A GEOMETRIC PROOF OF THE FLYPING THEOREM

THOMAS KINDRED

Abstract. In 1898, Tait asserted several properties of alternating knot diagrams. These assertions became known as Tait's conjectures and remained open until the discovery of the Jones polynomial in 1985. The new polynomial invariants soon led to proofs of all of Tait's conjectures, culminating in 1993 with Menasco–Thistlethwaite's proof of Tait's flyping conjecture.

In 2017, Greene (and independently Howie) answered a longstanding question of Fox by characterizing alternating links geometrically. Greene then used his characterization to give the first *geometric* proof of part of Tait's conjectures. We use Greene's characterization, Menasco's crossing ball structures, and a hierarchy of isotopy and *re-plumbing* moves to give the first entirely geometric proof of Menasco–Thistlethwaite's flyping theorem.

1. Introduction

P.G. Tait asserted in 1898 that all reduced alternating diagrams of a given prime link in S^3 minimize crossings, have equal writhe, and are related by *flype* moves (see Figure 1) [T1898]. Tait's conjectures remained unproven until the 1985 discovery of the Jones polynomial, which quickly led to proofs of Tait's conjectures about crossing number and writhe. Tait's flyping conjecture remained open until 1993, when Menasco–Thistlethwaite gave its first proof [MT91, MT93], which they described as follows:

> The proof of the Main Theorem stems from an analysis of the [checkerboard surfaces] of a link diagram, in which we use geometric techniques [introduced in [Me84]]... and properties of the Jones and Kauffman polynomials.... Perhaps the most striking use of polynomials is... where we "detect a flype" by using the fact that if just one crossing is switched in a reduced alternating diagram of n crossings, and if the resulting link also admits an alternating diagram, then the crossing number of that link is at most $n-2$. Thus, although the proof of the Main Theorem has a strong

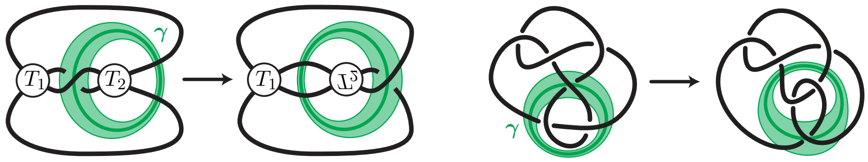


FIGURE 1. A *flype* along an annulus $A = \nu\gamma \subset S^2$.

> geometric flavor, it is not entirely geometric; the question remains open as to *whether there exist purely geometric proofs of this and other results* that have been obtained with the help of new polynomial invariants.

We answer part of Menasco–Thistlethwaite's question by giving the first entirely geometric proof of Tait's flyping conjecture:

Theorem 5.5 (Tait's flyping conjecture [MT91, MT93])**.** *All reduced alternating diagrams of a given prime link $L \subset S^3$ are related by flype moves and planar isotopy.*

(The version of Theorem 5.5 that we prove is a slightly stronger statement.) In the process, we obtain new geometric proofs of other parts of Tait's conjectures, which were first proven independently by Kauffman, Murasugi, and Thistlethwaite using the Jones polynomial, and were first proved geometrically by Greene:

Theorem 4.7 (Part of Tait's first conjecture [Gr17, Ka87, Mu87, Th87, Tu87])**.** *All reduced alternating diagrams of a given link $L \subset S^3$ have the same number of crossings.*

Theorem 5.6 (Tait's second conjecture [Gr17, M87ii, T88b])**.** *All reduced alternating diagrams of a given link $L \subset S^3$ have equal writhe.*

Like Menasco–Thistlethwaite's proof, ours stems from an analysis of checkerboard surfaces and uses the geometric techniques introduced in [Me84]. The most striking difference between our proof and the original proof in [MT93] is that we "detect flypes" via *re-plumbing* moves. Indeed, any flype move isotopes one checkerboard surface and *re-plumbs* the other (see Figure 2); it follows that the checkerboard surfaces from any flype-related diagrams are related pairwise by isotopy and such re-plumbing moves. The main idea behind our proof of the flyping theorem is to reverse this reasoning by establishing this plumb-equivalence geometrically. Thus, our proof of the flyping theorem is entirely geometric, not just in the formal sense that it does not use the Jones polynomial, but also in the more genuine sense that it conveys a geometric way of *understanding why* the flyping theorem is true.

To translate the question of flype-equivalence of link diagrams to a question about plumb-equivalence of spanning surfaces, we extend

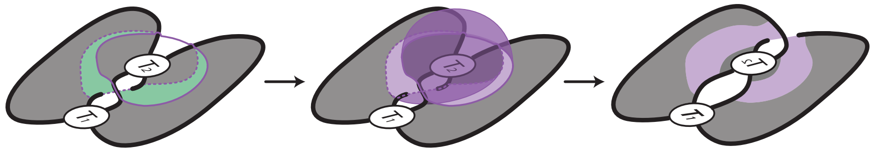


FIGURE 2. A flype isotopes one checkerboard surface (here, W) and re-plumbs the other.

recent insights of Greene and Howie [Gr17, Ho17][1] by establishing a new correspondence between prime alternating link diagrams on S^2 and pairs of essential definite spanning surfaces (see Conventions 2.3 and 2.14 and Definition 2.11):

Theorem 2.30. *Suppose B, W and B', W' are the respective checkerboard surfaces of prime alternating diagrams D and D' of a link $L \subset S^3$. Then D and D' are equivalent if and only if B and B' are isotopic in $S^3 \setminus \mathring{\nu} L$, as are W and W'.*

Corollary 2.31. *There is a bijective correspondence between equivalence classes of prime alternating link diagrams $D_{B,W}$ on S^2 and pairs B, W of isotopy classes of essential definite surfaces of opposite signs spanning the same prime link in S^3.*

Theorem 2.30 does not extend to non-prime or non-alternating diagrams. For a simple example, consider any two distinct positive 5-crossing diagrams of the unknot: both white checkerboard surfaces will be disks, and both black surfaces will be isotopic to $\natural_{i=1}^{5}$ . See Example 2.32 for a prime, non-alternating example.

To utilize this correspondence, we use Menasco's crossing ball structures in §§3-4 to describe a hierarchy of isotopy moves (Moves 1-9) and re-plumbing moves (Move 10) and prove:

Theorem 4.5. *If B, W are the checkerboard surfaces from a prime alternating diagram $D \subset S^2$ of a link $L \subset S^3$, then any essential positive-definite surface F spanning L is plumb-related to B (via Moves 1-10); likewise for essential negative-definite surfaces and W.*

Yet, it is not obvious that the re-plumbing Move 10 is always sort of re-plumbing move associated with flypes. In §5, however, we will prove that this is always the case when B' is in "9-good position," meaning that none of Moves $1 - 9$ are possible. (This is Theorem 5.4.) Therefore, with the setup from Theorem 2.30 and notation from Corollary 2.31, $D = D_{B,W}$ and $D_{B',W}$ are flype-related, as are $D_{B',W}$ and $D_{B',W'} = D'$. For expository reasons, we include some proofs in §§2-4 but postpone others until §§6-8.

[1] Those insights answered another longstanding question, this one from Ralph Fox: "What [geometrically] is an alternating knot [or link]?"

Thank you to Colin Adams for posing a question about flypes and checkerboard surfaces during SMALL 2005 which eventually led to the insight behind Figure 2. Thank you to Hugh Howards, Josh Howie, and Alex Zupan for helpful discussions. Thank you to Josh Greene for helpful discussions and especially for encouraging me to think about this problem.

2. Alternating diagrams and definite surfaces

2.1. **Basic definitions.** All links are in S^3 and all link diagrams are on S^2. We call a link L *prime* if L is not a trivial link of one or two components and any connect sum decomposition $L = L_1 \# L_2$ has $L_1 = \bigcirc$ or $L_2 = \bigcirc$. We call a link *diagram D prime* if D has more than one crossing and any connect sum decomposition $D = D_1 \# D_2$ has $D_1 = \bigcirc$ or $D_2 = \bigcirc$. Our extra assumptions that $L \neq \bigcirc \bigcirc$ and that D has more than one crossing are unconventional but convenient because they imply:

Fact 2.1. *Every prime link is nontrivial and nonsplit (i.e. the link complement is irreducible), and every prime link diagram is nontrivial, connected and reduced.*[2]

Let νL be a closed regular neighborhood of a link L with projection $\pi_L : \nu L \to L$.[3] One can define *spanning surfaces* F for L in two ways; in both definitions, F is compact and unoriented (orientable or not), and each component of F has nonempty boundary. First, F is an embedded surface in S^3 with $\partial F = L$. Alternatively, F is properly embedded in the link exterior $S^3 \backslash \mathring{\nu} L$ such that ∂F intersects each meridian on $\partial \nu L$ transversally in one point.[4] We use the latter definition throughout, except where noted otherwise.

The rank $\beta_1(F)$ of the first homology group of a spanning surface F counts the number of "holes" in F. When F is connected, $\beta_1(F) = 1 - \chi(F)$ counts the number of cuts along disjoint, properly embedded arcs required to reduce F to a disk. Thus:

Observation 2.2. *If α is a properly embedded arc in a spanning surface F and $F' = F \setminus \mathring{\nu}\alpha$, then $\beta_1(F') - |F'| = \beta_1(F) - |F| - 1$.*[5] *In particular, if F' connected, then $\beta_1(F') = \beta_1(F) - 1$.*

Convention 2.3. *Isotopies* of properly embedded surfaces and arcs are always taken to be *proper isotopies*.[6] Two properly embedded

[2] A diagram D is *reduced* if no crossing is nugatory, i.e. incident to fewer than four distinct regions of $S^2 \setminus D$.

[3] νX always denotes a closed regular neighborhood of X, usually taken in S^3.

[4] A *meridian* on $\partial \nu L$ is a circle $\pi_L^{-1}(x) \cap \partial \nu L$ for a point $x \in L$.

[5] $|X|$ denotes the number of connected components of X.

[6] For example, an isotopy of a spanning surface $F \subset S^3 \setminus \mathring{\nu} L$ is a homotopy $h_t : F \to S^3 \setminus \mathring{\nu} L$, $t \in I$,with $h_0(F) = F$ where each $h_t(F)$ is a spanning surface.

surfaces or arcs are *parallel* if they have the same boundary and are related by an isotopy which fixes this boundary.

A spanning surface F is (geometrically) *incompressible* if every simple closed curve in F that bounds a disk in $S^3\backslash\backslash(F \cup \nu L)$ also bounds a disk in F;[7] F is *∂-incompressible* if every properly embedded arc in F that is parallel into $\partial\nu L$ in $S^3\backslash\backslash(F \cup \nu L)$ is also ∂-parallel in F. If F is incompressible and ∂-incompressible, then F is *essential*. This geometric notion of essentiality is weaker than the algebraic notion of *π_1-essentiality*, which holds F to be essential if inclusion $F \hookrightarrow S^3 \setminus \mathring{\nu}L$ induces an injective map on fundamental groups and F is not a möbius band spanning the unknot. A standard innermost circle argument shows:

Fact 2.4. *If an incompressible surface F spans a split link L, then the boundary of each connected component of F lies in a single split component of L.*

Proposition 2.5. *Suppose $F_\pm$ are definite surfaces of opposite signs spanning a link L and $F_+ \cap F_-$ consists only of arcs, none of which are ∂-parallel in both F_+ and F_-. If F_- is essential, then no arc of $F_+ \cap F_-$ is ∂-parallel in F_+.*

Proof. If any arcs of $F_+ \cap F_-$ are ∂-parallel in F_+, choose an outermost one, β; it is parallel into $\partial\nu L$ through a disk $X \subset F_+\backslash\backslash F_- \subset S^3\backslash\backslash(F_- \cup \nu L)$. Since F_- is essential, β is ∂-parallel in F_-. Yet, we assumed that no arc of $F_+ \cap F_-$ is ∂-parallel in both F_+ and F_-. □

Proposition 2.6. *If an essential spanning surface F contains an arc β which is parallel in $S^3\backslash\backslash(F \cup \nu L)$ to an arc $\alpha \subset \partial\nu L\backslash\backslash\partial F$, then α is parallel in $\partial\nu L$ into ∂F.*

Proof. It suffices to prove this when L is nonsplit and nontrivial. Because F is essential, β is parallel in F to an arc $\alpha' \subset \partial F$. The arcs α and α' are both parallel in $S^3 \setminus \mathring{\nu}L$ to β, hence co-bound a disk in $S^3 \setminus \mathring{\nu}L$, and therefore are parallel in $\partial\nu L$. □

---

[7]For compact $X, Y \subset S^3$, $X\backslash\backslash Y$ denotes the metric closure of $X \setminus Y$. We describe a general construction under the additional assumptions that X and $X \setminus Y$ are manifolds of the same dimension. If, for each $x \in X \cap Y$, a generic local neighborhood νx has the property that $Z \cap \nu x$ is connected or empty for each component Z of $X \setminus Y$, then $X\backslash\backslash Y$ is the disjoint union of the closures in S^3 of the components of $X \setminus Y$ (hence, each component of $X\backslash\backslash Y$ embeds naturally in S^3, although $X\backslash\backslash Y$ as a whole need not). More generally, let $\{(U_\alpha, \phi_\alpha)\}$ be a maximal atlas for X. About each $x \in X$, choose a chart (U_x, ϕ_x) that is tiny enough that, for each component Z of $\overline{U_x} \setminus Y$ and a generic local neighborhood νx of x in U_x, $Z \cap \nu x$ is connected or empty; construct $\overline{U_x}\backslash\backslash Y$ as above, denote the components of $U_x \cap (\overline{U_x}\backslash\backslash Y)$ by U_α, $\alpha \in \mathcal{I}_x$, and denote each natural embedding $f_\alpha : U_\alpha \to U_x$. Then $\bigcup_{x\in X}\{(U_\alpha, \phi_x \circ f_\alpha)\}_{\alpha\in\mathcal{I}_x}$ is an atlas for $X\backslash\backslash Y$. Gluing all the maps f_α yields a natural map $f : X\backslash\backslash Y \to X \subset S^3$.

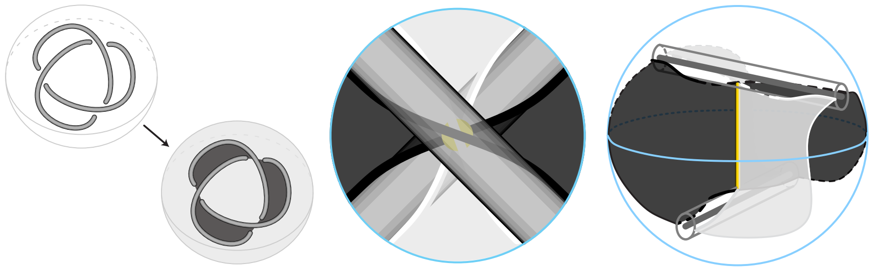


FIGURE 3. Constructing checkerboard surfaces; close-ups near a vertical arc (yellow) at a crossing

Given any diagram D of L, one may a color the complementary regions of D in the projection sphere S^2 black and white in checkerboard fashion.[8] See Figure 3. One may then construct spanning surfaces B and W for L such that B projects to the black regions, W projects to the white, and B and W intersect in *vertical arcs* which project to the the crossings of D. Call the surfaces B and W the *checkerboard surfaces* from D.

Fact 2.7. *Given a connected alternating diagram $D \subset S^2$, the following conditions are equivalent:*

(I) D is reduced.
(II) Both checkerboard surfaces B and W from D are essential.
(III) No vertical arc of $B \cap W$ is separating in B nor in W.

Proof. The implications (I) $\iff$ (III) and (II) $\implies$ (I) are straightforward. For (I) $\implies$ (II), see e.g. Theorem 9.8 of [Au56], Proposition 2.3 of [MT93], Theorems 2-3 of [Oz06], Theorem 3.15 of [Ho15], or Theorem 1.1 of [Ki23b]. □

Remark 2.8. In particular, by Fact 2.7, no vertical arc from a prime alternating diagram is ∂-parallel in either checkerboard surface.

2.2. Flype-related diagrams.

Definition 2.9. Suppose $D \subset S^2$ is a link diagram and $\gamma \subset S^2$ is a circle that intersects D transversally in three points, exactly one of them a crossing point c; we call the circle γ a *flyping circle* for D and the arc of $\gamma \backslash\backslash D$ with neither endpoint at c a *flyping arc* for D. Up to mirror symmetry, D and γ appear as shown far left in Figure 1; in particular, D intersects the two disks of $S^2 \setminus \mathring{\nu}\gamma$ in tangles T_1 and T_2. The move $D \to D'$ shown left in Figure 1 is called a *flype*: this move fixes the tangle T_1, switches which pair of strands cross within $\nu\gamma$, and changes T_2 by reflecting the underlying projection and reversing

[8] That is, so that regions of the same color meet only at crossing points.

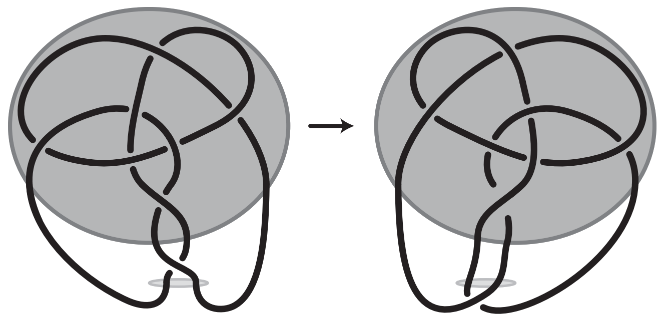


FIGURE 4. An *entire flype* of a diagram of the knot 8_{17}

all crossing information. Two link diagrams on S^2 are *flype-related* if they are related by a sequence of flype moves and planar isotopy.[9]

Observation 2.10. *If $D \to D'$ is a flype, then D and D' represent the same link L and have the same number of crossings. Also, if D and D' are oriented then they have the same writhe.[10] Further, if D is alternating (resp. prime), then so is D'.*

Definition 2.11. If the tangle T_1 in Figure 1 contains no crossings, then (up to planar isotopy) the associated flype has the effect of changing D to its mirror image and reversing all crossings. We call such a flype an *entire flype*. One may think of this move as leaving D unchanged and viewing it from the opposite side of S^2 in S^3. Figure 4 shows an example. We regard two link diagrams D and D' as *equivalent*, denoted $D \equiv D'$, if they are related by planar isotopy and possibly an entire flype.[11]

2.3. **Definite surfaces.** Given a(n unoriented) spanning surface F for an *oriented* link L, the *oriented euler number* $e(F, L)$ is the algebraic self-intersection number of the closed surface in the 4-ball obtained by pushing $\text{int}(F)$ into the 4-ball and capping off ∂F with a Seifert surface in S^3 (using the orientation on L). The *unoriented euler number* of F, denoted $e(F)$, is the average value of $e(F, L)$ over all orientations of L. Alternatively, $-e(F)$ can be computed by

---

[9] Every arc in $S^2 \backslash\backslash D$ with endpoints on adjacent edges of D is a flyping arc.

