi sum**. The associated decomposition is a **deplumbing**. See Figure 5.

The operation $F \to F' = (F \setminus U) \cup W$ is called **replumbing**. See Figure 6.

A great deal is known about *oriented* Murasugi sums $F = F_1 * F_2$. Murasugi first used plumbing to compute Alexander polynomials of alternating knots inductively, and thereby determined the genera of alternating knots [Mu58]. (Crowell independently obtained the same result [Cr59]; for a recent, elementary proof which also uses plumbing, see [Ki22].) Harer showed that every fiber surface in S^3 can be

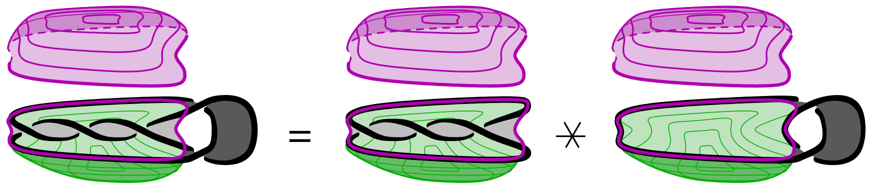


FIGURE 5. Plumbing, also called Murasugi sum

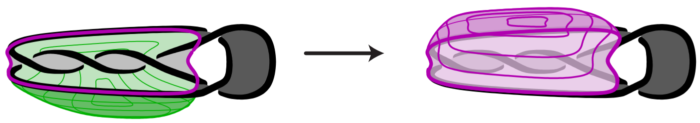


FIGURE 6. Replumbing

constructed by plumbing Hopf bands and performing twisting operations introduced by Stallings [Ha82, St78]. Harer conjectured further that plumbing *and deplumbing* Hopf bands suffices, as Giroux-Goodman later confirmed using contact topology [GG06]. Gabai proved that there are several geometric properties which F possesses if and only if F_1 and F_2 do [Ga83, Ga85]:

Theorem 2.10. *[Gabai* [Ga83, Ga85]*] If $F_1 * F_2 = F$ is a Murasugi sum of Seifert surfaces with each $\partial F_i = L_i$ and $\partial F = L$, then:*

(1) F is essential if F_1 and F_2 are essential.[8]
(2) F has minimal genus if and only if F_1 and F_2 both have minimal genus.
(3) L is a fibered link with fiber F if and only if each L_i is fibered with fiber F_i.
(4) $S^3 \setminus \mathring{\nu} L$ has a nice codimension 1 foliation if and only if both $S^3 \setminus \mathring{\nu} L_i$ do.

See [Ga85] for details. Much more on (i) shortly.

Recently, Baader-Graf described a simple geometric method of fiber-detection, leading to a new proof of (iii) [BG16]. Torisu extends (iii) to a statement about tight contact structures [To00]. Saito–Yamamoto prove that for any oriented plumbing $F = F_1 * F_2$ of fiber surfaces, the arc complex for the open book decomposition of S^3 with page F has translation distance at most two [SY10]. Extending (ii), Kobayashi proves that a minimal genus Seifert surface $F = F_1 * F_2$ is isotopically unique if and only if F_1 is also unique and F_2 is fibered, or vice-versa [Ko89].

[8] The converse is false. See Figure 10.

Oriented plumbing has also proven to be a valuable tool for studying polynomial and homological knot invariants. For example, Hongler–Weber [HW04, HW05] used Menasco–Thistlethwaite's flyping theorem [MT91, MT93, Tait] to show that every oriented *alternating* link decomposes in a unique way under Murasugi sum, and they used this decomposition to extend results of Kobayashi–Kodama [KK88] and Murasugi–Przytycki [MP89], which also used oriented plumbing, regarding the term of the HOMFLY-PT polynomial of maximum z-degree. Costa–Hongler used similar techniques to study Conway polynomials of *periodic* alternating links [CH18].

Perhaps the most remarkable application of oriented plumbing is Ni's plumbing-to-product formula for knot Floer homology,

$$\widehat{HFK}(K, g; \mathbb{F}) \cong \widehat{HFK}(K_1, g_1; \mathbb{F}) \otimes \widehat{HFK}(K_2, g_2; \mathbb{F}), \tag{1}$$

where $\mathbb{F}$ is any field and g, g_1, g_2 denote 3-genus [Ni06]. Juhász obtained a new proof of (1) which led to a simplified proof of the fact that knot Floer homology detects fibered knots [Ju08].

Rudolph constructed interesting oriented plumbings in the contexts of quasipositivity [Ru89] and the slice-ribbon conjecture [Ru02].

If $F_1 * F_2 = F_3$ is a plumbing of Seifert surfaces with plumbing cap X, then $|\partial X \cap L_3| = 2n$ for some n; Goda established the following inequality among the handle numbers of the sutured manifolds S_{F_i} [Go92]:[9]

$$h(S_{F_1}) + h(S_{F_2}) - n + 1 \leq h(S_{F_3}) \leq h(S_{F_1}) + h(S_{F_2}).$$

Thus, handle number is additive under boundary connect sum and is subadditive under plumbing, with defect bounded by the complexity of the plumbing.

For any $s \in \mathbb{Q}$, let $K(s)$ denote the 3-manifold obtained from S^3 by performing Dehn surgery along L_3 with surgery slope s. With n as above, Li showed that $K(s)$ has a taut foliation for all slopes $1 - n < s < n - 1$ [Li03].

Ozbagci–Popescu-Pampu generalized the notion of Murasugi sum to *smooth oriented manifolds of arbitrary dimension* in such a way that Gabai's theorem still holds [OP16]. Their paper is also an excellent survey of prior literature.

Perhaps the best-studied class of plumbings are the *arborescent surfaces*, obtained by plumbing together essential unknotted annuli and möbius bands according to the pattern of a tree, not just in the oriented case [Sa94, Ga86b, KK88] but also in the unoriented case in a magnificent treatise by Bonahon–Siebenmann [BS10].

[9] The handle number $h(W)$ of a compression body W is the minimal number of 2-handles needed to construct W. The handle number of a sutured manifold (M, γ) is $\min\{h(W) : (W, W') \text{ is a Heegaard splitting of } (M, \gamma)\}$.

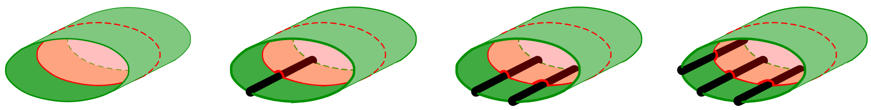


FIGURE 7. Caps for F are compressing disks for ∂S_F.

Unoriented plumbings appear less often in the literature than oriented ones. Recently, the author used replumbings of definite surfaces to give the first purely geometric proof of Menasco–Thistlethwaite's flyping theorem [Ki21, MT91, MT93, Tait]. Earlier, the author considered used replumbing moves in the context of Khovanov homology [Ki18]. The following theorem of Ozawa, extending part (i) of Gabai's theorem to the unoriented case, concludes this survey:

Theorem 2.11 ([Oz11])**.** *If $F = F_1 * F_2$ is a Murasugi sum of π_1-essential spanning surfaces F_i, then F is π_1-essential.*

As a corollary, Ozawa obtains the following fact (see Remark 2.16):

Theorem 2.12 ([Oz11])**.** *If x is a homogeneously adequate state, then F_x is π_1-essential.*

2.3. Caps and height. Again in §2.3, view $F \subset S^3$, rather than in the link exterior.

Definition 2.13. Using Notation 2.3, a **cap** for F is the image $W = \phi_F(\widetilde{W})$ of a compressing disk for ∂S_F.[10] See Figure 7.