[10] The *writhe* w_D is the number of positive crossings in D minus the number of negative crossings. Equivalently, w_D is the blackboard framing of D: if one embeds L in νS^2 according to D (see §3.1, e.g.) and takes a co-oriented pushoff $\widehat{L}$ in either direction normal to S^2, then $w_D = \text{lk}(L, \widehat{L})$.

[11] Any entire flype $f : D \to D'$ extends to an orientation-reversing homeomorphism $S^2 \to S^2$. Conversely, given any orientation-reversing homeomorphism $\iota : S^2 \to S^2$, the diagram D' obtained from $\iota(D)$ by reversing all crossing information is related to D by planar isotopy and an entire flype.

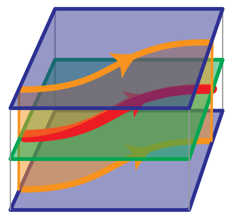


FIGURE 5. An curve γ on F, with $\widetilde{\gamma} = p^{-1}(\gamma)$ on $\widetilde{F}$.

summing the component-wise boundary slopes of F.[12] We denote $-e(F) = s(F)$ and call this the *slope* of F.

Given surface F spanning a link L, take νF in the link exterior $S^3 \backslash\backslash \nu L$ with projection $p : \nu F \to F$ such that $p^{-1}(\partial F) = \nu F \cap \partial \nu L$. Denote the frontier $\widetilde{F} = \partial \nu F \backslash\backslash \partial \nu L$ and[13] transfer map $\tau : H_1(F) \to H_1(\widetilde{F})$.[14] The *Gordon-Litherland pairing* [GL78]

$$\langle \cdot, \cdot \rangle : H_1(F) \times H_1(F) \to \mathbb{Z}$$

is the symmetric, bilinear mapping given by the linking number

$$\langle a, b \rangle = \mathrm{lk}(a, \tau(b)).$$

Any projective homology class $g = [\gamma] \in H_1(F)/\pm$ has a well-defined *self-pairing* $|\!| g |\!| = \langle g, g \rangle$; the *framing* of γ in F is given by $\frac{1}{2} |\!| g |\!|$.

Given an ordered basis $\mathcal{B} = (a_1, \dots, a_m)$ for $H_1(F)$, the *Goeritz matrix* $G = (x_{ij}) \in \mathbb{Z}^{m \times m}$ given by $x_{ij} = \langle a_i, a_j \rangle$ represents $\langle \cdot, \cdot \rangle$ with respect to $\mathcal{B}$.[15] The signature of G is called the *signature of* F and is denoted $\sigma(F)$. Gordon-Litherland showed that the quantity $\sigma(F) - \frac{1}{2}s(F)$ is independent of F, and in fact equals the Murasugi invariant $\xi(L)$, which is the average signature of L across all orientations.

They also showed that $\sigma(F)$ is the signature of the 4-manifold obtained by pushing the interior of F into the interior of the 4-ball B^4, while fixing ∂F in $\partial B^4 = S^3$, and taking the double-branched cover of B^4 along this surface. In particular, when L is a knot, $\xi(L)$ is the signature of L and of the 4-manifold obtained as a double-branched cover of B^4 along any perturbed Seifert surface.

[12] If $L_1, \dots, L_m$ are the components of ∂F and each $\widehat{L_i}$ is a co-oriented pushoff of L_i in F, then the boundary slope of F along each L_i equals the *framing* of L_i in F, given by the linking number $\mathrm{lk}(L_i, \widehat{L_i})$, and $-e(F) = \sum_{i=1}^m \mathrm{lk}(L_i, \widehat{L_i})$. Further, denoting $\widehat{L} = \bigcup_{i=1}^m \widehat{L_i}$ and *total linking number* $\mathrm{lk}(L) = \sum_{i<j} \mathrm{lk}\,(L_i, L_j)$, we have $-e(F, L) = \mathrm{lk}(L, \widehat{L}) = -e(F) + 2\mathrm{lk}(L)$.

[13] Thus, the restriction $p : \widetilde{F} \to F$ is a 2:1 covering map, $\widetilde{F}$ is orientable, and $\widetilde{F}$ is connected if and only if F is connected and nonorientable.

[14] Given any $g \in H_1(F)$, choose an oriented multicurve $\gamma \subset \mathrm{int}(F)$ representing g, denote $\widetilde{\gamma} = \partial(p^{-1}(\gamma))$, and orient $\widetilde{\gamma}$ following γ; then, $\tau(g) = [\widetilde{\gamma}]$.

[15] That is, any $y = \sum_{i=1}^m y_i a_i$ and $z = \sum_{i=1}^m z_i a_i$ satisfy

$$\langle y, z \rangle = \begin{bmatrix} y_1 & \cdots & y_m \end{bmatrix} G \begin{bmatrix} z_1 & \cdots & z_m \end{bmatrix}^T.$$

A spanning surface F is *positive-definite* if $\langle g, g\rangle > 0$ for all nonzero $g \in H_1(F)$ [Gr17]. Equivalently, F is positive-definite iff $\sigma(F) = \beta_1(F)$. *Negative-definite* surfaces are defined analogously.

Proposition 2.12. *If F_+ and F_- are positive- and negative-definite spanning surfaces for the same link L, then*

$$s(F_+) - s(F_-) = 2(\beta_1(F_+) + \beta_1(F_-)).$$

Proof. Definiteness implies that $\beta_1(F_\pm) = \pm\sigma(F_\pm)$, and [GL78] gives $s(F_\pm) = 2(\sigma(F_\pm) - \xi(L))$. Thus:

$$\begin{aligned} s(F_+) - s(F_-) &= 2(\sigma(F_+) - \xi(L)) - 2(\sigma(F_-) - \xi(L)) \\ &= 2(\beta_1(F_+) + \beta_1(F_-)). \end{aligned}$$

Note that this holds even if L is non-prime, since slopes and signatures are additive under connect sum and split union. □

Greene used definiteness to characterize nonsplit alternating links:

Theorem 1.1 of [Gr17]**.** *If B and W are positive- and negative-definite spanning surfaces for a link L in a homology $\mathbb{Z}/2\mathbb{Z}$ sphere with irreducible complement, then L is an alternating link in S^3, and it has an alternating diagram D whose checkerboard surfaces are isotopic to B and W. Moreover, D is reduced if and only if neither B nor W has a projective homology class with self-pairing ± 1.*

The converse of the first sentence of the theorem is also true:

Fact 2.13 (Proposition 4.1 of [Gr17])**.** *A connected link diagram is alternating if and only if its checkerboard surfaces are definite and of opposite signs.*[16]

Convention 2.14. If D is a connected alternating link diagram, then its checkerboard surfaces B, W are labeled such that B is positive-definite and W is negative-definite. Likewise for D', B', and W'. We may abbreviate this setup by denoting $D = D_{B,W}$ and $D' = D_{B',W'}$.

Fact 2.4 and definite surfaces' incompressibility (Corollary 3.2 of [Gr17]) extend Theorem 1.1 of [Gr17] to *split* links in S^3 as follows:

Fact 2.15. *If B and W are positive- and negative-definite spanning surfaces for a link L, then L has an alternating diagram $D \subset S^2$ such that, for each connected component D_i of D, denoting the corresponding split component of L by L_i,*[17] *each checkerboard surface of D_i (ignoring the rest of D) is isotopic in $S^3 \setminus \mathring{\nu} L_i$ to a connected component of B or W.*

In particular, B and W have the same number of connected components, and this equals the number of split components of L.

[16] Murasugi proved the forward direction for Tait's second conjecture [M87ii].

[17] This correspondence follows from Theorem 1 (a) of [Me84].

Greene used Theorem 1.1 of [Gr17] and lattice flows to give a geometric proof of part of Tait's conjectures:

Theorem 1.2 of [Gr17]**.** *All reduced alternating diagrams of a given link have the same crossing number and writhe.*

We will give alternate proofs of both parts of this theorem. The part about crossing number will follow from Theorem 4.5 and will serve as an intermediate step in our proof of the flyping theorem. Later, we will deduce the part about writhe as a corollary of the flyping theorem, since flypes preserve writhe.

Remark 2.16. Theorem 1.2 of [Gr17] does not imply, a priori, that a reduced alternating diagram realizes the underlying link's crossing number, since it does not rule out the possibility that a non-alternating diagram could have fewer crossings. All existing proofs of this fact [Ka87, Mu87, Th87, Tu87] use the Jones polynomial.

Problem 2.17. Give an entirely geometric proof that any reduced alternating diagram realizes the underlying link's crossing number.

Thistlethwaite proved more generally that any *adequate* link diagram minimizes crossings. See Corollary 3.4 of [T88a] (or Corollary 5.14 of [Li97] for Lickorish's simpler proof). Thistlethwaite then deduced that any reduced alternating *tangle diagram* minimizes crossings. See Definition 2.2 and Theorem 3.1 of [Th91].

Problem 2.18. Prove Corollary 3.4 of [T88a] geometrically.

Problem 2.19. Give a geometric proof of Theorem 3.1 of [Th91].

2.4. **Intersections between definite surfaces.** Let F and F' be spanning surfaces for a link L with $F \pitchfork F'$. Orient L arbitrarily, and orient ∂F and $\partial F'$ so that each is homologous in νL to L.

2.4.1. *Standard and non-standard arcs.* Given an arc α of $F \cap F'$, take $\nu\partial\alpha$ in $\partial\nu L$, so that ∂F and $\partial F'$ each intersect each disk of $\nu\partial\alpha$ in an arc, giving $i(\partial F, \partial F')_{\nu\partial\alpha} \in \{0, \pm 2\}$. Following Howie [Ho17], we call α *standard* if $i(\partial F, \partial F')_{\nu\partial\alpha} = \pm 2$; we call α *non-standard* if $i(\partial F, \partial F')_{\nu\partial\alpha} = 0$. One can compute the slope difference $s(F) - s(F')$ by counting the arcs of $F \cap F'$ with signs:

$$s(F) - s(F') = i(\partial F, \partial F')_{\partial\nu L} = \sum_{\text{arcs } \alpha \text{ of } F \cap F'} i(\partial F, \partial F')_{\nu\partial\alpha} \tag{2.1}$$

Procedure 2.20. Let S, T be connected spanning surfaces for a link L such that $S \cap T$ consists entirely of standard arcs and $|S \cap T| = \beta_1(S) + \beta_1(T)$. Extend S, T through νL so that $\partial S = L = \partial T$ and collapse $S \cup T$ along each arc of $\text{int}(S) \cap \text{int}(T)$. This gives a 2-sphere[18]

[18]This uses connectedness and the assumption that $|S \cap T| = \beta_1(S) + \beta_1(T)$.

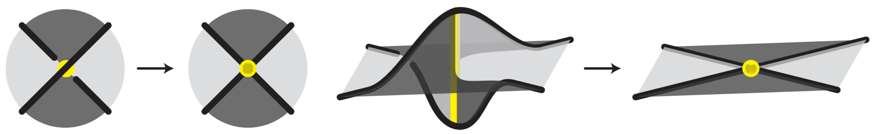


FIGURE 6. Collapsing $S \cup T$ along a standard arc

Q on which L collapses to a connected 4-valent graph; recovering crossing information gives a connected link diagram $D_{S,T} \subset Q$ whose checkerboard surfaces are S and T. See Figure 6.

Remark 2.21. In Procedure 2.20, the initial configuration of S and T, up to isotopy of $S \cup T$ in $S^3 \setminus \mathring{\nu} L$, uniquely determines the diagram D (up to planar isotopy and perhaps an entire flype).

Proposition 2.22. *Suppose $F_\pm$ are positive- and negative-definite surfaces spanning a nonsplit link L such that $F_+ \cap F_-$ consists only of arcs α with $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} = +2$. Then:*

(A) $|F_+ \cap F_-| = \beta_1(F_+) + \beta_1(F_-)$.
(B) $F_\pm$ give an alternating diagram D_{F_+,F_-} via Procedure 2.20.
(C) If F_+ and F_- are essential, then D is reduced.

Proof. Fact 2.15 implies that F_+ and F_- are connected, so the hypotheses regarding $F_+ \cap F_-$ and Proposition 2.12 imply

$$|F_+ \cap F_-| = \frac{1}{2}|\partial F_+ \cap \partial F_-| = \frac{1}{2}\left(s(F_+) - s(F_-)\right) = \beta_1(F_+) + \beta_1(F_-).$$

Hence, the pair $F_\pm$ determines a connected diagram D of L via Procedure 2.20. The checkerboard surfaces of D are $F_\pm$, so D is alternating by Fact 2.13. Fact 2.7 implies (C). □

The proof of Lemma 3.4 of [Gr17] shows:

Fact 2.23. *If $F_+ \pitchfork F_-$ are definite surfaces of opposite signs spanning a link L, then any circle in $F_+ \cap F_-$ bounds disks in both $F_\pm$.*

Procedure 2.24. Suppose $F_\pm$ are definite surfaces of opposite signs spanning a link L with $F_+ \pitchfork F_-$. While fixing F_-, isotope F_+ according to the following hierarchy of moves:[19]

(1) If $F_+ \cap F_-$ contains circles, then (using Fact 2.23) choose an innermost one in F_-, and let $X_\pm$ denote the disks it bounds in $F_\pm$. Using the irreducibility of $S^3 \setminus L$, isotope X_+ past X_- as shown in Figure 7. Meanwhile, fix F_+ away from X_+.
(2) If any arc α of $F_+ \cap F_-$ is parallel in $F_- \backslash\backslash F_+$ into ∂F_- *and* in $F_+ \backslash\backslash F_-$ into ∂F_+, then remove α as shown in Figure 8, top.

[19]That is, perform (1) whenever possible, perform (2) whenever possible unless (1) is possible, and perform (3) whenever possible unless (1) or (2) is possible.

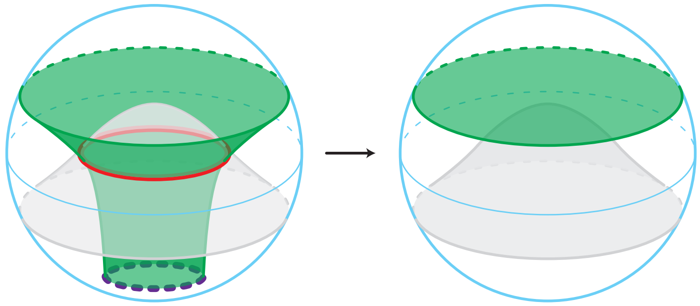


FIGURE 7. Removing a circle γ of intersection between positive- and negative-definite surfaces F_+ and F_-. The dashed purple circle bounds a disk in F_+.

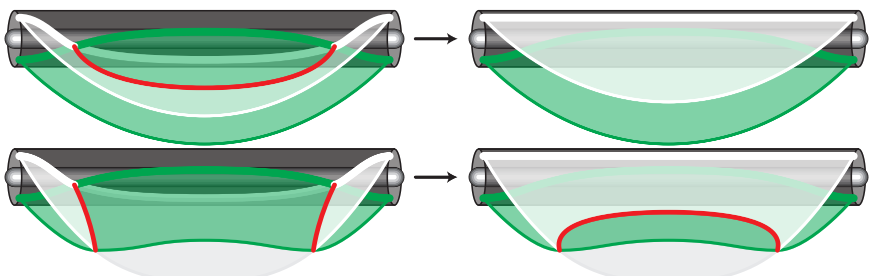


FIGURE 8. Removing adjacent points of $\partial F_+ \cap \partial F_-$ of opposite sign

(3) If arcs $\alpha_+ \subset \partial F_+ \backslash\backslash \partial F_-$ and $\alpha_- \subset \partial F_- \backslash\backslash \partial F_+$ are parallel in $\partial \nu L$, then push F_+ near α_+ past α_- as in Figure 8, bottom.

The reader may be puzzled as to why we include (2) in Procedure 2.24, when the same move can be achieved by (3) followed by (1). The reason, as we will see in Lemma 2.27, is that, when F_+ and F_- are essential, parts (1) and (2) ensure that $F_+ \cap F_-$ consists only of standard arcs, so (3) is ultimately superfluous; nevertheless, we find (3) useful in the leadup to the proof of Lemma 2.27 in §6.2. This will allow us to strengthen Remark 2.21 (see Theorem 2.30) by analyzing how an isotopy of F_+ can affect the standard arcs of $F_+ \cap F_-$.

2.4.2. *Isotopy of arcs in surfaces.* Given checkerboard surfaces B, W from a prime alternating diagram of a link L and an arbitrary essential positive-definite surface F spanning L, we will later analyze how isotoping F can affect $F \cap B$ and $F \cap W$. The next two lemmas anticipate this analysis. See §6 for their proofs and those of all other lemmas that appear in §2 without their proofs.

For both lemmas, let X be an abstract connected surface (not necessarily compact) with $\chi(X) < 0$, and let $u, v \subset X$ be systems of

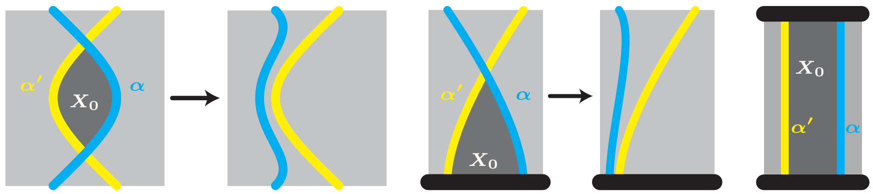


FIGURE 9. Isotopic arcs $\alpha, \alpha' \subset X$ cut off a "bigon," "triangle," or "rectangle" $X_0 \subset X \setminus\setminus(\alpha \cup \alpha')$.

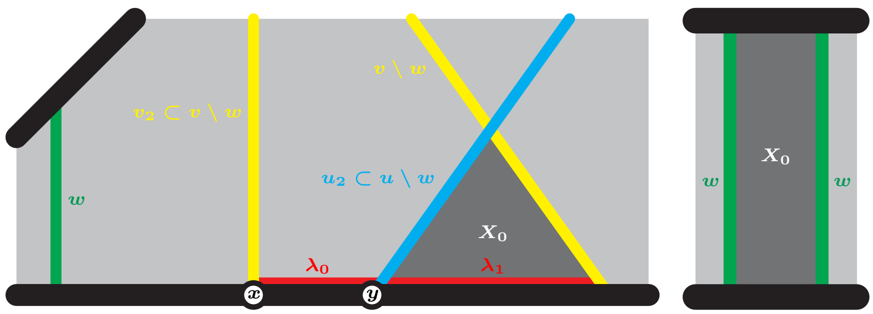


FIGURE 10. Permissible triangles and rectangles of $X \setminus\setminus(u \cup v)$ in condition (2.2) of Lemma 2.26

properly embedded, non-∂-parallel arcs. Let w denote the union of the arcs of u that lie in v, and assume that $u \setminus w \pitchfork v$.

Lemma 2.25. *If an arc u_1 of $u \setminus w$ is isotopic in $X \setminus w$ to an arc v_1 of $v \setminus w$, then:*

(A) Some compact disk X_0 of $X\setminus\setminus(\alpha \cup \beta)$ is a bigon, triangle, or rectangle with $|\partial X_0 \cap \alpha| = 1 = |\partial X_0 \cap \beta|$: see Figure 9.

(B) Using only the moves shown in Figure 9, both of which decrease $|\alpha \cap \beta|$, one can isotope α in $X \setminus w$ until $\alpha \cap \beta = \varnothing$.

(C) If $\alpha \cap \beta \neq \varnothing$ and no disk of $X \setminus\setminus(\alpha \cup \beta)$ is a bigon, then each endpoint of α is incident to exactly one triangle of $X\setminus\setminus(\alpha \cup \beta)$.

Now we consider u and v all together:

Lemma 2.26. *Given u, v, w as throughout §2.4.2, if*

$$\text{each disk } X_0 \subset X \setminus\setminus(u \cup v) \text{ with } |\partial X_0 \cap u| = 1 = |\partial X_0 \cap v| \text{ is the sort of triangle or rectangle shown in Figure 10,} \tag{2.2}$$

and if $u \setminus w$ and $v \setminus w$ are isotopic in $X \setminus w$,[20] *then $u = v = w$.*

---

[20]Situating the isotopy between u and v in $X \setminus w$ rather than in $X \setminus\setminus w$ prohibits their endpoints from sliding across w. An equivalent hypothesis is that u and v are related by a proper isotopy in X which fixes w.

2.4.3. *How definite surfaces of opposite signs intersect.*

Lemma 2.27. *Suppose $F_\pm$ are positive- and negative-definite surfaces spanning a link L, and α is an arc of $F_+ \pitchfork F_-$. Then:*

(A) $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} \neq -2$.
(B) *If α is nonseparating on F_-, then $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} = 2$.*
(C) *In particular, if L is prime, both $F_\pm$ are essential, and α is not ∂-parallel in both $F_\pm$, then $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} = 2$.*

Lemma 2.27 (C) implies that, when applying Procedure 2.24 to two essential surfaces $F_\pm$ whose boundary is prime, move (3) is never used. This in turn implies:

Fact 2.28. *Let $F_+ \pitchfork F_-$ be essential definite surfaces of opposite signs spanning a prime link L. Apply Procedure 2.24 to $F_\pm$. Let F'_+ denote the surface obtained from F_+, and let st_{F_+} and $st_{F'_+}$ denote the unions of the standard arcs of $F_+ \cap F_-$ and of $F'_+ \cap F_-$. Then:*

(A) $st_{F_+} = st_{F'_+} = F'_+ \cap F_-$, *and*
(B) *the alternating diagram $D_{F'_+, F_-}$ associated to F'_+, F_- by Proposition 2.22 (B) is determined by the isotopy class of $F_+ \cup F_-$, regardless of how Procedure 2.24 is carried out.*

Lemma 2.29. *Suppose $F_\pm$ are essential definite surfaces of opposite signs spanning a prime link L such that $F_+ \cap F_-$ consists only of standard arcs. If $\alpha_\pm \subset F_\pm \backslash\backslash F_\mp$ are arcs which are parallel in $S^3 \backslash \mathring{\nu} L$, then both endpoints of $\alpha_\pm$ lie on the same arc v_0 of $F_+ \cap F_-$, and each $\alpha_\pm$ is parallel in $F_\pm \backslash\backslash F_\mp$ into v_0.*

Theorem 2.30. *Suppose B, W and B', W' are the checkerboard surfaces of prime alternating diagrams D and D' of a link L. Then $D \equiv D'$ if and only if B is isotopic to B' and W is isotopic to W'.*[21]

See §6.2 for the proof.

Corollary 2.31. *There is a bijective correspondence between equivalence classes of prime alternating link diagrams on S^2 and pairs of isotopy classes of essential definite surfaces of opposite signs spanning the same prime link in S^3.*

Example 2.32. The diagrams $D = D_{B,W}$ and $D' = D_{B',W'}$ of the $(3,4)$ torus knot obtained by closing the braid diagrams shown left and right in Figure 11 are distinct. Yet, their checkerboard surfaces are isotopic. By symmetry, it suffices to check this for B and

[21]A third equivalent condition, which we will not need, is that there is an orientation-preserving homeomorphism $f : S^3 \to S^3$ that restricts to homeomorphisms $B \to B'$ and $W \to W'$ (any pairwise homeomorphism of (S^3, L) that respects meridians on $\partial\nu L$ can be extended to an ambient isotopy).

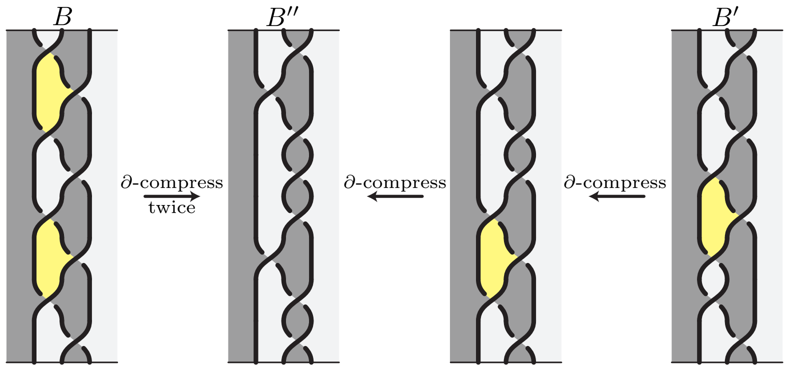


FIGURE 11. Both closed-up surfaces B and B' are isotopic to B'' with two negative crosscaps attached.

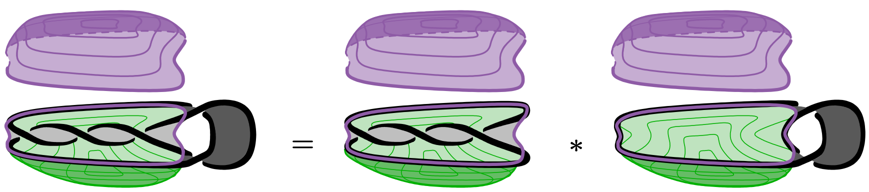


FIGURE 12. A plumbing cap and its shadow for a spanning surface, and the associated de-plumbing.

B'. Indeed, each admits a sequence of two positive meridional ∂-compressions[22] (each ∂-compression disk comes from a yellow region in the figure) to the black checkerboard surface B'' shown center-left in the figure, hence is isotopic to $B''\natural$$\natural$.

Question 2.33. To what classes of link diagrams does Theorem 2.30 extend?

2.5. **Generalized plumbing.**

2.5.1. *Basic definitions.* Let F be a spanning surface for a nonsplit link L. A *plumbing cap* for F is an embedded disk $V \subset S^3 \setminus \mathring{\nu}L$ with $V \cap (F \cup \partial\nu L) = \partial V$ such that:

- ∂V bounds a disk $\widehat{U} \subset F \cup \nu L$,
- $\widehat{U} \cap F$ is a disk U called the *shadow* of V, and
- denoting the 3-balls of $S^3 \backslash\backslash (\widehat{U} \cup V)$ by Y_1, Y_2, neither subsurface $F_i = F \cap Y_i$ is a disk.

If the first two properties hold but the third fails, we call V a *fake plumbing cap* for F; we still call U the shadow of V.

The decomposition $F = F_1 \cup F_2$ is a *plumbing decomposition* or *de-plumbing* of F along U and V, denoted $F = F_1 * F_2$. See Figure

[22] Defined in [AK13], this is a ∂-compression that takes a spanning surface to a spanning surface; it corresponds to de-summing a .

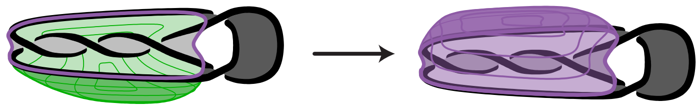


FIGURE 13. Re-plumbing a spanning surface replaces a plumbing shadow with its cap.

12. The reverse operation, in which one glues F_1 and F_2 along U to produce F, is called *generalized plumbing* or *Murasugi sum*.

If V is a plumbing cap for F with shadow U, then one can construct another spanning surface $F' = (F \backslash\backslash U) \cup V$; we call the operation of changing F to F' *re-plumbing*. See Figure 13. Call the analogous operation along a fake plumbing cap a *fake re-plumbing*; this is an isotopy move. Two spanning surfaces are *plumb-related* if there is sequence of re-plumbing and isotopy moves between them.

2.5.2. *Re-plumbing in S^3 and isotopy through B^4.*

Proposition 2.34. *Let L be a link in $S^3 = \partial B^4$, let $F_1, F_2 \subset S^3$ be compact embedded surfaces with $\partial F_i = L$, and let F_i' be properly embedded surfaces in B^4 obtained by perturbing int(F_i), while fixing $\partial F_i = L \subset S^3$. If $F_1 \setminus \mathring{\nu} L$ and $F_2 \setminus \mathring{\nu} L$ are plumb-related, then:*

(A) F_1' and F_2' are related by an ambient isotopy of B^4 which fixes $S^3 \supset L$ pointwise.
(B) There is an isomorphism $\phi : H_1(F_1) \to H_1(F_2)$ satisfying $\langle \alpha, \beta \rangle_{F_1} = \langle \phi(\alpha), \phi(\beta) \rangle_{F_2}$ for all $\alpha, \beta \in H_1(F_1)$.
(C) F_1 and F_2 have the same slope: $s(F_1) = s(F_2)$.[23]
(D) If F_1 is definite, then F_2 is definite and of the same sign.
(E) In particular, if F_1 is a checkerboard surface from a reduced alternating diagram, then so is F_2.

Proof. Part (A) follows from the observation that any re-plumbing move can be realized as an isotopy through B^4 in which one fixes the entire surface except the plumbing shadow and pushes the plumbing shadow through B^4 to the plumbing cap. Part (B) follows from (A) and Theorem 3 of [GL78], which states that the Gordon-Litherland pairing on F_i corresponds to the intersection pairing on the 2-fold branched cover of B^4 with branch set F_i'. Parts (C)-(E) then follow immediately, using [Gr17]. □

2.5.3. *Flyping caps.* Let D be a prime alternating link diagram with checkerboard surfaces B, W. Say that a plumbing cap V for B is a *flyping cap* (relative to W) if V appears as in Figure 14, left-center. There is then a corresponding flype move, as shown in Figure 14. (The resulting link diagram might be equivalent to D.)

[23]The component-wise slopes may differ, but their sums will be equal.

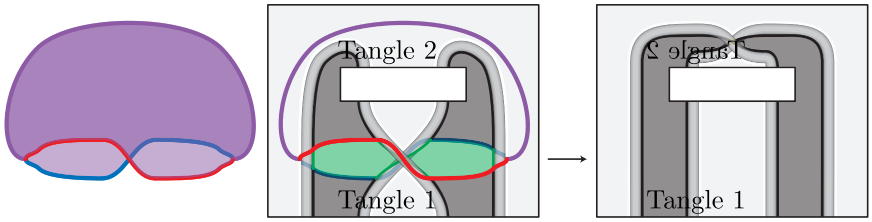


FIGURE 14. A flyping cap and the associated flype move

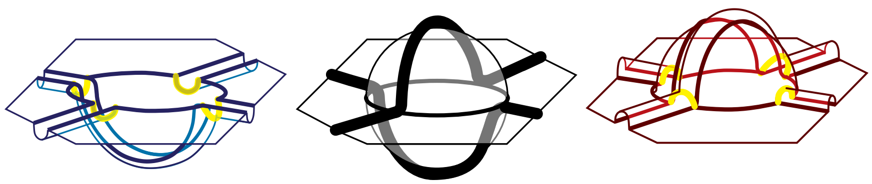


FIGURE 15. A link near a crossing ball with S_- and S_+.

Proposition 2.35. *Given $D = D_{B,W}$, let V be an flyping cap for B, $D \to D' = D_{B',W'}$ the flype move corresponding to V, and B'' the surface obtained by re-plumbing B along V. Then B' and B'' are isotopic, as are W' and W. Hence, $D' \equiv D_{B'',W}$.* [24]

Proof. Figure 2 shows the isotopies $B'' \to B'$ and $W \to W'$. □

Conversely, if $D \to D'$ is a flype move along a circle $\gamma \subset S^2$, then B (or W) has a flyping cap V with $V \cap W \subset \nu\gamma$ (resp. $V \cap B \subset \nu\gamma$).

3. Crossing ball setup and isotopy moves

Given a prime alternating diagram D of a link L and an arbitrary essential positive-definite F surface spanning L, §3 uses the crossing ball structures introduced in [Me84] to define and study a hierarchy of isotopy moves on F relative to D.

3.1. **Crossing ball setup.** Here is the setup for all of §§3-5, 7-8:

- D is a prime alternating diagram of a link L with crossings $c_1, \dots, c_n$; $\pi : \nu S^2 \to S^2$ denotes projection;[25] and (for §3.1 only) $Y_\pm$ are the 3-balls of $S^3 \backslash\backslash S^2$.
- Insert disjoint closed *crossing balls* C_t in $\mathring{\nu} S^2$, with each C_t centered at c_t. Denote $C = \bigsqcup_t C_t$, and embed L in $(S^2 \setminus \text{int}(C)) \cup \partial C$ by perturbing the arcs of $D \cap C$ following the crossing data, so that L appears near each C_t as shown center

---

[24] An analogous statement holds for flyping caps for W.

[25] The assumption that D is prime and alternating implies that D is reduced and, by Theorem 1 (b) of [Me84], that L is prime, hence nontrivial and nonsplit.

in Figure 15. For §3.1 only, call the arcs of $L \cap S^2$ and $L \cap \partial C \cap Y_\pm$ *edges*, *overpasses* and *underpasses*, respectively.

- Take $\nu L \subset \mathring{\nu} S^2$ with projection $\pi_L : \nu L \to L$. Denote the two 3-balls of $S^3 \backslash\backslash (S^2 \cup C \cup \nu L)$ by $H_\pm$, so that each $\text{int}(H_\pm) = Y_\pm \setminus (\nu L \cup C)$. Also denote $\partial H_\pm = S_\pm$. See Figure 15.
- Denote each *vertical arc* $v_t = \pi^{-1}(c_t) \cap C_t \setminus \mathring{\nu} L$; let $v = \bigcup_t v_t$.
- For each edge $e \subset L$, call the cylinder $E = \pi_L^{-1}(e) \cap \partial \nu L$ an *edge* (of $\partial \nu L$); the rectangles $E_\pm = E \cap Y_\pm$ are its *top* and *bottom*. For each over/underpass $e_\pm$ of L, call $E_\pm = \pi_L^{-1}(e_\pm) \cap \partial \nu L$ an *over/underpass* (of $\partial \nu L$); $E_+ \cap Y_+$ and $E_+ \backslash\backslash Y_+$ are the *top* and *bottom* of the overpass, while $E_- \cap Y_-$ and $E_- \backslash\backslash Y_-$ the *bottom* and *top* of the underpass. Say that an edge E and a crossing ball C_t are *incident* if they intersect; say that two edges (resp. crossing balls) are *adjacent* if there is a crossing ball (resp. an edge) incident to both of them.[26] Assume that $\pi_L^{-1}(\partial(L \cap \partial C)) = \partial \nu L \cap \pi^{-1}(\partial C \cap S^2)$: then these meridia, highlighted yellow in Figure 15, cut $\partial \nu L$ into its edges, overpasses, and underpasses.
- For each t, $\partial C_t \cap S^2 \setminus \mathring{\nu} L$ consists of four arcs, two β_1, β_2 in black regions of $S^2 \setminus D$ and two ω_1, ω_2 in white. A core circle in $\alpha \cup \beta \cup (\partial \nu L \cap C_t)$ bounds a disk $B_t \subset C_t$ such that $\pi(B_t)$ is disjoint from the white regions of $S^2 \setminus D$ and intersects D only at c_t. Likewise, ω_1, ω_2 yield a disk $W_t \subset C_t$; note that $B_t \cap W_t = v_t$. A properly embedded disk $X \subset C_t \setminus \mathring{\nu} L$ that contains v_t is called a *positive* (resp. *negative*) *crossing band* if there is an isotopy of $(X, \partial X \cap \partial \nu L, \partial X \cap \partial C_t)$ through $(C_t, \partial \nu L, \partial C_t)$ to B_t (resp. W_t). See Figure 3.
- Denote the union of the black and white regions of $S^2 \setminus \text{int}(C \cup \nu L)$ by $\widehat{B}$ and $\widehat{W}$. Then $B = \widehat{B} \cup \bigcup_t B_t$ and $W = \widehat{W} \cup \bigcup_t W_t$ are the *checkerboard surfaces* from D. Note that $B \cap W = v$.
- Denote each:[27][28][29][30]

$$\begin{aligned}
S_0 &= S^2 \setminus \text{int}(C \cup \nu L);\\
S_{\pm E} &= S_\pm \cap \partial \nu L \backslash\backslash (\pi^{-1} \circ \pi(C));\\
S_{\pm B} &= \widehat{B} \cup S_{\pm E} \text{ and } S_{\pm W} = \widehat{W} \cup S_{\pm E};\text{ and}\\
C_t^\pm &= S_\pm \cap (\pi^{-1} \circ \pi(C_t)) \text{ with } C^\pm = \textstyle\bigcup_t C_t^\pm.
\end{aligned}$$

[26] Note that any edge or crossing ball is therefore said to be adjacent to itself.

[27] The n-punctured sphere S_0 equals $\widehat{B} \sqcup \widehat{W} = S_+ \cap S_-$.

[28] $S_{\pm E}$ respectively consist of the upper/lower halves of all edges (of $\partial \nu L$).

[29] Each component of S_{+B} is a disk comprised of a disk of $\widehat{B}$ together with the top halves of all incident edges; similarly for S_{-B} and $S_{\pm W}$.