A **cap system** for F is a union $\mathcal{W} = \bigcup_i W_i$ of caps $W_i = \phi_F(\widetilde{W_i})$ for F with disjoint interiors, such that $\widetilde{\mathcal{W}} = \bigcup_i \widetilde{W_i}$ is a disjoint union of properly embedded disks which cuts S_F into balls, and $\partial\widetilde{\mathcal{W}}$ contains $\widetilde{L}$ and cuts $\widetilde{F}$ into disks.

Notation 2.14. If X is a cap for F, then $\widetilde{X}$ denotes the (unique) properly embedded disk in S_F satisfying $\phi_F(\widetilde{X}) = X$. Likewise, for a cap system $\mathcal{W}$, $\widetilde{\mathcal{W}}$ denotes (unique) lift which is comprised of disjoint, properly embedded disks.

Example 2.15. If B and W are the checkerboard surfaces from a connected link diagram, then B is a cap system for W, and W is a cap system for B.

Remark 2.16. If $\mathcal{W}$ is a cap system for F, then $F \cup \mathcal{W}$ cuts S^3 into polyhedra and cuts $S^3 \setminus L$ into ideal polyhedra. In [FKP13, FKP14],

[10] Note that if W is a cap for F, then ∂W may well intersect ∂F. Moreover, if $\partial W \cap \partial F = \varnothing$, then ∂W cannot be contractible in F, or else $\partial\widetilde{W}$ would be contractible in ∂S_F.

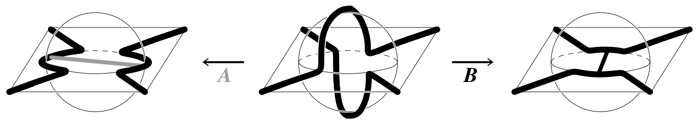


FIGURE 8. Smoothings in the crossing ball setting

Futer-Kalfagianni-Purcell use such polyhedral decompositions to establish deep relationships between essential surfaces, hyperbolic geometry, and colored Jones polynomials. In particular, they obtain an independent proof of Theorem 2.12 in the case that x is all-A or all-B.

To extend Example 2.15 to a more general class of examples, it will be helpful to use the crossing ball structures introduced by Menasco in [M84]. Given a diagram D of a link L, insert a tiny ball C_i at each crossing and perturb D to get an embedding of L in $(S^2 \setminus C) \cup \partial C$, where $C = \bigsqcup_i C_i$. Then the states of D correspond to the submanifolds $x \subset (L \cup \partial C) \cap S^2$ that contain $L \cap S^2$. See Figure 8.

In this setting, $\partial C_i \cap S^2 \setminus L$ consists of four arcs on the equator of ∂C_i for each i. The union of $L \cap \partial C_i$ with either opposite pair of arcs forms a simple closed curve on ∂C_i, which bounds a **crossing band** in C_i. Thus, a spanning surface F of L is a state surface of D if and only if (it can be isotoped such that) $F \cap C_i$ is comprised of crossing bands and $F \setminus C$ is comprised of disks.

Definition 2.17. Let F_x be a state surface from a connected link diagram D, with the crossing ball structure described above, i.e. $F \cap C_i$ is comprised of crossing bands and C cuts F into disks.

Suppose that $\text{int}(F_x) \cap S^2 \setminus C = \varnothing$, i.e. each state circle of x is capped with a disk disjoint from $S^2 \cup C$. Let y be the opposite state of D, i.e. x and y have opposite smoothings at each crossing, and let $\mathcal{W}_y$ be the union of the crossing bands associated to y. Then $(S^2 \setminus C) \cup \mathcal{W}_y$ is a cap system for F_x.

This works more generally, provided that any intersections between $\text{int}(F_x)$ and $S^2 \setminus C$ are (transverse) arcs, or (nontransverse) disks bounded by arcs. Then we call $(S^2 \setminus C) \cup \mathcal{W}_y$ the **flat** cap system for (this positioning of) F_x.

Capping structures $\mathcal{W}$ are useful for determining, e.g., whether F is π_1-essential, by either finding an algebraic compressing disk X or proving that none exists.

Here is the idea. Hypothesize an algebraic compressing disk X, and assume that X has been chosen to intersect $\mathcal{W}$ transversally and minimally. Then $|X \cap \mathcal{W}| = \frac{1}{2}|\partial X \cap \mathcal{W}|$. Hence, the latter quantity is also minimized, so no arc of ∂X is parallel through F to

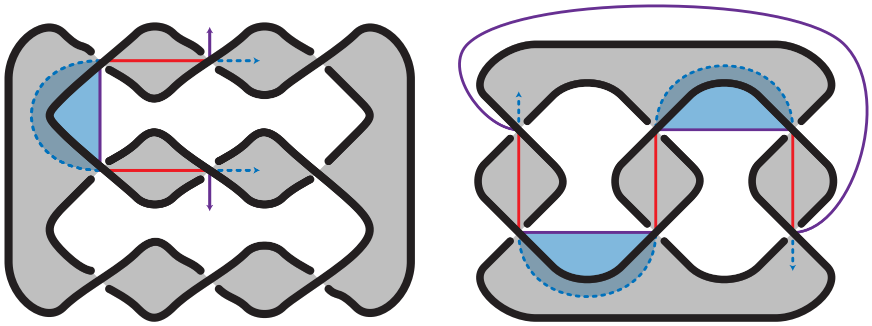


FIGURE 9. π_1-essential checkerboard surfaces for the $(-3,3,-3)$ and $(-2,2,-2)$ pretzel links

$\mathcal{W}$. Characterize the possible outermost disks of $X \setminus\setminus \mathcal{W}$ and "work inward."

In some examples, there are no outermost disks. For example:

Exercise 2.18. [FKP13] Both checkerboard surfaces of any reduced alternating diagram of any prime non-split link are π_1-essential.

Remark 2.19. Futer-Kalfagianni-Purcell define a polyhedral decomposition to be *prime* if no pair of faces meets along more than one edge. When the decomposition from $F \cup \mathcal{W}$ is prime, F is incompressible, since no outermost disk is possible.

Even when a given polyhedral decomposition is not prime, i.e. outermost disks of $X \setminus\setminus \mathcal{W}$ are possible, it is sometimes possible to refine it to produce a prime decomposition. See [FKP13, FKP14] for details. Alternatively, one can keep $\mathcal{W}$ and "work inward" in X according to the following notion of *height*.

Definition 2.20. Given a finite system $A = \bigsqcup_{i\in I} \alpha_i$ of disjoint properly embedded arcs in a disk X, let T be the tree with one vertex for each disk of $X \setminus A$ in which two vertices are adjacent whenever the corresponding disks abut.

Define the **height** of each disk of $X \setminus A$ recursively as follows. Let $T_0 = T$. Outermost disks of $X \setminus A$, corresponding to leaves in T_0, have height 0. For $i \geq 1$, Let T_i be the tree obtained by deleting each leaf of T_{i-1} and its edge. Disks of $X \setminus A$ that correspond to leaves in T_i, have height i.

Example 2.21. The surface shown left in Figure 9 is π_1-essential; its flat cap system admits no disk of height 1.