[30] The top of the overpass at C_t and the two disks of $\partial C_t \cap S_+$ comprise C_t^+.

- F is an essential positive-definite spanning surface for L.[31] Each crossing band in F contains an arc of v; denote the union of such arcs by v_F. Let $D_{F,W}$ denote the diagram that F, W determine via Theorem 2.30.

Remark 3.1. The combinatorial setup established above can also be constructed from B, W (assuming only that these are essential definite surfaces of opposite signs spanning a prime link L and that $B \cap W = v$ is comprised of standard arcs) by taking C to be a regular neighborhood of v in $S^3 \setminus\setminus \mathring{\nu} L$.

3.2. **Fair position, flyping circles, and push-through moves.**

Definition 3.2. F is in *fair position* if:[32]

(a) $F \cap W$ is comprised entirely of standard arcs;
(b) F is transverse in S^3 to B, W, ∂C, and $v \setminus v_F$;
(c) ∂F is transverse on $\partial \nu L$ to each meridian;
(d) whenever $\partial F \cap C_t \neq \varnothing$, $F \cap C_t$ is a crossing band;
(e) no arc of $F \cap \partial C \cap S_\pm$ is parallel in ∂C into $\partial C \cap \partial S_0$;
(f) $B \cup W$ cuts each component of $F \cap C$ into disks;
(g) each crossing band in F is disjoint from S_+; and
(h) $S_+ \cup S_-$ cuts F into disks.

Lemma 3.3. *F can be isotoped into fair position.*

The proof of Lemma 3.16 appears in §7, as do the proofs of all lemmas that appear in §3 without their proofs.

Lemma 3.4. *If F is in fair position, then:*

(A) balls comprise $(C \setminus \mathring{\nu} L)\setminus\setminus F$ and $H_\pm \setminus\setminus F$;
(B) arcs comprise $\partial F \cap S_\pm$, $F \cap S_0$, and $F \cap \partial C \cap S_\pm$; and
(C) each component X of $F \cap C$ is either a crossing band or a saddle disk as in Figure 16.[33]

Notation 3.5. Assume that F is in fair position.

- Each circle $\gamma \subset F \cap S_\pm$ bounds a disk $F_\gamma \subset F \cap H_\pm$.
- The arcs of $v \cup (F \cap W)$ induce a cell decomposition of W under which we may refer to *bigons*, *triangles*, etc.

Definition 3.6. A *flyping circle* for F is a circle γ of $F \cap S_+$ that appears as in Figure 17, left, where $\pi(\gamma)$ is a flyping circle for D.

---

[31] F is connected because L is prime, hence nonsplit; recall Fact 2.4.

[32] Later, we define increasingly restrictive *k-good* positions for F, $k = 0, 1, \dots, 10$, and 0-good position will be equivalent to fair position.

[33] In particular, X must intersect each of B and W in a single arc. Namely, if X is a crossing band in a crossing ball C_t, then $X \cap B = v_t = X \cap W$, and if X is a saddle disk, then $\beta = X \cap B$ and $\omega = X \cap W$ appear as in Figure 16, right.

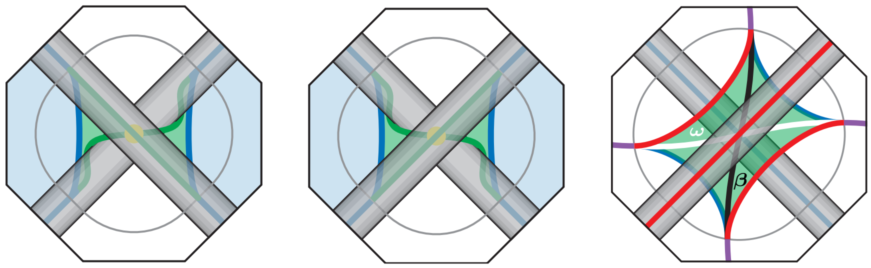


FIGURE 16. Positive (left) and negative (center) crossing bands and a saddle disk (right) in a surface F in fair position.

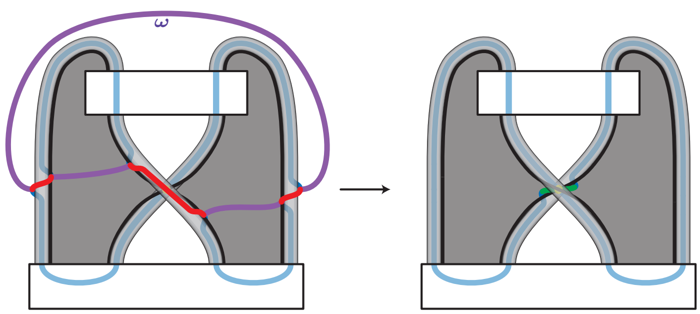


FIGURE 17. A flyping circle ω gives a flype-type re-plumbing.

Then the arc $\omega = \gamma \cap \widehat{W}$ is a *flyping arc* for F, and there is a *flype-type re-plumbing* move $F \to F'$ as shown in Figure 17, where F' is in fair position and $F' \cap S_+ = F \cap S_+ \setminus \gamma$.[34]

Lemma 3.7. *If F is in fair position and $F \cap S_+$ contains only flyping circles, then $D_{F,W}$ is related to D by a sequence of flypes that each preserve the isotopy class of W.*

Proposition 3.8. *If F is in fair position, then every crossing band in F is positive (as shown left in Figure 16).*

Proof. If F has a negative crossing band, say at C_t, then v_t is a non-standard arc of $F \cap W$ violating condition (a) of Definition 3.2. □

Proposition 3.8 and condition (g) in Definition 3.2 require each crossing band in F to appear as in Figure 16, left. This creates an asymmetry between $F \cap S_-$ versus $F \cap S_+$ which will be strategically useful. (We will sharpen this asymmetry further when we define Moves 7-9.) The idea is that pushing $F \cap (S_+ \cup S_-)$ into S_- near crossing bands (where F "looks nice") increases the likelihood that

[34]Because flyping circles for F lie in S_+ and those for D lie in S^2, we will find no need to distinguish these explicitly in the sequel.

figures/Example1A1.{ps,eps,pdf} not found (or no BBox)
figures/Example1C1.{ps,eps,pdf} not found (or no BBox)

FIGURE 18. Left: F is in 9-good position. Right: $D_{F,W}$.

the circles of $F \cap S_+$ will enable simplifying moves on F. This strategy will eventually bear fruit in the form of the re-plumbing Move 10. To get a sense of this, consider:

Example 3.9. In Figure 18, left, where F is in fair position,[35][36] each of the four (red-purple) circles of $F \cap S_+$ gives a Move 10, in fact a flype-type re-plumbing. The diagram on the right is $D_{F,W}$. Note:

- The circles of $F \cap S_+$ are more salient than those of $F \cap S_-$.
- One could isotope the arc β of $\partial F \cap S_-$ past ∂B into S_+, thus decreasing $|F \cap S_0|$, but then the circles of $F \cap S_+$ would be less illuminating. We will carefully define Moves 1-9, especially Moves 5 and 7, so as not to include this tempting move.
- The top-right flype could be achieved by means of isotopy, but this isotopy would not fix v_F. We prefer to define Moves 1-9 so that each fixes v_F (where F "looks nice").

Definition 3.10. Suppose F is in fair position and α is a properly embedded arc in $S_\pm \backslash\backslash F$ such that

(a) both endpoints of α lie on the same circle γ of $F \cap S_\pm$,
(b) α lies in a disk Y of $S_{\pm B}$ or $S_{\pm W}$,
(c) $|\alpha \cap S_0| = 1$,
(d) α's endpoints lie on the interiors of arcs γ', γ'' of $\gamma \cap Y \backslash\backslash \partial S_0$,
(e) no arc of $\gamma \cap S_0$ intersects both γ' and γ'',[37] and
(f) $\pi(\alpha) \cap \pi(\partial F \cap S_\mp) = \varnothing$.

Suppose a properly embedded arc $\beta \subset F_\gamma$ with $\partial\beta = \partial\alpha$ is parallel to α through a properly embedded disk $X \subset H_\pm \backslash\backslash F$.[38] Isotope F near β through X past α. We call this a **push-through move**.

There are three possible pictures of the situation, depending on how many endpoints of α lie in S_0 versus on $\partial\nu L$; see Figure 19.

Proposition 3.11. *If F admits a push-through move along $\alpha \subset S_{\pm W}$ and $\partial\alpha \subset \partial\nu L$, then the endpoints of α lie on the same edge.*

Proof. Such a move creates two non-standard arcs of $F \cap W$. Lemma 2.27 (C) implies that these arcs, and thus $\alpha \cap S_0$, are ∂-parallel in W. The result follows because D is prime. □

---

[35] In fact, F is in 9-good position; see §3.3.
[36] Color guide: $F \cap S_0$, $F \cap S_+ \setminus S_0$, $F \cap S_- \setminus S_0$, $F \cap C$.
[37] In particular, $\gamma' \cap \gamma'' = \varnothing$.
[38] Lemma 3.4 (A) guarantees the existence of β and X.

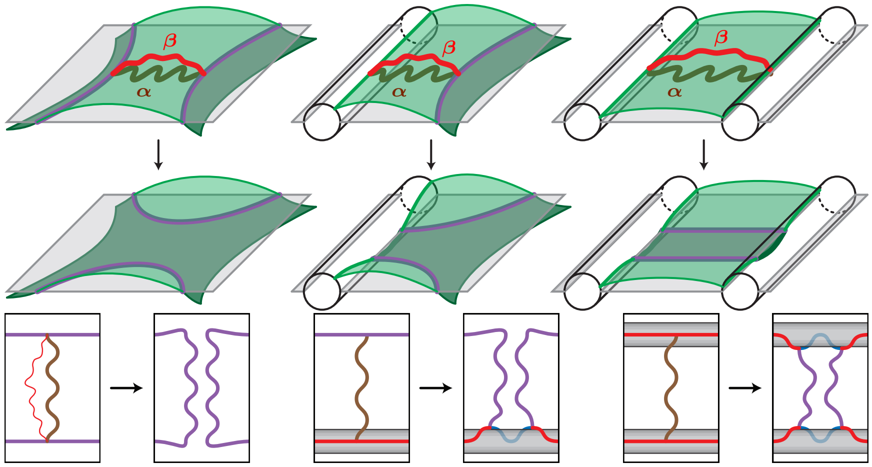


FIGURE 19. Push-through moves (Moves 7, 8, and 9)

Definition 3.12. If F is in fair position, then we define the following measures of *complexity* for F:

$$\begin{aligned}
&\boldsymbol{|}\,F\,\boldsymbol{|}_1 = |v\backslash\backslash F| = \left|\begin{matrix}\text{crossing balls without}\\ \text{crossing bands}\end{matrix}\right| + \left|\begin{matrix}\text{saddle}\\ \text{disks}\end{matrix}\right|,\\
&\boldsymbol{|}\,F\,\boldsymbol{|}_2 = |F \cap S_0|,\\
&\boldsymbol{|}\,F\,\boldsymbol{|}_3 = |F \cap S_0| - 2|F \cap S_+|.
\end{aligned} \tag{3.1}$$

3.3. **Hierarchy of isotopy moves on F.** In §§3.3-4.1 we describe several moves on F, denoted Move 1 through Move 10, subject to the following rule of hierarchy, which will ensure that each move preserves fair position:[39][40]

Convention 3.13. For each Move k defined in the sequel, $1 \leq k \leq 10$, we perform Move k only if F is in fair position and admits none of Moves $1, \dots, k-1$.

Definition 3.14. For $0 \leq k \leq 10$, F is in *k-good position* (relative to B, W) if F is in fair position and admits no Move ℓ with $\ell \leq k$.

Moves 1-9 will serve two main purposes. First, Moves 1-6 will simplify how the arcs of $F \cap W$ interact with v. (They will also simplify $F \cap B$.) Second, Moves 7-9 will increase the number of circles of $F \cap S_+$ and thus simplify these circles *individually*. In fact, we will see that in 9-good position each innermost circle of $F \cap S_+$

[39] Moves 1-9, defined in §3.3, are isotopies; Move 10 in §4.1 is a re-plumbing.

[40] Unlike the hierarchy described in Procedure 2.24, where it turns out that all (1)'s always precede all (2)'s which (vacuously) precede all (3)'s, we will see that there are situations where some Move k enables a previously impossible Move ℓ for some $\ell < k$. Lemma 3.25 will somewhat constrain this behavior.

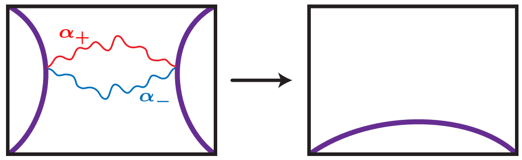


FIGURE 20. Move 1

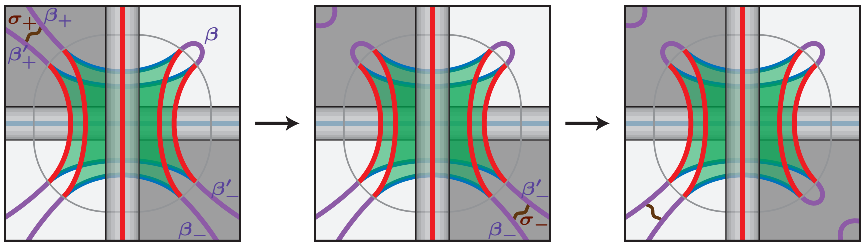


FIGURE 21. Move 2

enables a re-plumbing (Move 10), which we will eventually discover is always a flype-type re-plumbing.

Move 1. Suppose $\alpha \subset S_0$ is an arc with $\alpha \cap F = \partial\alpha = \{x, y\}$, where x, y lie on distinct arcs of $F \cap S_0$ but on the same circles $\gamma_+ \subset F \cap S_+$ *and* $\gamma_- \subset F \cap S_-$; suppose $\alpha_\pm \subset F_{\gamma_\pm}$ are properly embedded arcs with $\partial\alpha_\pm = \{x, y\}$ such that the circle $\gamma = \alpha_+ \cup \alpha_-$ bounds a disk $X \subset S^3 \setminus \mathring{\nu} L$ with $X \cap F = \partial X$ and $X \cap S_0 = \alpha$.[41] Then X is parallel in $S^3 \backslash\backslash (F \cup \nu L)$ to a disk $F_0 \subset F$; isotope F near F_0 past X.[42]

Figure 20 shows the effect of Move 1 near α. The next property motivates conditions (e)-(f) in Definition 3.10:

Observation 3.15. *If F is in 1-good position and $F \to F'$ is a push-through move, then F' is in fair position.*

Move 2. If $F \cap \widehat{W}$ contains an arc whose endpoints are both on the same crossing ball, then take ω to be an outermost such arc in $\widehat{W}$, and denote the circles of $F \cap S_\pm$ containing ω by $\gamma_\pm$. Each $\gamma_\pm \cap \partial C$ consists of two arcs incident to ω, each of which is incident to an arc of $\gamma_\pm \cap \widehat{B}$; let $\beta_\pm$ and $\beta'_\pm$ denote these arcs of $\gamma_\pm \cap \widehat{B}$. Choose $+$ or $-$ so that $\beta_\pm \neq \beta'_\pm$,[43] construct a properly embedded arc $\sigma_\pm \subset \widehat{B} \backslash\backslash F$ with one endpoint on each of $\beta_\pm$ and $\beta'_\pm$, and perform a push-through move along $\sigma_\pm$, as shown in Figure 21.

[41]Lemma 3.4 (A) guarantees the existence of $\alpha_\pm$ and X.
[42]Recall that F is incompressible and $S^3 \setminus L$ is irreducible.
[43]We may have $\beta_+ = \beta'_+$ or $\beta_- = \beta'_-$ but not both, by 1-good position.

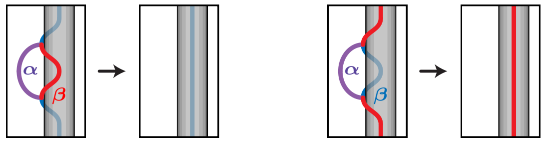


FIGURE 22. Move 3

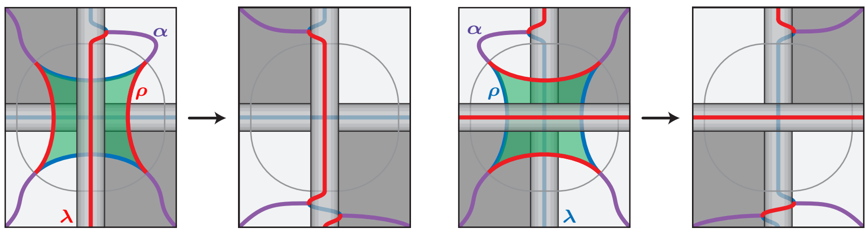


FIGURE 23. Move 4

Lemma 3.16. *With F in fair position, the following are equivalent:*

(I) No arc of $F \cap \widehat{W}$ is parallel in $\widehat{W}$ into ∂C.
(II) No arc of $F \cap W \backslash\backslash v$ is parallel in $W \backslash\backslash v$ into v.[44]
(III) F is in 2-good position.

Lemma 3.17. *If F is in 2-good position, then F admits no push-through move along any arc $\alpha \subset \widehat{W}$.*

Move 3. Suppose an arc α of $F \cap S_0$ is parallel in $S_0 \backslash\backslash F$ to an arc $\alpha' \subset \partial\nu L$. Proposition 2.6 implies that α' is parallel on $\partial\nu L$ to an arc $\beta \subset \partial F$. If $\text{int}(\beta) \cap \partial S_0 \neq \varnothing$, then push $(F_{\alpha\cup\beta}, \beta)$ through $(H_\pm, \partial\nu L)$ past (S_0, α') as shown in Figure 22.

Proposition 3.18. *If F is in 3-good position, then each circle γ of $F \cap S_+$ satisfies $|\gamma \cap S_0| \geq 2$, so $\boldsymbol{|}\, F \,\boldsymbol{|}_3 \geq 0$.*

Proof. Assume instead that $|\gamma \cap S_0| < 2$. Then Lemma 3.4 (B)-(C) implies that $\gamma \cap \partial C = \varnothing$ and $\gamma \not\subset S_0$. Further, since D is connected and nontrivial, $\gamma \not\subset \partial\nu L$. Therefore, F appears near γ as in Figure 22 and, contrary to assumption, admits a Move 3 near γ. $\square$

Lemma 3.19. *Given that F is in 2-good position, F is in 3-good position if and only if no arc of $F \cap \widehat{B}$ is ∂-parallel in B.*

Move 4. Suppose an arc α of $F \cap \widehat{W}$ is incident to (i) an arc λ of $\partial F \cap S_\pm$ that traverses the over/underpass at a crossing C_t and (ii) an arc ρ of $F \cap S_\pm \cap \partial C_t$ (at the same crossing).[45] Isotope F nearby as shown in Figure 23.