In general, with this setup, one can try to determine whether or not a surface F is π_1-essential by characterizing possible disks of height 0, then of height 1, and so on. For example:

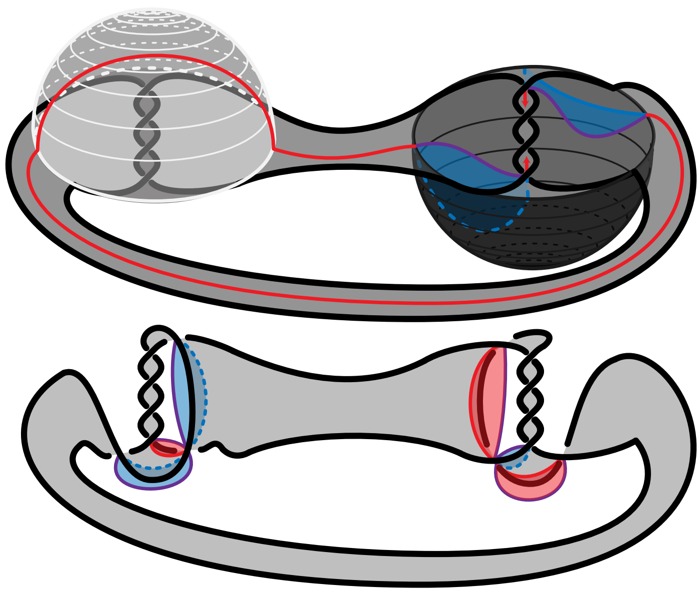
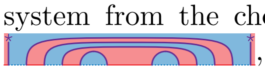


FIGURE 10. Top: an essential Seifert surface F obtained by plumbing an annulus onto a compressible surface. Bottom: F as a checkerboard surface.

Example 2.22. The surface shown right in Figure 9 is π_1-essential; its flat cap system admits disks of height 0 and 1 (shown), but not of height 2.[11]

Example 2.23. Consider the surface F in Figure 10. The flat cap system from the checkerboard picture admits a disk of height 4, , as shown in Figure 11, so it is not immediately clear whether F is essential. By contrast, the flat cap system from the top picture admits disks of height 0, but none of height 1, confirming that F is indeed π_1-essential.

Example 2.23 demonstrates that it is possible to plumb an essential Seifert surface onto a compressible one in a way that yields an essential surface. In fact, the next example shows that it is possible to plumb two compressible Seifert surfaces in a way that yields an essential surface.

Example 2.24. The surface shown in Figure 12, obtained by plumbing two compressible Seifert surfaces, is essential. Indeed, the flat cap system from Figure 12 admits disks of height 0, but not of height 1.

[11] Here is another proof that this surface must be π_1-essential: compressing this surface would yield a disjoint union of a disk and an annulus, but this is impossible since each pair of link components has nonzero linking number.

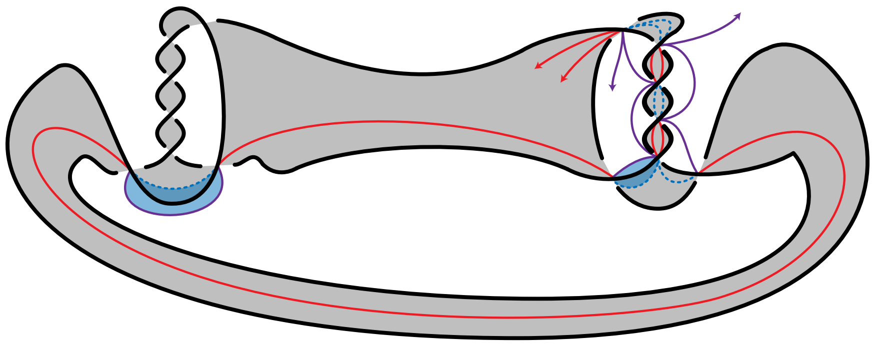


FIGURE 11. The flat cap system from the checkerboard picture in Figure 10 admits a disk of height 4.

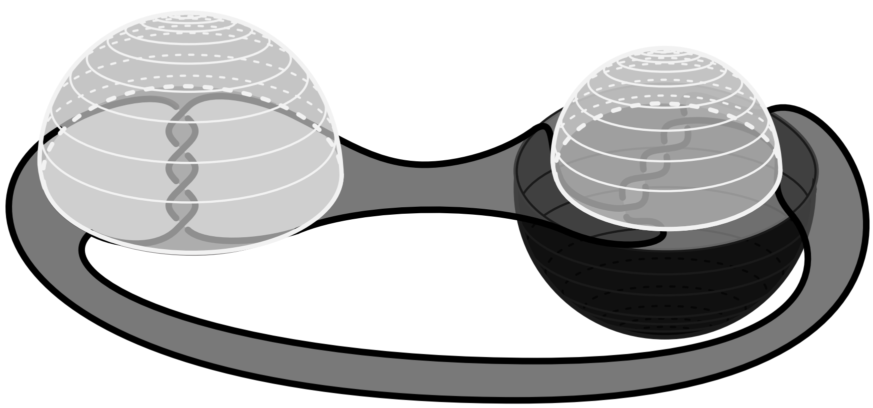


FIGURE 12. An essential Seifert surface obtained by plumbing two compressible Seifert surfaces.

3. Geometric essentiality under unoriented plumbing

In this section, we use cap systems and height to prove that the surface F_2 constructed in Figure 4 is geometrically essential, giving our first main result:

Theorem 3.1. *A Murasugi sum of geometrically essential surfaces need not be geometrically essential.*

Proof. By plumbing a Hopf band onto the surface in Figure 4 as shown in Figure 13, one can obtain a geometrically compressible surface. Indeed, the boundary of a compressing disk and its intersection with the projection plane are colored in the rightmost part of the figure. The theorem now follows the following proposition. $\square$

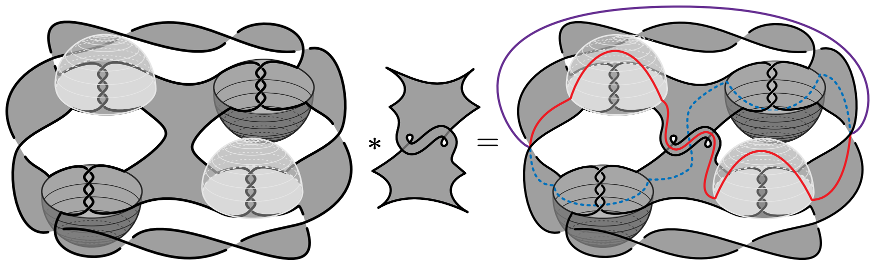


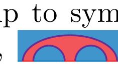
FIGURE 13. A geometrically inessential plumbing of geometrically essential surfaces

Proposition 2.8. *The surface F_2 constructed in Figure 4 is geometrically essential but π_1-inessential.*

Proof. Certainly F_2 is algebraically compressible, as the plumbed-on annuli do not obstruct the algebraic compressing disk from Figure 3. To see that F_2 is geometrically essential, isotope F_2 to appear as the checkerboard surface shown in Figure 14, and consider the resulting flat cap system $\mathcal{W}$. Denote $\text{int}(F_2) \cap \mathcal{W} = v$, so that $F_2 \cap \mathcal{W} = L \cup v$; v consists of vertical arcs, one at each crossing.

Suppose for contradiction that F_2 is (geometrically) compressible. Choose a compressing disk X for F_2 which minimizes $|X \pitchfork \mathcal{W}|$. Then $X \cap \mathcal{W}$ consists entirely of arcs, each with endpoints on distinct arcs of v. Likewise, each arc of $\partial X \setminus \backslash v$ has endpoints on distinct arcs of v. Conversely, each point of $\partial X \cap v$ is an endpoint of such an arc.