[44] That is, there are no bigons in $W \backslash\backslash (F \cup v)$.
[45] Note that the endpoint x shared by α and λ satisfies $i(\partial F, \partial W) = +1$.

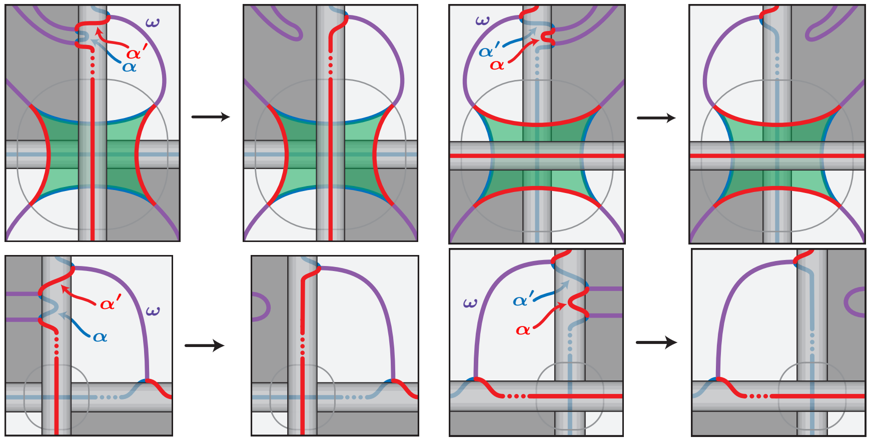


FIGURE 24. Move 5

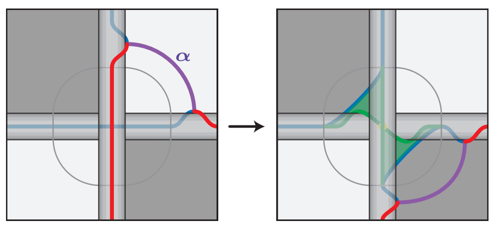


FIGURE 25. Move 6.

Move 5. Suppose that an arc α of $\partial F \cap S_\pm$ lies entirely on an edge E and is parallel in E into ∂B, and that one of the arcs α' of $\partial F \cap S_\mp$ incident to α lies entirely in E and is incident to an arc ω of $F \cap \widehat{W}$ whose other endpoint lies either:

- on a crossing ball incident to E or
- on an edge E' adjacent to E[46] at a crossing C_t with $v_t \not\subset F$.

Isotope F near α as shown in Figure 24.

Lemma 3.20. *If F is in 5-good position and an arc α' of $F \cap W \setminus v_F$ is isotopic in $W \setminus v_F$ into $\widehat{W} \cup v$, then $\alpha' \subset \widehat{W}$.*[47]

Lemma 3.21. *If F is in 5-good position and admits a push-through move along an arc $\alpha \subset S_\pm \backslash\backslash F$, then α intersects B, not W.*

Lemma 3.22. *If F is in 5-good position and $\gamma \subset F \cap S_+$ is a flyping circle which traverses the overpass at C_t, then $|F \cap C_t| \neq 1$.*[48]

[46] Lemma 3.19 implies that $E' \neq E$.
[47] Note: in $W \setminus v_F$, α' is isotopic into $\widehat{W} \cup v$ if and only if it is isotopic into $\widehat{W}$.
[48] In fact, $F \cap C_t = \varnothing$, but we will not need this.

Move 6. Suppose an arc α of $F \cap \widehat{W}$ is incident to arcs of $\partial F \cap S_+$ and $\partial F \cap S_-$ that traverse the overpass and underpass at the same crossing. Isotope F near α as shown in Figure 25.

Lemma 3.23. *With F in fair position, the following are equivalent:*

(I) No arc of $F \cap \widehat{B}$ is ∂-parallel in B, and no arc of $F \cap \widehat{W}$:
- *(a) is parallel in S_0 into ∂C,*
- *(b) has endpoints on a crossing ball and incident edge, nor*
- *(c) has endpoints on edges that are adjacent at a crossing ball where F* does not *have a crossing band.*

(II) No disk of $B \backslash\backslash (v \cup F)$ is a bigon, and no disk X of $W \backslash\backslash (v \cup F)$ satisfies $|\partial X \cap v| = 1 = |\partial X \cap F|$.[49]

(III) F is in 6-good position.

Move 7. Perform a push-through move along an arc $\alpha \subset \widehat{B} \backslash\backslash F$ whose endpoints lie on the same circle of $F \cap S_+$.

Move 8. Perform a push-through move along an arc $\alpha \subset S_{+B} \backslash\backslash F$ whose endpoints $x \in \widehat{B}$ and $y \in \partial \nu L$ lie on the same circle of $F \cap S_+$.

Move 9. Perform a push-through move along an arc $\alpha \subset S_{+B} \backslash\backslash F$ whose endpoints $x, y \in \partial \nu L$ lie on the same circle of $F \cap S_+$.

When F is in 9-good position, circles of $F \cap S_-$ may admit push-through moves, but those of $F \cap S_+$ must not, due to Lemma 3.21.

Lemma 3.24. *Moves 1-9 all preserve fair position and fix or decrease $|\!| F |\!|_1$, Moves 1-7 each lead to a lexicographical decrease in $(|\!| F |\!|_1, |\!| F |\!|_2, |\!| F |\!|_3)$,*[50] *and Moves 8-9 both decrease $|\!| F |\!|_3$.*

Lemma 3.25. *Suppose that F is in 2-good position, and $F = F_0 \to \cdots \to F_r$ is a sequence of Moves 1-9. Then:*

- *(A) Neither Move 1 nor Move 2 appears in the sequence.*
- *(B) The isotopy $F_0 \to F_r$ restricts to an isotopy $F_0 \cap W \to F_r \cap W$ in W which fixes $v_{F_0} \subset v_{F_r}$.*
- *(C) If F is in 6-good position, then the sequence $F_0 \to F_r$ fixes $F \cap W$ and involves only Moves 3 and 7-9.*
- *(D) If F is in 7-good position, then $F_0 \to F_r$ uses only Moves 8-9.*

Lemma 3.26. *Any sequence of Moves 1-9 terminates, giving an isotopy $F \to F'$ where F' is in 9-good position with $|\!| F' |\!|_1 \leq |\!| F |\!|_1$.*

[49] Such X is either a bigon, triangle, or rectangle.

[50] Namely, Move 1 decreases $|\!| F |\!|_1$ (and $|\!| F |\!|_2$); Move 2 fixes $|\!| F |\!|_1$ and $|\!| F |\!|_2$ and leads to Move 1 (that is, although Move 2 itself fixes complexity, it is always possible to follow Move 2 either with a Move 1 or with a second Move 2 and then a Move 1, and in either case, this sequence of moves decreases complexity); Moves 4 and 6 decrease $|\!| F |\!|_1$; Moves 3 and 5 fix $|\!| F |\!|_1$ and decrease $|\!| F |\!|_2$; and Move 7 fixes $|\!| F |\!|_1$ and $|\!| F |\!|_2$ while decreasing $|\!| F |\!|_3$.

4. Plumb-equivalence of essential positive-definite surfaces

In §§4-5, we will discover that when F is in 9-good position, $F \cap S_+$ consists entirely of flyping circles; this collection of circles instantly reveals the sequence of flype moves that takes D to $D_{F,W}$. Our path to this discovery is indirect. In §4, we analyze innermost circles of $F \cap S_+$ when F is in 9-good position and discover that any such circle enables a re-plumbing, which we define as Move 10. A priori, Move 10 can be much more complicated than flype-type re-plumbing. Nevertheless, Move 10 allows us to deduce that F and B are plumb-related; this gives a new proof of part of Tait's first conjecture and helps set the stage for the proof of our main result in §5. Section 8 contains the proofs of all lemmas that appear without their proofs in §4.

4.1. Innermost circles in 9-good position.

In §4.1, keeping the setup from §3.1, we assume that F is in 9-good position with $F \cap S_+ \neq \varnothing$ and consider an arbitrary innermost disk T_+ of $S_+ \backslash\backslash F$. Denote $\partial T_+ = \gamma_0$ and orient γ_0 so that it runs counterclockwise around T_+ when viewed from H_+, and denote $T_- = S_- \cap (\pi^{-1} \circ \pi(T_+))$.

Lemma 4.1. *Consider an arc ρ of $\gamma_0 \cap \partial C$, denote the incident arcs of $\gamma_0 \cap \widehat{B}$ and $\gamma_0 \cap \widehat{W}$ by β and ω. Let C_t denote the crossing ball containing ρ, B_0 and W_0 the disks of $\widehat{B}$ and $\widehat{W}$ containing β and ω, E the edge incident to B_0, W_0, and C_t, and C_s the other crossing ball incident to E. Then $\beta \cup \rho \cup \omega$ appears as in Figure 26, left:*

(A) $\gamma_0 \cap C_t^+ = \rho$,[51]
(B) $\gamma_0 \cap E = \varnothing$,
(C) C_s lies in Y_1 and contains a crossing band in F, and
(D) both endpoints of $\beta \cup \rho \cup \omega$ lie on $\partial \nu L$.

Next, we describe how γ_0 gives a re-plumbing move $F \to F'$ such that $\big| F' \big|_1 < \big| F \big|_1$. We then deduce that all essential positive-definite spanning surfaces for L are plumb-equivalent.

Take an annular neighborhood A of $\pi(\gamma_0)$ in S^2, such that A intersects only the crossing balls that $\pi(\gamma_0)$ intersects, $\partial A \cap C = \varnothing$, and each arc of $F \cap S_0 \cap A$ lies on γ_0 or has an endpoint on ∂C. Denote $\partial A = \gamma_1 \cup \gamma_2$ where $\gamma_1 \subset \pi(T_+)$, denote $S^2 \backslash\backslash A = S_1 \sqcup S_2$ with each $\partial S_i = \gamma_i$, denote each ball $\pi^{-1}(S_i) = \widehat{Y}_i$, and denote the annular prism $\pi^{-1}(A) = \widehat{P}$.

Viewing $\nu S^2 \equiv S^2 \times [-1, 1]$, choose $0 < r < R < 1$ such that $C \cup \nu L \subset S^2 \times [-r, r]$, and denote $P = \widehat{P} \cap (S^2 \times [-R, R])$ and $Y_i = \widehat{Y}_i \cap (S^2 \times [-R, R])$, $i = 1, 2$. While fixing $F \cap (S_+ \cup S_- \cup C)$, isotope F_{γ_0} into $\big(\pi^{-1} \circ \pi(T_+)\big) \cap \big(S^2 \times [0, R]\big)$ so that $\pi|_{F_{\gamma_0}}$ is injective; adjust

[51] In particular, γ_0 does not traverse the overpass at C_t.

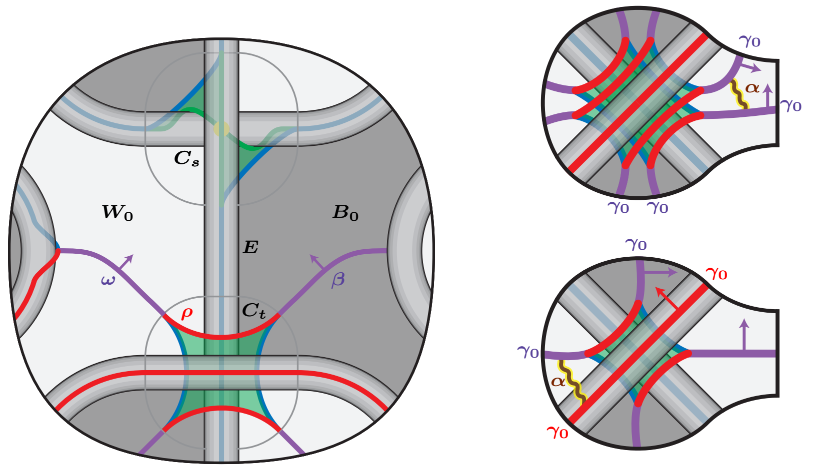


FIGURE 26. F near $\rho \subset \gamma_0 \cap C$ (arrows point into T_+; the sign of β's endpoint on $\partial\nu L$ is unspecified).

all other disks X of $F \cap H_+$ so that $X \cap Y_1 = \varnothing$, $X \cap P \subset \pi^{-1}(\partial X)$, and $\pi|_{X\backslash\backslash P}$ is injective; and adjust each disk X of $F \cap H_-$ so that $X \subset S^2 \times [-R, 0]$ and $\pi|_X$ is injective.[52]

Denote the arcs of $\gamma_0 \cap \widehat{W}$ by $\omega_1, \dots, \omega_m$, indexed following γ_0's orientation. Each ω_i has a dual arc $\alpha_i \subset A \cap \widehat{W}$.[53] Denote the rectangles of $A\backslash\backslash(\alpha_1 \cup \cdots \cup \alpha_m)$ by $A_1, \dots, A_m$ with each $\partial A_i \supset \alpha_i \cup \alpha_{i+1}$, taking indices modulo m. Denote each prism $\pi^{-1}(A_i) \cap P = P_i$.

Lemma 4.2. *With the setup above, each prism P_i intersects F in one of the three ways indicated in the left column of Figure 27.*[54]

For each i, let F_i denote the component of $F \cap P_i$ which intersects γ_0. Observe that each F_i is a disk, and that F_i and F_j intersect in an arc when $i \equiv j \pm 1 \pmod m$ and are disjoint when $i \not\equiv j, j \pm 1 \pmod m$. Denote $F_A = F_1 \cup \cdots \cup F_m$. The disk $F_{\gamma_0} \cap Y_1$ attaches to F_A along its boundary; therefore, F_A is an annulus, and the following subsurface of F is a disk:

$$U = (F_{\gamma_0} \cap Y_1) \cup F_A.$$

[52] We do this so Figure 27 will be generic; some of the complication is for the benefit of [Ki23b].

[53] The arc α_i has one endpoint on γ_1 and one on γ_2, with $|\omega_i \cap \alpha_i| = 1$.

[54] The green arcs top-left describe a disk $X_i \subset P_i \setminus \nu L$ (∂X_i is shown thick, and $X_i \cap S_+$ is shown thin) which is parallel through a ball $Z_i \subset P_i$ into $\pi^{-1}(\gamma_2)$ (Z_i contains the overpass in P_i); F intersects P_i as shown and in an arbitrary number of additional disks in Z_i, each containing a saddle disk.

figures/LastMoveM.{ps,eps,pdf} not found (or no BBox)

FIGURE 27. Move 10 within each prism P_i.

There is a properly embedded disk $V \subset \mathring{\nu}S^2 \backslash\backslash (F \cup \nu L)$ which intersects Y_1 in a disk (in H_-) and intersects each prism P_i as indicated in the right column of Figure 27. Note that $\partial V \cap F = \partial U \cap F \subset \pi^{-1}(\partial A)$ and that $(\partial V \cap \partial \nu L) \cup (\partial U \cap \partial \nu L)$ is a system of meridia and inessential circles on $\partial \nu L$.[55] Thus V is a(n a priori possibly fake) plumbing cap for F, and U is its shadow, so F is plumb-related to $F' = (F\backslash\backslash U) \cup V$.[56]

Move 10. With the setup above, replace F with $F' = (F\backslash\backslash U) \cup V$. In each prism P_i, this changes $F \to F'$ as shown in Figure 27.

Note that when F is in 9-good position any *flype-type* re-plumbing $F \to F'$ is a Move 10.

4.2. **Properties of Move 10.**

Lemma 4.3. *Any Move 10 $F \to F'$ leaves F' in fair position.*

Proposition 4.4. *Given any sequence $F \to F'$ of Moves 1-10 that involves at least one Move 10, we have $\left| F \right|_1 > \left| F' \right|_1$. Hence, any sequence $F \to F'$ of Moves 1-10 terminates.*

Proof. By Lemmas 3.24 and 4.3, Moves 1-10 all preserve fair position, and none of Moves 1-9 increase $\left| F \right|_1$. Further, Move 10 removes a saddle disk or creates a crossing band in each prism P_i, hence strictly decreases $\left| F \right|_1$. The second claim follows immediately. □

In §5, we will prove that when F is in 9-good position $F \cap S_+$ contains only flyping circles; hence, Move 10 is always a flype-type re-plumbing, and thus (by Lemma 3.7) $D_{F,W}$ is flype-related to D. A symmetric argument will then complete our proof of the flyping theorem. For now, though, only this conclusion is at hand:

Theorem 4.5. *If B, W are the checkerboard surfaces from a prime alternating diagram $D \subset S^2$ of a link L, then any essential positive-definite surface F spanning L is plumb-related to B (via Moves 1-10); likewise for essential negative-definite surfaces and W.*

Proof. Put F in fair position and apply Moves 1-10. By Proposition 4.4, this terminates, giving a sequence of isotopy and re-plumbing moves from F to B. □

Proposition 2.34 and Theorem 4.5 imply:

[55] Inessential circles arise only in prisms of type II.

[56] In each prism P_i of type I, we have $F' \cap Z_i = F \cap Z_i$, using Note 54's notation.

Corollary 4.6. *If B and B' are essential definite surfaces of the same sign spanning L, then $\beta_1(B) = \beta_1(B')$ and $s(B) = s(B')$.*

Facts 2.7 and 2.13, Lemma 2.27, Theorem 4.5, and Corollary 4.6 give a new proof of part of Tait's first conjecture:

Theorem 4.7 (Part of Tait's first conjecture [Gr17, Ka87, Mu87, Th87, Tu87])**.** *All reduced alternating diagrams of any link $L \subset S^3$ have the same number of crossings.*

Proof. Assume first that L is prime. Consider two reduced alternating diagrams D_i of L, $i = 1, 2$, with checkerboard surfaces B_i, W_i. Each arc α of $B_i \cap W_i$ satisfies $i(\partial B_i, \partial W_i)_{\nu\partial\alpha} = +2$. Also, $s(B_1) = s(B_2)$ and $s(W_1) = s(W_2)$. Thus,

$$2c(D_1) = i(\partial B_1, \partial W_1) = s(B_1) - s(W_1) = s(B_2) - s(W_2) = 2c(D_2).$$

The general case now follows, as the number of crossings is additive under diagrammatic connect sum and disjoint union. □

Lemma 4.8. *If $F_0 \to F_1$ is a Move 10, then:*

(A) F_1 is in 3-good position; and
(B) if no prism is of type I, then F_1 is in 9-good position.