Now consider the possibilities for the disks of $X \setminus \backslash \mathcal{W}$, starting with those of height 0, each of whose boundary consists of just two arcs, one in $\mathcal{W} \setminus \backslash v$ and one in $\partial X \setminus \backslash v \subset F \setminus \backslash v$. Up to symmetry, there are three types of height 0 disks, two of each type in each ball of $S^3 \setminus (F_2 \cup \mathcal{W})$; all twelve possible disks appear in Figure 14.

Up to symmetry, there are three types of height 1 disks that abut a single outermost disk, ; Figure 15 shows all three types. There are also two possible configurations, up to symmetry, for height 1 disks which abut two outermost disks,  (or the same with colors reversed); Figure 16 shows both types.

There are no geometric compressing disks in which every subdisk has height at most 1. (There is, however, such an *algebraic* compressing disk!) There are also no disks of height 2, since such a disk could not abut any type of disk shown in Figure 15, nor on the left in Figure 16, hence must abut the type shown right in Figure 16, which gives the contradiction  shown in Figure 17. Thus, F is incompressible.

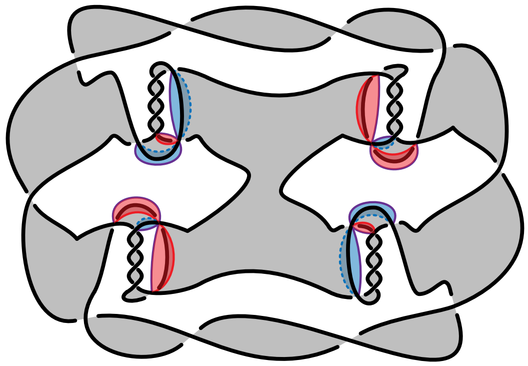


FIGURE 14. Possible height 0 disks in the proof of Proposition 2.8

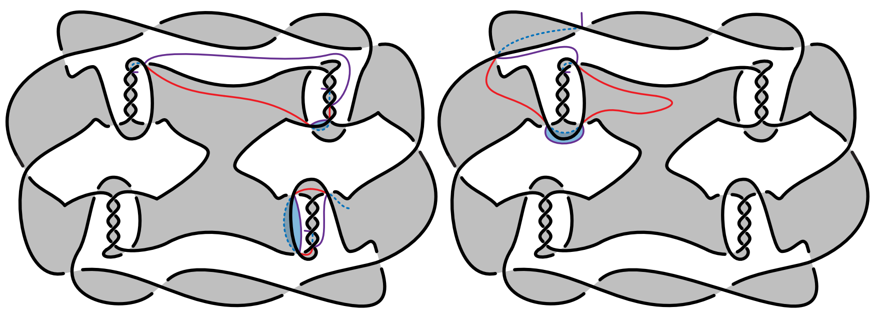


FIGURE 15. Height 1 disks abutting a single outermost disk

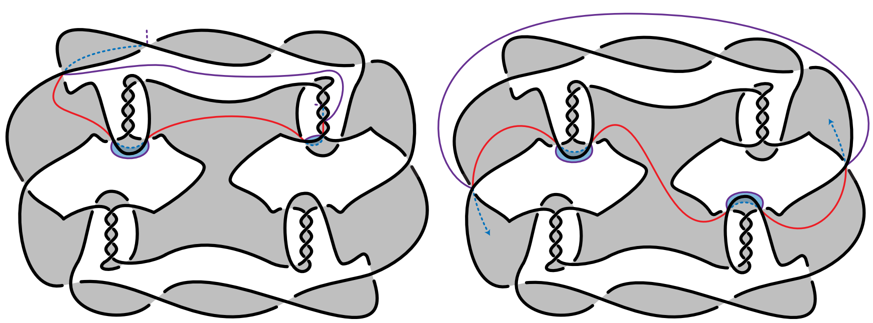


FIGURE 16. Height 1 disks abutting two outermost disks

Finally, we must adapt the argument above to show that F_2 is not ∂-compressible. Suppose otherwise. As before, choose a ∂-compressing disk X for F_2 which minimizes $|X \pitchfork \mathcal{W}|$, and consider

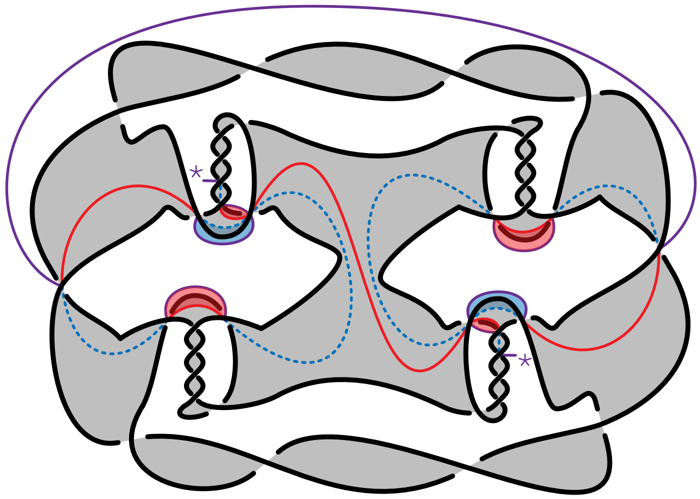


FIGURE 17. The main contradiction in the proof of Proposition 2.8.

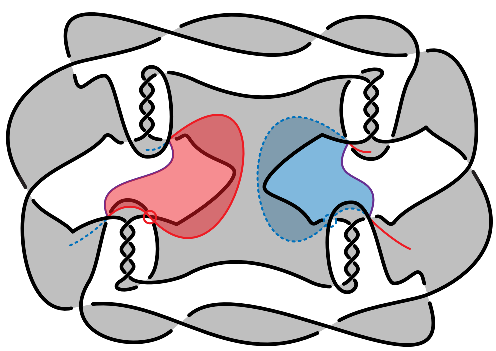
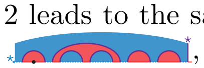


FIGURE 18. Possible height 0 disks in a ∂-compressing disk

the possible configurations of the disks of $X \backslash\backslash \mathcal{W}$, starting with those of height 0.

In addition to the types from Figure 14, there is, up to symmetry, one additional type of possible outermost disk of $X \backslash\backslash \mathcal{W}$. See Figure 18. Yet, such a disk cannot abut a disk of height 1. In fact, the only types of height 1 are still those in Figures 15-16. Considering disks of height 2 leads to the same contradiction as before, with an additional case , shown in Figure 19. □

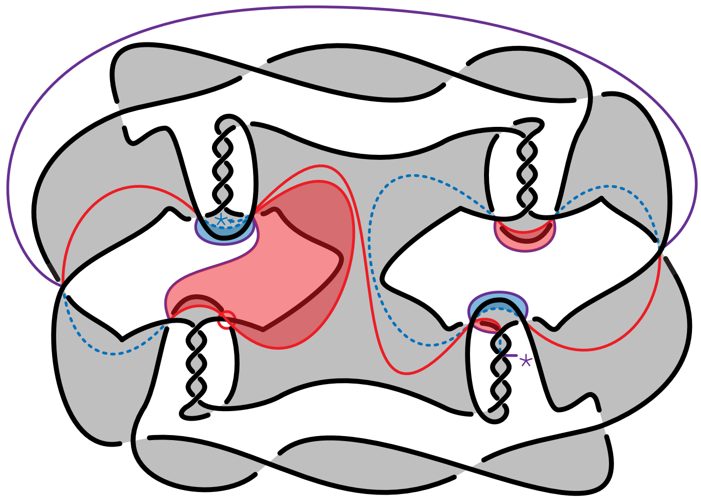


FIGURE 19. The final contradiction in the proof of Proposition 2.8.