Proof. Recall that F_1 is in fair position by Lemma 4.3, so applying Lemma 3.23 to F_0 and Lemmas 3.16 and 3.19 to F_1 confirms (A) (see Figure 27). Part (B) follows from Lemmas 3.23 and 4.2. □

In any sequence of Moves 1-10 that uses Move 10 at least once and ends in 10-good position, the *final* move in the sequence is a Move 10 with no prisms of type I, i.e. a flype-type re-plumbing:

Lemma 4.9. *If $F = F_0 \to F_1$ is a Move 10 along γ_0 and $F_1 \to F_2$ is a sequence of Moves 1-9 leaving F_2 in 10-good position, then:*

(A) no prism in the Move 10 is of type I,
(B) γ_0 is the only circle of $F \cap S_+$, and
(C) γ_0 is a flyping circle.

Therefore, if F is in 9-good position with no saddle disks, then $D_{F,W}$ and D are flype-related:

Lemma 4.10. *If F is in 9-good position and $F \cap C = v_F$, then every circle γ of $F \cap S_+$ is a flyping circle; thus $D_{F,W}$ is related to D by a sequence of flypes that preserve the isotopy class of W.*

Proof. Lemma 4.8 (B) implies that any sequence $F = F_0 \to \cdots \to F_r$ of Moves 1-10 uses only Move 10. Each Move 10 $F_i \to F_{i+1}$ fixes each circle of $F_i \cap S_+$ except the one it removes, and we may perform this sequence so that γ is the last remaining circle. Lemma 4.9 (C) now confirms the first claim. Lemma 3.7 then confirms the rest. □

5. Main results

We will show that 9-good position prohibits $F \cap C$ from containing saddle disks, i.e. forces $F \cap C = v_F$. Lemma 4.10 will then imply that $D_{F,W}$ and D are flype-related. The proof of the flyping theorem will then follow.

5.1. **Bad position.** Assuming by way of contradiction that F is in 9-good position and $F \cap C \neq v_F$, Lemma 3.20 implies that $F \cap W \setminus v_F$ is not isotopic in $W \setminus v_F$ into $\widehat{W}$; we will prove that there must then be an innermost circle γ_0 of $F \cap S_+$ such that, even after we perform Move 10 $F \to F'$ along γ_0, $F' \cap W \setminus v_{F'}$ still is not isotopic in $W \setminus v_{F'}$ into $\widehat{W}$. This will imply, however, that by performing Moves 1-10 such that each Move 10 proceeds along *such a circle* γ_0, we will never reach 10-good position, contradicting Proposition 4.4. This strategy motivates the following definition.

Definition 5.1. Say that F is in *bad position* if F is in 9-good position, $F \cap C \neq v_F$, and, after each possible Move 10 $F \to F'$, $F' \cap W \setminus v_{F'}$ is isotopic in $W \setminus v_{F'}$ into $\widehat{W}$.

Sublemma 5.2. *Suppose F is in bad position and γ_0 is an innermost circle of $F \cap S_+$. Then:*

- *(A) For every arc α_0 of $F \cap W \setminus v_F$, either α_0 is isotopic in $W \setminus v_F$ into $\widehat{W}$ or α_0 has an endpoint on γ_0;*
- *(B) Each arc α of $F \cap \widehat{W}$ has $\partial\alpha \subset \partial C$ or $\partial\alpha \subset \partial\nu L$ or lies on an innermost circle of $F \cap S_+$.*
- *(C) $\gamma_0 \cap \partial C \neq \varnothing$;*
- *(D) $|F \cap S_+| \geq 3$; and*

Proof. For (A), if α_0 is not isotopic in $W \setminus v_F$ into $\widehat{W}$, then the Move 10 along γ_0 must change α_0. Recalling Lemma 4.2 and Figure 27, this requires α_0 and γ_0 to intersect, which further requires α_0 to have an endpoint on γ_0. Part (A) implies (B).

For (C), if $\gamma_0 \cap \partial C \neq \varnothing$, then the Move 10 $F \to F'$ along γ_0 has no type I prisms, hence fixes every arc of $F \cap W$ that intersects v and, by Lemma 4.8 (B), leaves F' in 9-good position. This contradicts the assumption of bad position. Part (D) follows from (C), using Lemmas 3.4 (C) and 4.1 (A). □

Lemma 5.3. *F cannot be in bad position.*

Proof. Assume otherwise. Choose a circle γ_1 of $F \cap S_+$ and a disk X of $S_+ \setminus\setminus \gamma_1$ for which $\text{int}(X) \cap F = \gamma_0$ is a nonempty collection of innermost circles of $F \cap S_+$.[57] We claim that $\gamma_1 \cap C = \varnothing$. If not, take

[57]Here, Sublemma 5.2 (A) implies that the circles of $F \cap S_+$ are mutually nested and thus that γ_0 is a single innermost circle, but this is less clear in [Ki23b].

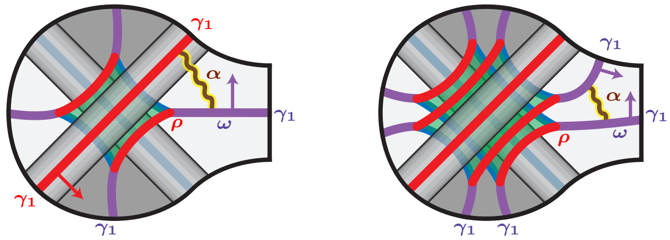


FIGURE 28. γ_0 and γ_1 near C_t in the proof of Lemma 5.3

an arc ω of $\gamma_1 \cap \widehat{W}$ incident to C, so that $\partial\omega \subset \partial C$ by Sublemma 5.2 (C)-(D). Consider the crossing ball C_t and arc ρ of $\gamma_1 \cap \partial C_t$, both incident to ω, for which an arrow pointing from ρ into X points toward the overpass at C_t. See Figure 28. Since $|\gamma_0 \cap \partial C_t| \leq |\gamma_0|$ by Lemma 4.1 (A), F admits a push-through move near C_t along an arc $\alpha \subset S_{+W}$, violating Lemma 3.21. This confirms that $\gamma_1 \cap C = \varnothing$.

Bad position requires γ_0 to intersect some disk C_s^+ of C^+, and γ_1 must traverse the overpass at C_s, due to Lemma 4.1 (A) and the fact that $\gamma_1 \cap \partial C = \varnothing$. Ergo, $|F \cap C_s| = 1$, contradicting Lemma 3.22. $\square$

Theorem 5.4. *If F is in 9-good position, then $F \cap C = v_F$. Hence, $F \cap S_+$ contains only flyping circles, so $D_{F,W}$ is related to D by a sequence of flypes (that preserve the isotopy class of W).*

Proof. By Lemma 4.10, it suffices to prove that $F \cap C = v_F$. Suppose instead that at least one arc of $F \cap W \setminus v_F$ intersects C; by Lemma 3.20, no such arc is isotopic in $W \setminus v_F$ into $\widehat{W} \cup v$. By Lemma 5.3 there is a Move 10 $F = F_0 \to F_1$ after which $F_1 \cap W \setminus v_{F_1}$ still is not isotopic in $W \setminus v_{F_1}$ into $\widehat{W} \cup v$. By Lemma 3.26, there is then a sequence $F_1 \to F_2$ of Moves 1-9 for which F_2 is in 9-good position, and by Lemmas 4.8 (A) and 3.25 (B), this sequence restricts to an isotopy $F_1 \cap W \to F_2 \cap W$ in W which fixes v_{F_1}. Thus, $F_2 \cap W \not\subset \widehat{W} \cup v$, so $F_2 \cap C \neq v_{F_2}$. Therefore, repeating this process gives an infinite sequence of Moves 1-10, contradicting Proposition 4.4. $\square$

5.2. **Proof of Tait's conjectures.** Using Convention 2.14 and the notation introduced there, we have:

Theorem 5.5 (Tait's flyping conjecture [MT91, MT93])**.** *Any two reduced alternating diagrams $D = D_{B,W}$ and $D' = D_{B',W'}$ of the same prime link $L \subset S^3$ are related by a sequence of flypes $D \to \cdots \to D'' \to \cdots \to D'$ in which $D \to \cdots \to D''$ preserves the isotopy class of W and $D'' \to \cdots \to D'$ preserves the isotopy class of B'.*

Proof. Denote $D'' = D_{B',W}$. Use Lemmas 3.3 and 3.26 to isotope B' into 9-good position relative to B, W; Theorem 5.4 gives the needed sequence $D \to D''$. Isotope W' into 9-good position relative to B', W; Theorem 5.4 gives the needed sequence $D'' \to D'$. $\square$

Since writhe is invariant under flypes (recall Observation 2.10) and additive under diagrammatic connect sum and disjoint union, we obtain a new geometric proof of Tait's second conjecture:

Theorem 5.6 (Tait's second conjecture [Gr17, M87ii, T88b])**.** *All reduced alternating diagrams of a given link $L \subset S^3$ have equal writhe.*

We again remark that Problems 2.17-2.19 remain open.

6. Proofs of technical lemmas from §2

It remains to prove several results from §§2-4. We prove those from §2 in this section, those from §3 in §7, and those from §4 in §8.

6.1. **Operations on definite surfaces.** We will prove Lemmas 2.25, 2.26, 2.27 and 2.29 and Theorem 2.30 in §6.2. First, in §6.1, we lay some groundwork.

Proposition 6.1. *If F_1 and F_2 are definite surfaces of the same sign, and $F = F_1 \natural F_2$, then F is definite and of the same sign.*

Proof. If G_i be a Goeritz matrix for F_i, $i = 1, 2$, then $G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}$ is a Goeritz matrix for F with $\sigma(G) = \sigma(G_1) + \sigma(G_2)$. $\square$

Proposition 6.2. *If S is a compact subsurface of a definite surface F and every component of $F \setminus S$ intersects ∂F, then S is definite.*[58]

Proof. We will prove that the map $j_* : H_1(S) \to H_1(F)$ induced by inclusion is injective. Let $g \in H_1(S)$ with $j_*(g) = 0 \in H_1(F)$. Choose an oriented multicurve $\gamma \subset \text{int}(S)$ representing g. Then $\gamma = \partial F'$ for some orientable subsurface $F' \subset F$. If $F' \subset S$, then $g = 0 \in H_1(S)$ and we are done. If not, then F' intersects a component F_1 of $F \setminus S$; in fact, $F' \supset F_1$, because $\gamma \subset S$. This gives the following contradiction:

$$\varnothing = \partial F' \setminus \gamma = F' \cap \partial F \supset F_1 \cap \partial F \neq \varnothing. \qquad \square$$

In particular, Proposition 6.2 immediately implies:

Sublemma 6.3. *If α is a system of disjoint properly embedded arcs in a definite surface F, then $F \setminus \mathring{\nu}\alpha$ is definite.*

Next, consider the operation of *adding (half) twists*, shown in Figure 29. It works like this. Let F be a spanning surface for a link L, $\alpha \subset F$ a properly embedded arc, and m an integer. Let A be an unknotted annulus or möbius band whose core circle has framing $\frac{m}{2}$, and let $\alpha' \subset A$ be a co-core. Construct $F \natural A$ in such a way that α and

[58]This extends Lemma 3.3 of [Gr17]: If S is a compact subsurface of a definite surface F and ∂S is connected, then S is definite.

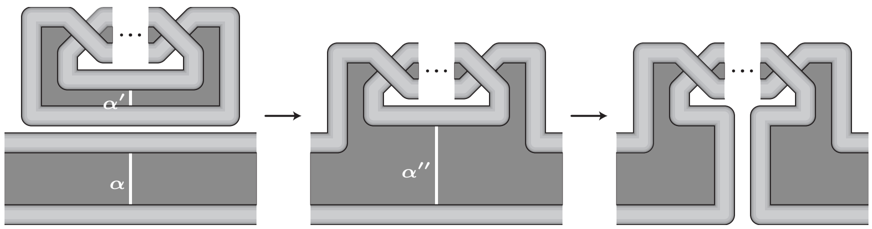


FIGURE 29. Adding twists to a spanning surface

α' are glued at their endpoints to form an arc $\alpha'' \subset F\natural A$. Depending on the sign of m, the surface $F' = (F\natural A) \setminus \mathring{\nu}\alpha''$ is said to be obtained from F by *adding* $\left|\frac{m}{2}\right|$ *positive or negative twists along* α.

Proposition 6.4. *If F' is obtained by adding positive twists to a positive-definite surface F, then F' is positive-definite.*[59]

Indeed, if G is a positive-definite symmetric matrix and G' is obtained by increasing a diagonal entry of G, then G' is also positive-definite. Alternatively, here is a geometric proof:

Proof. Let A be an unknotted annulus or möbius band with m half-twists for some $m > 0$. Then A is also positive-definite, as are $F\natural A$ and F', by Proposition 6.1 and Sublemma 6.3. □

Proposition 6.5. *Suppose $F_\pm$ are definite surfaces of opposite signs spanning a link L and α is a non-standard arc of $F_+ \cap F_-$. Denote $F'_+ = F_+ \setminus \mathring{\nu}\alpha$, $L' = \partial F'_+$, and $F'_- = F_- \setminus \mathring{\nu}\alpha$. Then the following are equivalent:*

(I) α is separating on F_+;
(II) α is separating on F_-;
(III) L' has one more split component than L.

Proof. Sublemma 6.3 implies that F'_+ and F'_- are definite spanning surfaces of opposite sign, and both span L' because α is non-standard (see Figure 31, bottom), so L' is alternating by the first part of Fact 2.15. The conclusion now follows from the last part of Fact 2.15. □

Proposition 6.6. *A positive-definite surface F spanning a prime alternating link L is essential if and only if every nonzero $a \in H_1(F)$ satisfies $\langle a, a\rangle \geq 2$.*

Proof. Take an essential negative-definite spanning surface W for L, and let $D = D_{F,W}$. If D is reduced, then both conditions are satisfied, the first by Fact 2.7 and the second by Corollary 5.2 of [Gr17].[60] Conversely, if D has a nugatory crossing c, then, since W

[59]Likewise for adding negative twists to a negative-definite surface.

[60]The proof of Lemma 4 of [Ki23a] gives an alternate, self-contained proof that the second condition holds.

figures/ArcIsotopy1.{ps,eps,pdf} not found (or no BBox)
figures/ArcIsotopy2.{ps,eps,pdf} not found (or no BBox)

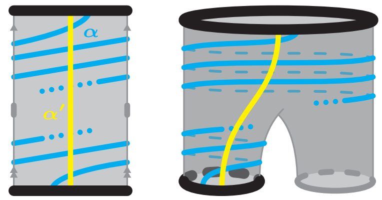


FIGURE 30. Left: options for $Y \subset (I \times I) \setminus\setminus (A \cup V)$. Right: transverse, isotopic arcs α, α' cutting off no bigon lie in a pair of pants.

is essential, c is incident to distinct disks of $W \setminus\setminus F$, hence to a single disk of $F \setminus\setminus W$, and so neither condition is satisfied. $\square$

Proposition 6.7. *Let F be a positive-definite surface spanning a prime alternating link L, and let $\alpha \subset F$ be a properly embedded arc such that $F' = F \setminus \mathring{\nu}\alpha$ spans a prime alternating link L'. If F is essential, then F' is also essential.*

Proof. By Sublemma 6.3, F' is positive-definite. By Proposition 6.6, all nonzero $c \in H_1(F)$ satisfy $\langle c, c \rangle \geq 2$; thus, so do all nonzero $c \in H_1(F')$. Ergo, by Proposition 6.6 (as L' is prime and alternating), F' is essential. $\square$

6.2. **How definite surfaces of opposite signs intersect.**

Proposition 6.8. *After one completes Procedure 2.24, each component α of $F_+ \cap F_-$ is an arc with $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} = +2$.*[61]

Proof. Procedure 2.24 (1) removes all circles of $F_+ \cap F_-$, and (2) and (3) ensure that any remaining points $x, y \in \partial F_+ \cap \partial F_-$ on the same component $\partial\nu L_i$ of $\partial\nu L$ have the same sign, $i(\partial F_+, \partial F_-)_{\nu x} = i(\partial F_+, \partial F_-)_{\nu y}$. This sign must be positive, since definiteness gives:

$$\left| \partial F_+ \cap \partial\nu L_i \right| \geq 0 \geq \left| \partial F_- \cap \partial\nu L_i \right| . \qquad \square$$

Proposition 6.9. *If $F_\pm$ are definite surfaces of opposite signs spanning a link L and α is an arc of $F_+ \cap F_-$ that is ∂-parallel in both F_+ and F_-, then α is non-standard.*

Proof. Procedure 2.24 eventually removes α via move (2), and just before it does, α is non-standard, but none of the prior moves in the construction change α, so α is non-standard initially too. $\square$

[61]Procedure 2.24 terminates, as (1)-(3) all decrease $|F_+ \cap F_-| + |\partial F_+ \cap \partial F_-|$.