4. Algebraic and geometric essence

Recall that a *cap* is the image $X = \phi_F(\widetilde{X})$ of a compressing disk for ∂S_F, and that X is a *plumbing cap* if ∂X bounds a disk in F. More generally, define the following types of caps:

Definition 4.1. Let X be a cap for F. Call X **geometric** if ∂X does not self-intersect, **singular** if it does. Call X **∂-contractible** if ∂X is contractible in F, **∂-essential** if it does not.

Note: a geometric ∂-contractible cap is also called a plumbing cap.

Definition 4.2. Let X be a cap for F which is not parallel to F,[12] and denote the set of self-intersection points of ∂X by x. We say that X is **acceptable** if it admits none of the simplifying moves shown in Figures 20 and 21, i.e. if:[13]

- no arc of $X \cap \partial\nu L$ is parallel in $\partial\nu L$ to $F \cap \nu L$;
- no arc of $\partial X \setminus\setminus \partial\nu L$ that contains at most one point in X is parallel in F to $F \cap \nu L$; and
- no two arcs of $\partial X \setminus\setminus x$ are parallel in $F \setminus \nu L$.

Notation 4.3. Denote the set of all caps for F by $\mathrm{Cap}(F)$. Likewise, denote these sets of caps for F as follows.

[12]That is, ∂X either intersects L or is an essential curve in the interior of F.

[13]It may be too onerous to require that a *cap system* be comprised entirely of acceptable caps. For example, the flat cap system $\phi_F(\bigcup_i \widetilde{U}_i)$ for a non-alternating checkerboard is never acceptable, and the simplifying isotopy removes the property that $\bigcup_i \partial\widetilde{U}_i \supset \widetilde{L}$.

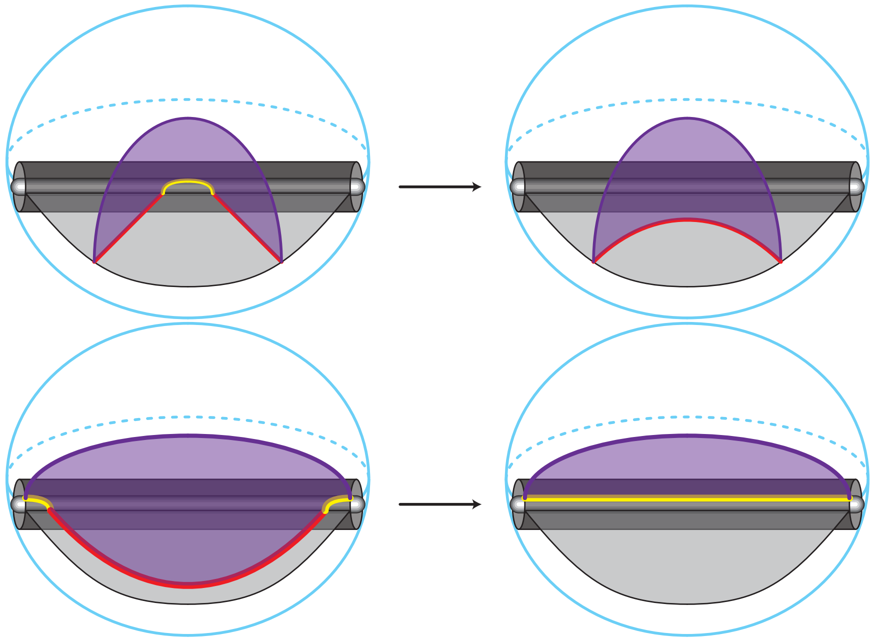


FIGURE 20. A geometric cap is *acceptable* if it cannot be simplified by either of these moves.

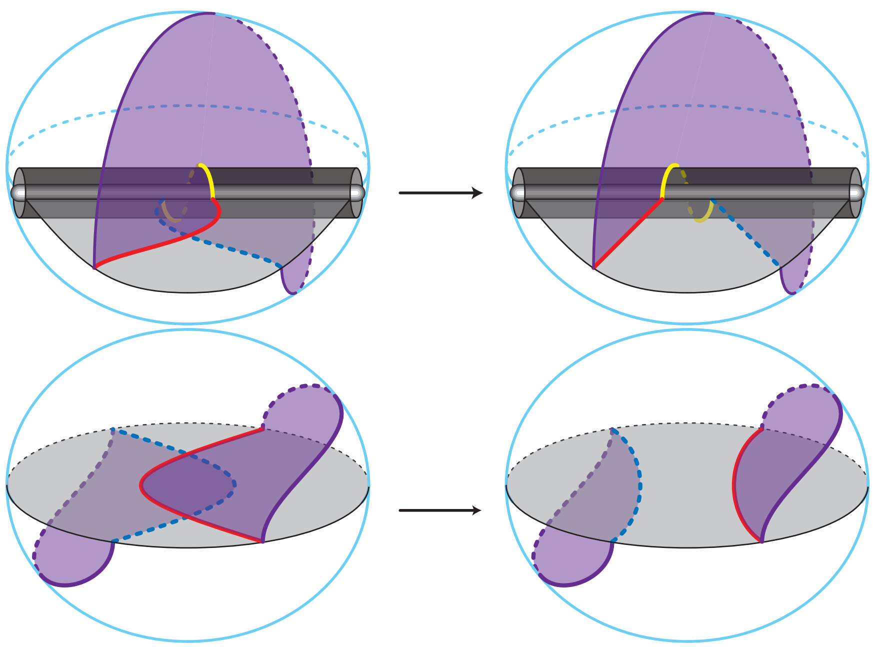


FIGURE 21. A cap is *acceptable* if it cannot be simplified by these moves or those in Figure 20.

- Geometric caps: $\mathrm{Cap}_g(F)$;
- ∂-essential caps: $\mathrm{Cap}_e(F)$;
- Plumbing caps: $\mathrm{Cap}_p(F)$;

Definition 4.4. The (algebraic) **essence** of F is

$$\mathrm{ess}(F) = \min_{X \in \mathrm{Cap}_e(F)} |\partial X \cap L|.$$

We will show in Theorem 5.5 that $\mathrm{ess}(F)$ is well-behaved under plumbing. Theorem 5.6 is a related result for $\min\{\mathrm{ess}(F), \mathrm{ess}_c(F)\}$, albeit with an extra condition on the complexity of the plumbing. First, two remarks:

Remark 4.5. F is π_1-essential if and only if $\mathrm{ess}(F) \geq 2$.

Remark 4.6. $\mathrm{ess}(F) = 1$ if and only if F is a möbius band spanning the unknot.

Although we will mainly be interested in essence as defined "algebraically" above, there is a related "geometric notion," which we now introduce and discuss briefly. Recall that a cap X is *geometric* if ∂X does not self-intersect, $\mathrm{Cap}_g(F)$ denotes the set of geometric caps X for F, and $\mathrm{Cap}_e(F)$ denotes the set of ∂-essential caps for F. Note that $\mathrm{Cap}_g(F) \setminus \mathrm{Cap}_e(F) = \mathrm{Cap}_p(F)$ is the set of all plumbing caps for F.