Proof of Lemma 2.25. Let $0 < \varepsilon \ll 1$ and take a proper isotopy $f_t : I \to X \setminus w$, $-\varepsilon \leq t \leq 1+\varepsilon$, such that, denoting each $f_t(I) = \alpha_t$, we have $\alpha_0 = u_1$ and $\alpha_1 = v_1$. Denote $f : I \times [-\varepsilon, 1+\varepsilon] \to X$ where each restriction $f|_{I\times\{t\}} \equiv f_t$. Assume that f is generic in the sense that $f^{-1}(u_1) = A'$ and $f^{-1}(v_1) = V'$ are 1-submanifolds of $I \times [-\varepsilon, 1+\varepsilon]$ with $A' \pitchfork V'$. Denote $A = A' \cap (I \times (0,1])$ and $V = V' \cap (I \times [0,1))$, let A_H and V_H denote the set of points in A and V with horizontal tangent lines, assume that f has been chosen (subject to the preceding requirements) to minimize the lexicographical quantity $(|A|+|V|, |A_H|+|V_H|)$. Then A (resp. V) is comprised of arcs, each with at least one endpoint on $I \times \{1\}$ (resp. $I \times \{0\}$), and A_H (resp. V_H) consists of one point on each arc of A (resp. V) whose endpoints both lie on $I \times \{1\}$ (resp. $I \times \{0\}$). Taking outermost disks carefully twice gives a disk Y of $(I \times I) \setminus\setminus (A \cup V)$ with $|\partial Y \cap A'| = 1 = |\partial Y \cap V'|$ (see Figure 30, left). Setting $X_0 = f(Y)$ then confirms (A); this implies (B). The existence part of (C) follows by induction on $|u_1 \cap v_1|$, using (B) (see Figure 30, right); uniqueness follows from the assumption that no arc of v is ∂-parallel. $\square$

Proof of Lemma 2.26. Assume that the arcs of u and v are indexed so that the isotopy from $u \setminus w$ to $v \setminus w$ in $F \setminus w$ sends each u_i to v_i. Suppose by way of contradiction that $u \neq v$. Choose an arc u_1 of $u \setminus w$. Lemma 2.25 (A) provides a compact disk X_1 of $(X \setminus w) \setminus\setminus (u_1 \cup v_1)$ with $|\partial X_1 \cap u_1| = 1 = |\partial X_1 \cap v_1|$. Since $X_1 \subset X \setminus w$ is compact, (2.2) implies that $u \cap \operatorname{int}(X_1) = \varnothing$ and, taking a disk X_0 of $X_1 \setminus\setminus v$ with $|\partial X_0 \cap u| = 1 = |\partial X_0 \cap v|$, that X appears near X_0 as in Figure 10 with $u_1 \equiv u_2$. In particular, X_1 is not a bigon, nor is any disk of $X \setminus\setminus (u_1 \cup v_1)$. Further, the arcs labeled u_2 and v_2 in the figure must correspond under the isotopy in $X \setminus w$, so *both* $u_1 \equiv u_2$ *and* $v_1 \equiv v_2$. Denote $x \in \partial u_2 \equiv \partial u_1$, $y \in \partial v_2 \equiv \partial v_1$, and $\lambda_0, \lambda_1 \subset \partial X$ as in Figure 10. Since no disk of $X \setminus\setminus (u_1 \cup v_1)$ is a bigon, Lemma 2.25 (C) implies that x abuts a compact disk X_2 of $(X \setminus w) \setminus\setminus (u_1 \cup v_1)$ with $|\partial X_2 \cap u_1| = 1 = |\partial X_2 \cap v_1|$. Hence, $\lambda_0 \subset \partial X_2$. Yet, $\lambda_0 \not\subset \partial X_0$, so $X_0 \neq X_2$, violating the uniqueness in Lemma 2.25 (C) at y. $\square$

Proof of Lemma 2.27. Apply moves (1)-(2) of Procedure 2.24 to F_+ and F_- until neither move is possible. Either this fixes F_+ and F_- near α, or it removes α. In the latter case, α was ∂-parallel in both F_+ and F_-, so $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} = 0$ by Proposition 6.9, confirming the first claim; the second and third claims then hold vacuously.

Instead, we may assume for the rest of the proof that F_+ and F_- admit neither move (1)-(2) of Procedure 2.24. Denote $F'_+ = F_+ \setminus \mathring{\nu}\alpha$ and $\partial F'_+ = L'$. Then F'_+ is positive-definite with $\beta_1(F_+) - |F_+| = \beta_1(F'_+) + 1 - |F'_+|$ by Sublemma 6.3 and Observation 2.2.

Suppose, contrary to (A), that $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} = -2$. Construct a surface F'_- by adding one negative half-twist to F_- along α; see

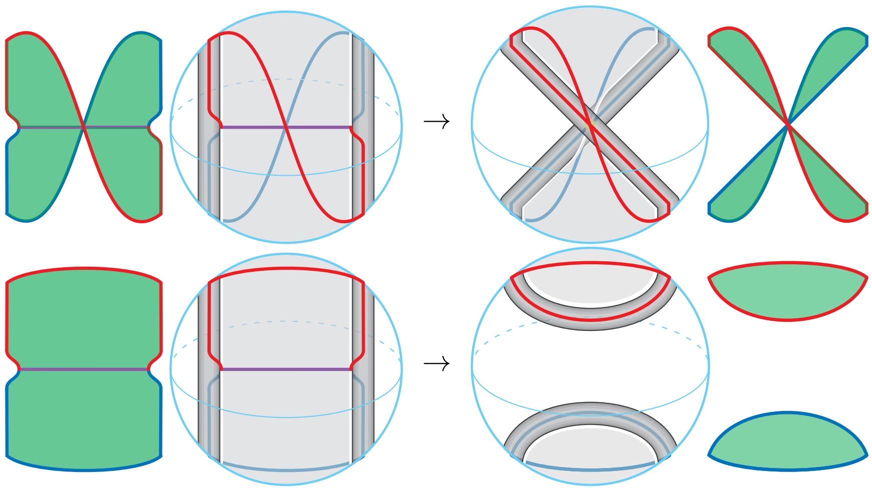


FIGURE 31. A positive-definite surface F_+ cannot intersect a negative-definite surface F_- along an arc α with $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} = -2$ nor along a nonseparating arc α with $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} = 0$.

Figure 31, top. Then F'_- also spans L' with $\beta_1(F'_-) = \beta_1(F_-)$, and F'_- is negative-definite by Proposition 6.4; hence, L' is alternating, by Fact 2.15. Moreover, since $|F'_-| = |F_-|$, Proposition 6.5 implies that $|F_+| = |F'_+|$, hence $\beta_1(F_+) = \beta_1(F'_+) + 1$, and thus:

$$\begin{aligned} s(F'_+) - s(F'_-) &= s(F_+) - s(F_-) + 2 && \text{using (2.1)} \\ &= 2(\beta_1(F_+) + \beta_1(F_-)) + 2 && \text{by Prop. 2.12} \\ &= 2(\beta_1(F'_+) + \beta_1(F'_-)) + 4. \end{aligned} \tag{6.1}$$

This contradicts Proposition 2.12.

For (B), assume by way of contradiction that α is nonseparating on F_- and $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} = 0$. The argument here is identical to the first case, except that we define $F'_- = F_- \setminus \mathring{\nu}\alpha$ (see Figure 31, bottom). The assumption that $|F'_-| = |F_-|$ then gives $\beta_1(F'_-) = \beta_1(F_-) - 1$, which again contradicts Proposition 2.12:

$$\begin{aligned} s(F'_+) - s(F'_-) &= s(F_+) - s(F_-) \\ &= 2(\beta_1(F_+) + \beta_1(F_-)) \\ &= 2(\beta_1(F'_+) + \beta_1(F'_-)) + 2. \end{aligned} \tag{6.2}$$

For (C), assume for contradiction that that L is prime (hence nonsplit), $F_\pm$ are essential, $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} \neq 2$, and α is not ∂-parallel in both $F_\pm$. Part (A) implies that $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} = 0$. Hence, by Proposition 6.8, when we apply Procedure 2.24 to $F_\pm$ until it terminates, the resulting sequence $F_+ = F_0 \to F_1 \to \cdots \to F_t$

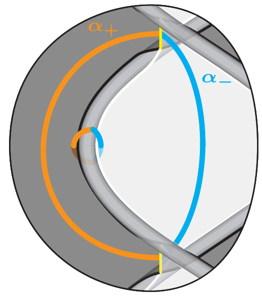


FIGURE 32. If arcs $\alpha_\pm \subset F_\pm$ with $\partial\alpha_+ = \partial\alpha_- \subset F_+ \cap F_-$ are not isotopic in $F_\pm$ to $F_+ \cap F_-$, then $\alpha_+ \cup \alpha_-$ is isotopic in $S^3 \setminus \mathring{\nu}L$ to a meridian on $\partial\nu L$.

features move (3) at least once. Consider the *last* move (3) $F_s \to F_{s+1}$ in this sequence. Observe that the following property holds for $i = t$ (because F_t, F_- determine an alternating link diagram, by Proposition 2.22, and *this diagram is prime* by Theorem 1 (b) of [Me84]) and therefore holds for *all* $i = s+1, \dots, t$ (since moves (1) and (2) from Procedure 2.24 do not affect this property):

(6.3) Each arc in $F_-\backslash\backslash F_i$ that separates F_- is ∂-parallel in F_-.

The step $F_s \to F_{s+1}$ involves two arcs α_1, α_2 of $F_s \cap F_-$ and one arc α_3 of $F_{s+1} \cap F_-$. The first two parts of this lemma imply without loss of generality that α_1 is non-standard and thus separating in F_-. Perturb α_1 in F_- so that it is disjoint from F_{s+1}. Then $\alpha_1 \subset F_-\backslash\backslash F_{s+1}$ is separating on F_-, hence ∂-parallel in F_- by (6.3), but this contradicts the hierarchy of the moves in Procedure 2.24. □

Proof of Lemma 2.29. Fact 2.15 implies that L is alternating. Since L is also nonsplit, both $F_\pm$ are connected by Fact 2.4. Moreover, Lemma 2.27 (A) implies that every arc α of $F_+ \cap F_-$, being standard, satisfies $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} = +2$. Thus, by Proposition 2.22, the pair $F_\pm$ determines a connected alternating diagram D of L, which is prime by Theorem 1 (b) of [Me84].

Note that each component of each $F_\pm\backslash\backslash F_\mp$ is a disk, corresponding to a checkerboard region of $S^2\backslash\backslash D$. Thus, if the endpoints of $\alpha_\pm$ lie on the same arc of $F_+ \cap F_-$, then each $\alpha_\pm$ is parallel in $F_\pm\backslash\backslash F_\mp$ to this arc. Assume instead that the endpoints of $\alpha_\pm$ lie on distinct arcs of $F_+ \cap F_-$. Denote the disks of $F_\pm\backslash\backslash F_\mp$ containing $\alpha_\pm$ by $X_\pm$. Then X_+ and X_- correspond to two oppositely colored disks of $S^2\backslash\backslash D$, and since D is prime these disks meet in at most one edge hence at most two crossings: $X_+ \cap X_- = v_0 \cup v_1$. Therefore, as shown in Figure 32, $\alpha_+ \cup \alpha_-$ is isotopic in $S^3 \setminus \mathring{\nu}L$ to a meridian on $\partial\nu L$, contrary to the assumption that α_+ and α_- are parallel in $S^3 \setminus \mathring{\nu}L$. □

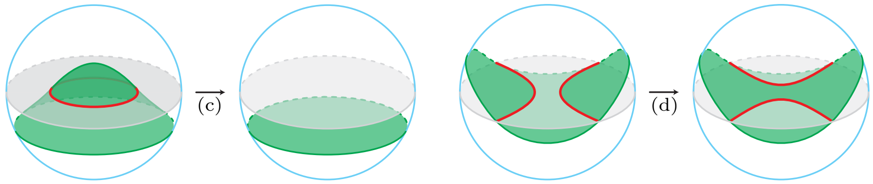


FIGURE 33. (c) and (d) in the proof of Proposition 6.11

Fact 2.28 and Lemma 2.29 imply:

Fact 6.10. *If $F_\pm$ are essential definite surfaces of opposite signs spanning a prime link L and $\alpha_\pm \subset F_\pm \backslash\backslash F_\mp$ are arcs which are parallel in $S^3 \setminus \mathring{\nu} L$ and whose endpoints lie on distinct components of $F_+ \cap F_-$, then at most one of these endpoints lies on a standard arc of $F_+ \cap F_-$.*

Proposition 6.11. *Suppose F_- is an essential negative-definite surface spanning a prime link L and $f_t : F_+ \to S^3 \setminus \mathring{\nu} L$, $t \in I$, is an isotopy of essential positive-definite spanning surfaces for L. Denote each $f_t(F_+) = F_t$. Assume generically that $F_t \pitchfork F_-$ for all but finitely many $t = t_1, \dots, t_r$, where $0 = t_0 < t_1 < \cdots < t_r < t_{r+1} = 1$, that there is only one non-transverse point p_i in each $F_{t_i} \cap F_-$, and that each p_i is non-degenerate. For each $t \neq t_1, \dots, t_r$, denote the union of the standard arcs of $F_t \cap F_-$ by st_{F_t}. Then st_{F_0} and st_{F_1} are isotopic in F_-.*

Proof. Choose some positive $\varepsilon \ll \min\{t_{i+1} - t_i\}_{i=1}^{r+1}$. Near each point $(p_i, t_i) \in (S^3 \setminus \mathring{\nu} L) \times (0, 1)$, f_t changes $F_{t_i - \varepsilon} \cap F_-$ to $F_{t_i+\varepsilon} \cap F_-$ via one of the following moves or its inverse:

(1) removing a simple closed curve (Figure 33, left);
(2) removing an arc that is ∂-parallel in both $F_\pm$ (Figure 8, top);
(3) merging two arcs near $\partial \nu L$ (Figure 8, bottom);
(4) (the sort of "saddle point" shown right in Figure 33).

We must check that each of these gives an isotopy in F_- from $st_{F_{t_i-\varepsilon}}$ to $st_{F_{t_i+\varepsilon}}$. For (1) this is trivial; likewise for (2), using Proposition 6.9. For (3), the two endpoints involved have opposite signs, so at least one of the un-merged arcs is non-standard, hence ∂-parallel in F_- by Lemma 2.27 (C); hence, the other un-merged arc is isotopic in F_- to the merged arc, and the former is standard if and only if the latter is.

For (4), let $U \subset S^3 \setminus \mathring{\nu} L$ denote the local neighborhood shown right in Figure 33. Note that the arcs of $F_t \cap F_- \cap U$ lie on distinct arcs of $F_t \cap F_-$ either for both $t = t_i \pm \varepsilon$ or for neither. In the former case, Fact 6.10 and Lemma 2.27 (C) imply, for both $t = t_i \pm \varepsilon$, that at least one of these arcs of $F_t \cap F_-$ is non-standard and thus ∂-parallel in F_-; hence, the second arcs of $F_{t_i \pm \varepsilon} \cap F_-$ that intersect U are isotopic in F_- to each other. In the latter case, this move either creates or removes a simple closed curve of $F_{t_i\pm\varepsilon}\cap F_-$. By Fact 2.23, this curve bounds a disk $X\subset F_-$, which guides the needed isotopy. □

Proof of Theorem 2.30. The forward implication is straightforward. For the converse, apply Procedure 2.24 to B' and W to get an isotopy $B'\to B''$ in $S^3\setminus\mathring{\nu}L$ after which $B''\cap W$ consists only of standard arcs. Proposition 6.11 gives an isotopy $f: B\cap W\to B''\cap W$ in W, and since W cuts B and B'' into disks, f extends to an isotopy $B\cup W\to B''\cup W$ in $S^3\setminus\mathring{\nu}L$. Remark 2.21 and Fact 2.28 imply that the pairs B,W (and B'',W) and B',W determine equivalent reduced alternating diagrams of L: $D=D_{B,W}\equiv D_{B',W}$. The same reasoning shows that $D_{B',W}\equiv D_{B',W'}=D'$, so $D\equiv D'$. □

7. Proofs of technical lemmas from §3

In §7, we adopt all setup from §3.1. We will prove Lemmas 3.3, 3.4, and 3.7 in §7.1, Lemmas 3.16 and 3.19 in §7.2, Lemmas 3.20 and 3.21 in §7.3, and Lemmas 3.23, 3.24, 3.25, and 3.26 in §7.4.

7.1. **Fair position.**

Proof of Lemma 3.3. Applying Procedure 2.24 to F,W gives (a). Perturbing F generically relative to B,W while fixing v_F and taking C to be a thin regular neighborhood of v in $S^3\setminus\setminus\mathring{\nu}L$ as described in Remark 3.1 gives (b)-(f), and adjusting F near C gives (g) also.

One may then isotope F as follows, while preserving (a)-(g), until $S_+\cup S_-$ cuts F into disks. If $S_+\cup S_-$ does not cut F into disks, then by a standard innermost circle argument, there is a circle $\gamma\subset F\setminus(S_+\cup S_-)$ that bounds a disk $X\subset(S^3\setminus(\nu L\cup S_+\cup S_-))\setminus\setminus F$ but bounds no disk in $F\setminus(S_+\cup S_-)$.[62] Since F is incompressible, γ bounds a disk $F_0\subset F$, and since L is nonsplit and $\mathrm{int}(X)\cap F=\varnothing$, the 2-sphere $X\cup F_0$ bounds a ball Y in $(S^3\setminus\nu L)\setminus\setminus F$. Isotope F near F_0 through Y past X. This isotopy fixes $(F\setminus F_0)\cap(S_+\cup S_-)$ and removes all of $F_0\cap(S_+\cup S_-)\neq\varnothing$, hence preserves (a)-(g) and decreases $|F\cap(S_+\cup S_-)|$. Ergo, any sequence of such moves terminates, and when it does, F is in fair position. □

Proof of Lemma 3.4. By Definition 3.2 (h), F intersects $C\setminus\mathring{\nu}L$ in disks, hence cuts it into balls; likewise with $H_\pm$. This proves (A).

For (B), each component of $\partial F\cap S_\pm$ is an arc because D is prime, hence nontrivial and connected; and no component γ of $F\cap S_0$ nor

[62] Choose a component X' of $F\setminus\setminus(S_+\cup S_-)$ that is not a disk; then choose any component of $\partial X'$ and take a parallel copy γ' of it in $\mathrm{int}(X')$. Note that γ' bounds no disk in X'. Yet, γ' does bound a disk Z in $S^3\setminus(S_+\cup S_-\cup\nu L)$, and γ' is 0-framed in F, so we may require that $Z\pitchfork F$ is comprised of circles, no arcs. Among all such choices for Z (given γ'), choose one which minimizes $|Z\cap F|$. Now choose an innermost disk $X\subset Z\setminus\setminus F$ and take $\partial X=\gamma$.

$F \cap \partial C \cap S_\pm$ is a circle, or else, by (h), γ would bound disks in F in both incident components of $S^3 \backslash\backslash (S_+ \cup S_- \cup \nu L)$, but F being a spanning surface, has no closed components.