Definition 4.7. The **geometric essence** of F is

$$\mathrm{ess}_g(F) = \min_{X \in \mathrm{Cap}_g(F) \cap \mathrm{Cap}_e(F)} |\partial X \cap L|.$$

Remark 4.8. Every spanning surface F satisfies $\mathrm{ess}(F) \leq \mathrm{ess}_g(F)$.

Remark 4.9. A spanning surface F is geometrically incompressible if and only if $\mathrm{ess}_g(F) \geq 1$.

Remark 4.10. A spanning surface F is geometrically essential if and only if $\mathrm{ess}_g(F) \geq 2$.

Theorem 3.1 implies that plumbing does not respect geometric essence. Yet, this notion also has advantages. One advantage is that any acceptable geometric cap X describes a possible surgery move on F, much as geometric compressing disks and ∂-compressing disks do (recall Definition 4.2).

One can surger F along an acceptable geometric cap X as follows. Viewing F as a properly embedded surface in the link exterior E, cut F along the n arcs of $\partial X \cap F$, and glue in two parallel copies of X. The resulting surface F' satisfies $\beta_1(F') = \beta_1(F) + n - 2$ and $|s(F) - s(F')| \leq 2n$, where $s(F)$ is the *slope* of F [Ki21].[14] Thus, the

[14] When L is a knot, $s(F)$ is the boundary slope of F. See [Ki21] for the general case.

effect of the surgery move on $\beta_1(F)$ and $s(F)$ gets "worse" as $|\partial X \cap L|$ increases. The precise effect of this surgery on $s(F)$ depends on the *slope* of X, which we define next.

5. Plumbing respects essence

The main goal of §5 is to prove Theorem 5.5. First, we need some technical results. For expository reasons, we also include a proof of a weaker version of Theorem 5.5 in which boundary connect sum replaces Murasugi sum (see Theorem 5.3).

Recall that a *fake plumbing cap* is a geometric cap X for F for which ∂X bounds a disk U in X, and F intersects one of the closed balls comprising $S^3 \setminus\setminus(X \cup U)$ in a disk; the latter condition is equivalent to the condition that $\partial\widetilde{X}$ bounds a disk in ∂S_F (recall Notation 2.14). Generalize this terminology as follows. Suppose $X = \phi_F(\widetilde{X})$ is an arbitrary cap for F. Say that X is **fake** if $\partial\widetilde{X}$ bounds a disk in ∂S_F.

Observation 5.1. *If $X = \phi_F(\widetilde{X})$ is a fake cap for F with $\partial\widetilde{X} \pitchfork \widetilde{L}$, then $|\partial X \cap L|$ is even.*

Observation 5.2. *If X is a fake cap for F, then ∂X is contractible in F.*

Theorem 5.3. *If $F = F_1 \natural F_2$ and F_1, F_2 are π_1-essential, then ess$(F) = \min_{i=1,2}$ ess(F_i).*

Proof. Certainly ess$(F) \leq k$.[15] For the reverse inequality, let W be a plumbing cap which decomposes F as $F_1 \natural F_2$, and choose $X \in \text{Cap}(F)$ to lexicographically minimize $|\partial X \cap L|$ and $|X \cap W|$. If $X \cap W = \varnothing$, then X is a non-plumbing cap for F_1 or F_2, and we are done.

Assume instead that $X \cap W \neq \varnothing$. Since $|\partial W \cap L| = 2$, there is an outermost disk Y of $W \setminus\setminus X$ with $|\partial Y \cap L| \leq 1$. Surger X along Y, and denote the resulting caps by X_1, X_2. They are not fake, by the minimality of $|X \cap W|$. Theorem 2.11 implies that F is π_1-essential, so each $|\partial X_i \cap L| \geq 2$. Further,

$$|\partial X_1 \cap L| + |\partial X_2 \cap L| = |\partial X \cap L| + 2|\partial Y \cap L| \leq |\partial X \cap L| + 2,$$

so each $|\partial X_i \cap L| \leq |\partial X \cap L|$. This, the fact that $|X_i \cap W| < |X \cap W|$, and the lexicographical minimality of $|\partial X \cap L|$ and $|X \cap W|$ imply that ∂X_1 and ∂X_2 are both contractible in F. Therefore, ∂X too is contractible in F, contrary to assumption. □

Before proving the main theorem of this section, we need to establish a technical lemma. The setting is similar to that of Theorem

[15] Perhaps surprisingly, this inequality fails to extend to plumbing more generally, but the opposite inequality does extend.

5.3 and its proof, except with plumbing in place of boundary connect sum.

Namely, suppose that W is a geometric plumbing cap for F, that $U \subset F$ is the disk with $\partial U = \partial W$, and that $B_i, i \in \mathbb{Z}/2\mathbb{Z}$ are the closed 3-balls comprising $S^3 \backslash\backslash (W \cup U)$. Denote each $F \cap B_i = F_i$, so that $F = F_0 * F_1$ along $W \cup U$. Let each $\widehat{B}_i$ denote a copy of B_i in which F has been deleted from its interior, so that each $B_i \cup \widehat{B}_{i+1}$ is a 3-sphere containing a copy of F_i. Identify these copies with F_0, F_1 themselves, and identify the hemispheres of each $\partial \widehat{B}_i$ with U and W.

Given a cap X for F, isotope X to intersect W minimally and transversally, such that W contains no self-intersection points of ∂F and no points of $\partial F \cap L$. Assume that $X \cap W \neq \varnothing$. Orient X. Label the arcs of $X \cap W$ as $\alpha_1, \dots, \alpha_\ell$ and the disks of $X \backslash\backslash W$ as $X_0, \dots, X_\ell$, such that each $\alpha_j = \partial X_j \cap \partial X_{j'}$ for some $j' < j$. For each $j = 0, \dots, \ell$, denote $I_j = \{t : \ \alpha_t \subset \partial X_j\}$. Note that $I_0 = \{1\}$ and $\min I_j = j$ for each $j \geq 1$. Define $\rho : \{0, \dots, \ell\} \to \mathbb{Z}/2\mathbb{Z}$ so that each $X_j \subset B_{\rho(j)}$.

Let $\{\beta_1, \dots, \beta_\ell\}$ be a collection of disjoint properly embedded arcs in U, where each β_j has the same endpoints as α_j. For each $i \in \mathbb{Z}/2\mathbb{Z}$, let $Y_i = \bigsqcup_{j=1}^{\ell} Y_{i,j} \subset \widehat{B}_{i+1}$ be a system of disjoint properly embedded disks with each $\partial Y_{i,j} = \alpha_j \cup \beta_j$. Extend each disk X_j to a disk

$$Z_j = X_j \cup \bigcup_{t \in I_j} Y_{\rho(j),t} \subset B_{\rho(j)} \cup \widehat{B}_{\rho(j)+1}.$$

This disk Z_j is either a cap or a fake cap for $F_{\rho(j)}$. Each Z_j inherits an orientation from $X_j \subset X$ and lifts to a properly embedded, oriented disk $\widetilde{Z_j} \subset S_{F_{\rho(j)}}$.

Lemma 5.4. *With the setup above, if each ∂Z_j is contractible in $F_{\rho(j)}$, then ∂X is contractible in F.*

Proof. Assume there exist continuous maps $f_j : (D^2_j, \partial D^2_j) \to (F_{\rho(j)}, \partial Z_j)$ for each $j = 0, \dots, \ell$. Glue the disks D^2_j by identifying the arcs $f_j^{-1}(\beta_j)$ and $f_{j'}^{-1}(\beta_{j'})$. The resulting quotient space is a disk,

$$\bigsqcup_{i=0}^{\ell} D^2_j / \left(x \in \partial D^2_j \sim x' \in \partial D^2_{j'} \text{ if } f_j(x) = f_{j'}(x')\right).$$

Gluing the maps f_j along the arcs β_j gives a map from this disk and its boundary to F and ∂X, respectively. Thus, ∂X is contractible in F. $\square$

Now we are ready to prove the following generalization of Ozawa's plumbing theorem.

Theorem 5.5. *If $F = F_1 * F_2$ is a plumbing of π_1-essential spanning surfaces, then ess$(F) \geq \min_{i=1,2}$ ess$(F_i) = n$.*

Proof. Let W be a plumbing cap which decomposes F into F_1 and F_2, and let U be its shadow. Choose $X \in \mathrm{Cap}_e(F)$ so as lexicographically to minimize $k = |\partial X \cap L|$, $\ell = |X \cap W|$, and $m = |\partial X \cap \partial U|$, provided that $X \pitchfork W$ and ∂U contains no points where ∂X intersects itself or L. If $\ell = 0$, then WLOG X is a cap for F_1 and ∂X is not contractible in F_1, so $\mathrm{ess}(F_1) \leq |\partial X \cap L| = \mathrm{ess}(F)$, and we are done. Assume instead that $\ell > 0$.

Set up $F_i \subset B_i \cup \widehat{B}_{i+1}$, $i \in \mathbb{Z}/2\mathbb{Z}$, as in §**??**, along with $\alpha_1, \dots, \alpha_\ell$; $X_0, \dots, X_\ell$; $I_1, \dots, I_\ell$; $\beta_1, \dots, \beta_\ell$; Y_i, $i \in \mathbb{Z}/2\mathbb{Z}$; and $Z_0, Z_1, \dots, Z_\ell$.

Decorate ∂X with $k + 2\ell$ markers as follows. First, mark ∂X by drawing a dot on each component of $\partial X \cap L$. Note that none of these points is an endpoint of $X \cap W$. Second, observe that near each endpoint of each arc of $X \cap W$, ∂X runs along U in one direction but not the other. See Figure 22. Mark ∂X near each of the 2ℓ endpoints of $X \cap W$ with an arrow that points in the direction where ∂X runs along U.

There are now $k + 2\ell$ markers, each of which lies on the boundary of exactly one of the disks $X_0, \dots, X_\ell$ of $X \setminus W$. The number of markers on each X_j equals the number of points of $|\partial Z_j \cap L|$.

There are now $k + 2\ell$ markers distributed among the $\ell + 2$ disks of $X \setminus W$. We claim that each X_j must have at least two markers.

To see why, suppose Z_j has fewer. Then Z_j must be a *fake cap* for the π_1-essential surface $F_{\rho(j)}$, hence must satisfy $|\partial Z_j \cap L| = 0$ (and not 1) by Observation 5.1. Assume WLOG that $\rho(j) = 0$. Then $\partial \widetilde{Z}_j$ bounds a disk $\widetilde{U}_j \subset \partial S_{F_0}$. Note that $\partial \widetilde{U}_j$ must intersect $p_{F_0}^{-1}(\partial U)$. Consider an outermost disk $\widetilde{V}$ of $\widetilde{U}_j \setminus p_{F_0}^{-1}(\partial U)$.

Let $V = p_{F_0}(\widetilde{V})$. Then V is disjoint from L and lies either in F or W; its interior is disjoint from X and L; and its boundary consists of an arc $\sigma \subset \partial U$ and an arc τ which lies in either F or W. If $\tau \subset F$, then pushing X near τ through V past σ decreases the lexicographically minimized quantity (k, ℓ, m). Assume instead that $\tau \subset W$. Then $V \in \mathrm{Cap}(X)$ and $|V \cap L| = 0$. Surgering X along V gives two disks X_i with $\partial X_i \subset F$ and $(\partial X_i \cap L, X_i \cap W) < (X \cap L, X \cap W)$. The lexicographical minimality of this quantity implies that both ∂X_i must be contractible in F, but this implies contrary to assumption that ∂X is also contractible in F.

Thus, each of $X_0, \dots, X_\ell$ has at least two markers. Since there are $k + 2\ell$ markers in total, each of $X_0, \dots, X_\ell$ has at most k markers. If $k < n$, then each X_j has fewer than n markers; hence each ∂Z_j is contractible in $F_{\rho(j)}$. Therefore, by Lemma 5.4, ∂X is contractible

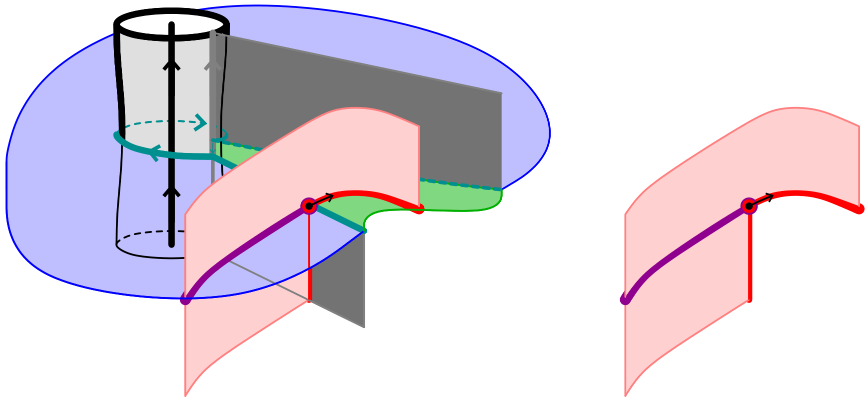


FIGURE 22. Mark ∂X near each endpoint of each arc of $X \cap W$ with an arrow that points in the direction where ∂X runs along U.

in F, contrary to the assumption that $X \in \mathrm{Cap}_e(F)$. Therefore, $\mathrm{ess}(F) = |\partial X \cap L| = k \geq n = \min_{i=1,2} \mathrm{ess}(F_i)$. □

Theorem 5.6. *If $F = F_1 * F_2$ is a plumbing along a disk U with $|\partial U \cap L| = n$, and if $\min_{0,1} \min\{ess(F_i), ess_c(F_i)\} \geq n$, then $ess_c(F) = n$.*

Proof. Let W be the plumbing cap for F with $\partial W = \partial U$. Then $|\partial W \cap L| = n$, so $\mathrm{ess}_c(F) \leq n$. Let $\mathrm{ess}_c(F) = k$. We must show that $k \geq n$. Choose a ∂-contractible cap X for F with $|\partial X \cap L| = k$ which lexicographically minimizes $\ell = |X \cap W|$ and $m = |\partial X \cap \partial U|$, provided that $X \pitchfork W$ and ∂U contains no points where ∂X intersects itself or L. Assume for contradiction that $k < n$. Decorate ∂X with $k + 2\ell$ markers as in the proof of Theorem 5.5. Observe that

$$k + 2\ell - n(\ell + 1) = (k - n) + \ell(2 - n) < 0,$$

so some disk X_j of $X \setminus\setminus W$ must have fewer than n markers. Hence, X_j extends as in that proof to a disk Z_j which is a *fake cap* for F. This gives a contradiction as in the middle of the proof of Theorem 5.5. □

References

[AK13] C. Adams, T. Kindred, *A classification of spanning surfaces for alternating links*, Alg. Geom. Topol. 13 (2013), no. 5, 2967-3007.

[BG16] S. Baader, C. Graf, *Fibred links in S^3*, Expo. Math. 34 (2016), no. 4, 423-435.

[BS10] F. Bonahon, L.C. Siebenmann, *New geometric splittings of classical knots and the classification and symmetries of arborescent knots*, Geom. Topol. Monographs 10 (2010), 365pp.

[Cl78] B.E. Clark, *Crosscaps and knots*, Internat. J. Math. Math. Sci. 1 (1978), no. 1, 113-123.

[CH18] A.F. Costa, C.V.Q. Hongler, *Murasugi decomposition and periodic alternating links*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 112 (2018), no. 3, 793-802.

[Cr59] R. Crowell, *Genus of alternating link types*, Ann. of Math. (2) 69 (1959), 258-275.

[FKP13] D. Futer, E. Kalfagianni, J. Purcell, *Guts of surfaces and the colored Jones polynomial*, Lecture Notes in Math., 2069, Springer, Heidelberg, 2013. x+170pp.

[FKP14] D. Futer, E. Kalfagianni, J. Purcell, *Quasifuchsian state surfaces*, Trans. Amer. Math. Soc. 366 (2014), 4323-4343.

[Ga83] D. Gabai, *The Murasugi sum is a natural geometric operation*, Low-dimensional topology (San Francisco, Calif., 1981), 131-143, Contemp. Math., 20, Amer. Math. Soc., Providence, RI, 1983.

[Ga85] D. Gabai, *The Murasugi sum is a natural geometric operation II*, Combinatorial methods in topology and algebraic geometry (Rochester, N.Y., 1982), 93-100, Contemp. Math., 44, Amer. Math. Soc., Providence, RI, 1985.

[Ga86a] D. Gabai, *Genera of the alternating links*, Duke Math J. Vol 53 (1986), no. 3, 677-681.

[Ga86b] D. Gabai, *Genera of the arborescent links*, Mem. Amer. Math. Soc. 59 (1986), no. 339, i-viii and 1-98.

[Gi14] D. Girão, *On the fibration of augmented link complements*, Geom. Dedicata 168 (2014), 207-220.

[GG06] E. Giroux, N. Goodman, *On the stable equivalence of open books in three-manifolds*, Geom. Topol. 10 (2006), 97-114.

[Go92] H. Goda, *Heegaard splitting for sutured manifolds and Murasugi sum*, Osaka J. Math. 29 (1992), no. 1, 21-40.

[GL78] C. McA. Gordon, R.A. Litherland, *On the signature of a link*, Invent. Math. 47 (1978), no. 1, 53-69.

[Gr17] J. Greene, *Alternating links and definite surfaces*, with an appendix by A. Juhasz, M Lackenby, Duke Math. J. 166 (2017), no. 11, 2133-2151.

[Ha82] J. Harer, *How to construct all fibered knots and links*, Topology 21 (1982), no. 3, 263-280.

[HW04] C.V.Q. Hongler; C. Weber, *On the topological invariance of Murasugi special components of an alternating link*, Math. Proc. Cambridge Philos. Soc. 137 (2004), no. 1, 95-108.

[HW05] C.V.Q. Hongler; C. Weber, *A Murasugi decomposition for achiral alternating links*, Pacific J. Math. 222 (2005), no. 2, 317-336.

[IT20] N. Ito, Y. Takimura, *A lower bound of crosscap numbers of alternating knots*, J. Knot Theory Ramifications 29 (2020), no. 1, 1950092, 15 pp.

[Ju08] A. Juhász, *Floer homology and surface decompositions*, Geom. Topol. 12 (2008), no. 1, 299-350.

[Ki18] T. Kindred, *Plumbing essential states in Khovanov homology*, New York J. Math. 24 (2018), 588-610.

[Ki20] T. Kindred, *Crosscap numbers of alternating knots via unknotting splices*, Internat. J. Math. 31 (2020), no. 7, 2050057, 30 pp.

[Ki21] T. Kindred, *A geometric proof of the flyping theorem*, preprint.

[Ki22] T. Kindred, *A simple proof of the Crowell-Murasugi theorem*, preprint.

[Ko89] T. Kobayashi, *Uniqueness of minimal genus Seifert surfaces for links*, Topology Appl. 33 (1989), no. 3, 265-279.

[KK88] K. Kobayashi, K. Kodama, *On the* $\deg_z P_L(v,z)$ *for plumbing diagrams and oriented arborescent links*, Kobe J. Math. 5 (1988), no. 2, 221-231.

[Li03] T. Li, *Boundary train tracks of laminar branched surfaces*, Topology and geometry of manifolds (Athens, GA, 2001), 269-285, Proc. Sympos. Pure Math., 71, Amer. Math. Soc., Providence, RI, 2003.

[M84] W. Menasco, *Closed incompressible surfaces in alternating knot and link complements*, Topology 23 (1984), no. 1, 37-44.

[MT91] W. Menasco, M. Thistlethwaite, *The Tait flyping conjecture*, Bull. Amer. Math. Soc. (N.S.) 25 (1991), no. 2, 403-412.

[MT93] W. Menasco, M. Thistlethwaite, *The classification of alternating links*, Ann. of Math. (2) 138 (1993), no. 1, 113-171.

[Mu58] K. Murasugi, *On the genus of the alternating knot. I, II*, J. Math. Soc. Japan 10 (1958), 94-105, 235-248.

[MP89] K. Murasugi, J. Przytycki, *The skein polynomial of a planar star product of two links*, Math. Proc. Cambridge Philos. Soc. 106 (1989), no. 2, 273-276.

[Ni06] Y. Ni, *Sutured Heegaard diagrams for knots*, Algebr. Geom. Topol. 6 (2006), 513-537.

[Oz11] M. Ozawa, *Essential state surfaces for knots and links*, J. Aust. Math. Soc. 91 (2011), no. 3, 391-404.

[OP16] B. Ozbagci, P. Popescu-Pampu, *Generalized plumbings and Murasugi sums*, Arnold Math. J. 2 (2016), no. 1, 69-119.

[Ru89] L. Rudolph, *Quasipositivity and new knot invariants*, Rev. Mat. Univ. Complut. Madrid 2 (1989), no. 1, 85-109.

[Ru02] L. Rudolph, *A non-ribbon plumbing of fibered ribbon knots*, Proc. Amer. Math. Soc. 130 (2002), no. 12, 3741-3743.

[Sa94] M. Sakuma, *Minimal genus Seifert surfaces for special arborescent links*, Osaka J. Math. 31 (1994), no. 4, 861-905.

[SY10] T. Saito, R. Yamamoto, *Complexity of open book decompositions via arc complex*, J. Knot Theory Ramifications 19 (2010), no. 1, 55-69.

[St78] J.R. Stallings, *Constructions of fibred knots and links*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, pp. 55-60, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978.

[Tait] P.G. Tait, *On Knots I, II, and III*, Scientific papers 1 (1898), 273-347.

[To00] I. Torisu, *Convex contact structures and fibered links in 3-manifolds*, Internat. Math. Res. Notices 2000, no. 9, 441-454.

[Ya03] R. Yamamoto, *Stallings twists which can be realized by plumbing and deplumbing Hopf bands*, J. Knot Theory Ramifications 12 (2003), no. 6, 867-876.

Department of Mathematics & Statistics, Wake Forest University, Winston-Salem North Carolina, 27109

thomas.kindred@wfu.edu

www.thomaskindred.com