For (C), consider a crossing ball C_t where F does not have a crossing band, and let γ be a component of $F \cap \partial C_t$. By (d), $\partial F \cap C_t = \varnothing$, so γ is a circle; (B) and (e) imply that ∂S_0 cuts γ into arcs, each of whose endpoints are on distinct arcs of $\partial C_t \cap S_0$. Since each disk of $\partial C_t \cap S_\pm$ contains only two arcs of $\partial C_t \cap S_0$, γ is uniquely determined up to isotopy of $(\gamma, \gamma \cap \partial S_0)$ in $(\partial C_t \setminus \nu L, \partial S_0)$. In particular, by (h), γ bounds a saddle disk of $F \cap C_t$. □

Proof of Lemma 3.7. Ordering the r circles of $F \cap S_+$ arbitrarily gives a sequence of flype-type re-plumbings $F = F_0 \to F_1 \to \cdots \to F_r$ where F_r is disjoint from S_+, hence (by fair position) isotopic to B. Theorem 2.30 implies that $D_{F_r,W} \equiv D$. Putting the sequence in reverse, each F_i is obtained by re-plumbing F_{i+1} along a flyping cap (relative to W), so by Proposition 2.35 each $D_{F_i,W}$ is related to $D_{F_{i+1},W}$ by a flype which preserves the isotopy class of W. Ergo, $D_{F,W}$ and D are related by a sequence of such flypes. □

7.2. **Properties of 1-, 2-, and 3-good position.**

Proposition 7.1. *If F is in fair position and no arc of $F \cap \widehat{W}$ is parallel in $\widehat{W}$ into ∂C, then no arc of $F \cap \widehat{B}$ is parallel in $\widehat{B}$ into ∂C.*

Proof. Assume instead that some arc β of $F \cap \widehat{B}$ is parallel in $\widehat{B}$ into ∂C. Taking β to be an outermost such arc in $\widehat{B}$, let γ denote the circle of $F \cap S_+$ containing β, and let ω, ω' denote the arcs of $\gamma \cap \widehat{W}$ incident to the arcs of $\gamma \cap C$ that are incident to β; see Figure 34, left. Construct properly embedded arcs $\sigma \subset \widehat{W} \backslash\backslash F$ and $\sigma_+ \subset F_\gamma$ with the same endpoints, one of each of ω, ω'. Then σ and σ_+ are parallel in $S^3 \setminus \mathring{\nu} L$, so Lemma 2.29 implies that σ is parallel through a disk $W_0 \subset W \backslash\backslash F$ to $F \cap W$. The disk W_0 must intersect v because $\omega \neq \omega'$. Consider an outermost disk W_1 of $W_0 \backslash\backslash v$: the arc $\alpha = \partial W_1 \cap \partial W_0$ is an arc of $F \cap W \backslash\backslash v$ which is parallel in W into v, so contrary to assumption $\alpha \cap \widehat{W}$ is parallel in $\widehat{W}$ into ∂C. □

Proposition 7.2. *Suppose F is in fair position and no arc of $F \cap W \backslash\backslash v$ is parallel in W into v. If $X \subset S^3 \backslash\backslash (F \cup \nu L)$ is a properly embedded disk such that $\partial X \subset F \setminus C$ intersects S_0 in a nonempty collection of points on mutually distinct arcs of $F \cap S_0$, then ∂X intersects both B and W.*

Proof. Denote $\partial X = \gamma$ and assume that $X \pitchfork B, W$. Incompressibility implies that γ bounds a disk $F_0 \subset \text{int}(F)$. If $\gamma \cap W = \varnothing$, then

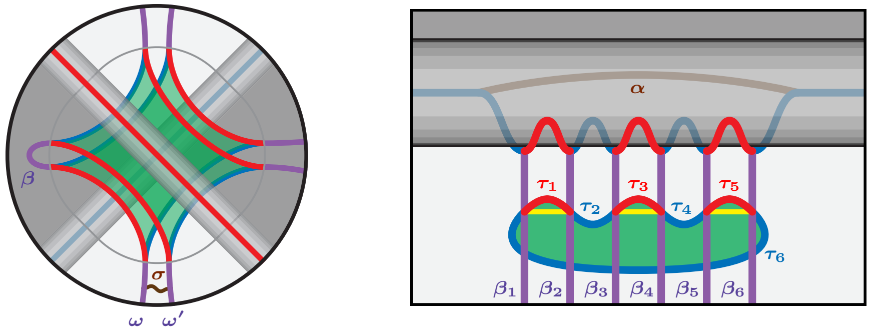


FIGURE 34. The situations in the proofs of Propositions 7.1 and 7.4

$F_0 \cap W$ is nonempty[63] and comprised of circles, violating Definition 3.2 (a). Assume instead that $\gamma \cap B = \varnothing$. Then $F_0 \cap B$ is nonempty and comprised of circles. Choose an innermost disk F_1 of $F_0 \backslash\backslash B$ in F_0. Lemma 3.4 (B) implies that $F_1 \cap v \neq \varnothing$; choose an outermost disk F_2 of $F_1 \backslash\backslash v$. Then $\partial F_2 \cap \partial F_1$ is an arc of $F \cap B \backslash\backslash v$ with both endpoints on the same vertical arc, violating Proposition 7.1. □

Proof of Lemma 3.16. The contrapositive of (III) $\implies$ (I) is clear, as is (I) $\iff$ (II), by fair position. Finally, if (I) and (II) hold, then these prohibit Move 2 and Proposition 7.2 prohibits Move 1. □

Proof of Lemma 3.17. Suppose otherwise. Then, using Definition 3.2 (a) to apply Lemma 2.29, α is parallel through a disk $W_0 \subset W \backslash\backslash F$ to an arc $\omega \subset F \cap W$. Condition (e) of Definition 3.10 implies that $\omega \cap v \neq \varnothing$. Taking an outermost disk $W_1 \subset W_0 \backslash\backslash F$, ∂W_1 consists of an arc of $F \cap W \backslash\backslash v$ and an arc in v which are parallel through $W \backslash\backslash v$. This violates the 2-good position of F, due to Lemma 3.16. □

Proposition 7.3. *If F is in 2-good position and $F \to F'$ is a push-through move, then F' is in 2-good position.*

Proof. By Observation 3.15, F' is in fair position. By Lemma 3.16, no arc of $F \cap \widehat{W}$ is parallel in $\widehat{W}$ into ∂C, and it suffices to prove that the same holds for F'. This is clear if the arc α guiding the push-through move lies in $S_{\pm B}$ (as $F' \cap \widehat{W} = F \cap \widehat{W}$) or has at least one endpoint on $\partial \nu L$ (as all arcs of $(F' \cap \widehat{W}) \setminus (F \cap \widehat{W})$ have an endpoint on $\partial \nu L$), and Lemma 3.17 implies that $\alpha \not\subset \widehat{W}$. □

Proposition 7.4. *Suppose F is in 3-good position, E is an edge, γ is a circle of $F \cap S_\pm$, and $\alpha \subset int(E_\pm) \backslash\backslash \partial F$ is an arc with $\partial \alpha \subset \gamma$, so that (by Proposition 2.6) α is parallel in $E \backslash\backslash \partial F$ to an arc $\alpha' \subset \partial F$. Then either α' intersects both ∂B and ∂W or it intersects neither.*

[63]Otherwise, each arc α of $F_0 \cap B$ would lie in a single arc of $F_0 \cap S_0$, which would contain both endpoints of α, contrary to assumption.

Proof. Assume by way of contradiction that $\alpha \subset S_-$, $\alpha' \cap \partial W \neq \varnothing$, and $\alpha' \cap \partial B = \varnothing$; the proofs with $\alpha \subset S_+$ and with ∂B and ∂W reversed are analogous. Denote the arcs of $F \cap S_0$ incident to α' by $\beta_1, \dots, \beta_{2m}$, indexed by their order along α' as in Figure 34, right, and note that $\beta_1, \dots, \beta_{2m}$ are distinct, because F admits no Move 3. For each $i = 1, \dots, 2m$, construct a properly embedded arc τ_i in the disk F_i of $F \cap H_\pm$ incident to β_{i-1} and β_i, taking indices modulo $2m$; do this so that each τ_i shares an endpoint with each $\tau_{i\pm1}$. The circle $\tau = \bigcup_i \tau_i \subset F$ bounds a disk $X \subset S^3 \backslash\backslash (F \cup \nu L)$ disjoint from B; yet, $\tau \cap S_0$ consists of one point on each of the mutually disjoint arcs $\beta_1, \dots, \beta_{2m}$, contradicting Proposition 7.2. $\square$

Proof of Lemma 3.19. One direction is trivial. For the other, suppose F is in 3-good position, but such an arc exists; choose one, β, which is outermost in $\widehat{B}$. Then β is parallel in $S_0 \backslash\backslash F$ to an arc α of $\partial B \backslash\backslash \partial F$, and $\partial\alpha$ are the endpoints of an arc $\alpha' \subset \partial F \cap E$. Denoting $\alpha'' = \alpha' \setminus \mathring{\nu}\partial\alpha$, $\alpha'' \cap \partial B = \varnothing$, as β is outermost, but $\alpha'' \cap \partial W \neq \varnothing$, as F admits no Move 3. This contradicts Proposition 7.4.[64] $\square$

7.3. **Properties of 5-good position.**

Sublemma 7.5. *If F is in 5-good position, then no arc of $F \cap \widehat{W}$ has endpoints on a crossing ball C_t and incident edge E.*

Proof. Suppose otherwise. Then there is an arc α of $F \cap W$ for which some arc α_0 of $\alpha \backslash\backslash v$ cuts off a triangle of $W \backslash\backslash (F \cup v)$. Denote $\partial\alpha_0 = \{x, y\}$ where $x \in v_t$ and $y \in E$. Since no Move 4 is possible, the arc λ of $\partial F \cap E \backslash\backslash \{y\}$ incident to C_t must intersect ∂S_0. Moreover, $\text{int}(\lambda) \cap \partial S_0 \subset \partial B$ (because α cuts off a triangle), and Definition 3.2 (a) gives $i(\partial F, \partial W)_{\nu y} = +1$, which implies that $|\text{int}(\lambda) \cap \partial S_0| \geq 2$ (compare with Figure 23). Ergo, contrary to assumption, F admits Move 5 between y and C_t. $\square$

Proposition 7.6. *If a properly embedded arc $\alpha' \subset W$ with $\alpha' \pitchfork v \neq \varnothing$ is isotopic in W to an arc $\alpha \subset \widehat{W}$, then some arc α_0' of $\alpha' \backslash\backslash v$ cuts off a bigon or triangle of $W \backslash\backslash (v \cup \alpha')$.*[65]

Proof. Isotope $(\alpha, \partial\alpha)$ in $(\widehat{W}, \partial\widehat{W} \cap \partial W)$ to minimize $|\alpha \pitchfork \alpha'|$. Now by Lemma 2.25 (A), there is a disk W_0 of $W \backslash\backslash (\alpha \cup \alpha')$ such that $\partial W_0 \cap \alpha$ and $\partial W_0 \cap \alpha'$ each consist of a single arc. The minimality of $\alpha \cap \alpha'$ and the assumption that $\alpha' \cap v \neq \varnothing$ imply that $W_0 \cap v \neq \varnothing$; since $\alpha \cap v = \varnothing$ it follows that there is an outermost disk W_1 of $W_0 \backslash\backslash v$ with $\partial W_1 \cap \alpha = \varnothing$. Take $\alpha_0' = \partial W_1 \cap \alpha'$. $\square$

---

[64] Alternatively, this contradicts Definition 3.2 (a) directly, since $i(\alpha'', \partial W) = 0$. We will actually *need* to use Proposition 7.4 in the proof of Proposition 7.7.

[65] That is, one endpoint of α_0' lies on a vertical arc $v_0 \subset v$ and the other lies either on v_0 or on an arc of $\partial W \backslash\backslash \partial v$ incident to v_0.

Proof of Lemma 3.20. By Proposition 7.6, either $\alpha' \subset \widehat{W}$ or an arc of $\alpha' \cap \widehat{W}$ has a form prohibited by Lemma 3.16 or Sublemma 7.5. $\square$

Proposition 7.7. *If F is in 5-good position, then no circle γ of $F \cap S_\pm$ intersects any edge E in more than one arc.*

Proof. Suppose otherwise. Then there is an arc $\alpha \subset S_{\pm E} \backslash\backslash \partial F$ whose endpoints lie on distinct arcs of $\gamma \cap E$. Proposition 2.6 implies that α is parallel through a disk $E_0 \subset E$ into ∂F. By assumption, E_0 must intersect ∂B or ∂W, so Proposition 7.4 implies that $E_0 \cap \partial W \neq \varnothing$; yet, the endpoints of any outermost arc of $E_0 \cap \partial W$ are points of $\partial F \cap \partial W$ of opposite sign, violating Definition 3.2 (a). $\square$

Proof of Lemma 3.21. Assume for simplicity that the circle $\gamma \subset F \cap S_\pm$ that contains $\partial\alpha$ lies in $F \cap S_+$, and assume for contradiction that $\alpha \subset S_{+W}$. Lemma 3.17 implies that $\partial\alpha \not\subset \widehat{W}$, while Definitions 3.10 (e)-(f) and 3.2 (a) imply that $\partial\alpha \not\subset \partial\nu L$. Hence, one endpoint of α lies on an arc γ' of $\gamma \cap \partial\nu L$, while the other endpoint lies on an arc γ'' of $\gamma \cap \widehat{W}$; see Figure 35, left.

The push-through move $F \to F'$ along α introduces two oppositely signed points $x_\pm$ of $\partial F' \cap \partial W$, and Lemma 2.27 (C) implies that the negative point x_- is an endpoint of an arc ω of $F' \cap W$ that cuts off a disk W_0 from W; denote $\partial\omega = \{x_-, z\}$. Note that $W_0 \cap v \neq \varnothing$ because $\gamma' \cap \gamma'' = \varnothing$ by Definition 3.10 (e), so there is an outermost disk W_1 of $W_0 \backslash\backslash v$ with $x_- \notin \partial W_1$. Denoting $\omega_1 = \partial W_1 \cap \omega$, Lemma 3.16 and Remark 2.8 imply that ω_1 cuts off a triangle of $W \backslash\backslash (v \cup \omega)$, and Sublemma 7.5 implies that ω_1 is one of the two arcs of $(F' \cap W \backslash\backslash v)$ not in $(F \cap W \backslash\backslash v)$. Since $z \in \omega_1$ and $x_- \notin \omega_1$, it follows that $z = x_+$. Yet, this implies that $\partial\omega = \{x_+, x_-\}$ and thus that ω comes from a *circle* of $F \cap W$, violating Definition 3.2 (a). $\square$

Proof of Lemma 3.22. Suppose otherwise. Then, because γ_1 is a flyping circle and $|F \cap C_t|$ is a single saddle disk X, there are at most two circles of $F \cap S_-$ that intersect both disks of $S_- \setminus (\pi^{-1} \circ \pi(\gamma_0))$, and one must both abut X and traverse the underpass at C_t. Yet, as shown right in Figure 35, this implies that F admits a push-through move near C_t along an arc in S_{-W}, contradicting Lemma 3.21. $\square$

7.4. **Properties of 6-good position.**

Proof of Lemma 3.23. The equivalence of (I) and (II) is straightforward (using Proposition 7.1), so it suffices to prove that (I) and (III) are equivalent. If (I) holds, then (a), the condition on $F \cap \widehat{B}$, and Lemmas 3.16 and 3.19 prohibit Moves 1-3, while (b) and (c) prohibit Moves 4-6. Conversely, if F is in 6-good position, then Definition 3.2

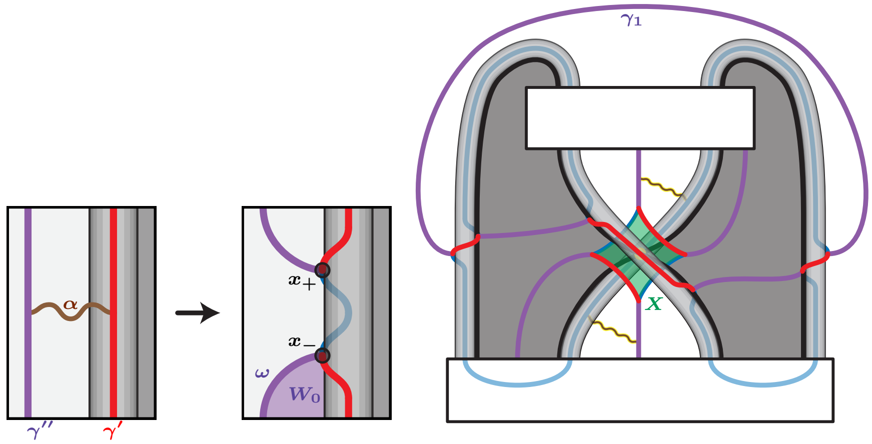


FIGURE 35. The situations in the proofs of Lemmas 3.21 and 3.22

(a) and Lemmas 3.16 and 3.19 give (a) and the condition on $F \cap \widehat{B}$, and Sublemma 7.5 gives (b); (c) is then straightfoward.[66] $\square$

Proof of Lemma 3.24. For the claim regarding fair position, but Proposition 7.3 takes care of Moves 7-9 and the other moves are easy to check (we rely here on Convention 3.13). The remaining claims are straightforward: note that a push-through move on a circle of $F \cap S_+$ via an arc $\alpha \subset S_+ \backslash\backslash F$ changes $\| F \|_3$ by $|\partial\alpha \cap \partial\nu L| - 2 \leq 0$. $\square$

Proof of Lemma 3.25. By Lemma 3.16, no arc of $F_0 \cap \widehat{W}$ is parallel in $\widehat{W}$ into ∂C; we claim that the same holds for F_1. This is obvious if $F_0 \to F_1$ is Move 3, 4, 6, 8, or 9, and since any Move 5 has the same effect as a push-through move followed by a Move 3, Proposition 7.3 confirms our claim if $F_0 \to F_1$ is Move 5 or Move 7. Thus, by Lemma 3.16, F_1 is in 2-good position. Moreover, any Move 3-9 $F_0 \to F_1$ restricts to an isotopy $F_0 \cap W \to F_1 \cap W$ in W which fixes $v_{F_0} \subset v_{F_1}$. Repeating this argument confirms (A) and (B).

For (C), observe that any Move 7, 8, or 9 $F_i \to F_{i+1}$ fixes $F_i \cap W = F_{i+1} \cap W$ and, by (A), preserves 2-good position. Hence, such a move gives rise to no arc of type (a) nor (b) nor (c) from Lemma 3.23 (I). The same reasoning applies to a Move 3 along an arc in $\widehat{B}$. For (D), observe also that by Definition 3.10 (e) no Move 8 nor 9 $F_i \to F_{i+1}$ can create an arc of $F_{i+1} \cap \widehat{B}$ that is ∂-parallel in B. $\square$

---

[66] If an arc α of $F \cap \widehat{W}$ has endpoints x, y on edges E, E' which are adjacent at a crossing ball C_t where F has no crossing band, then denote the arcs of $\partial F \cap S_\pm$ traversing the over/underpass at C_t by $\lambda_\pm$, and consider the disk W_0 of $\widehat{W} \backslash\backslash \alpha$ with $\partial W_0 \subset \alpha \cup E \cup E' \cup \partial C_t$. Any arc of $F \cap \text{int}(W_0)$ is isotopic in W_0 to α, so by passing to an outermost arc we may assume that $F \cap \text{int}(W_0) = \varnothing$. If α is incident to both λ_+ and λ_- then F admits Move 6; otherwise F admits Move 5.

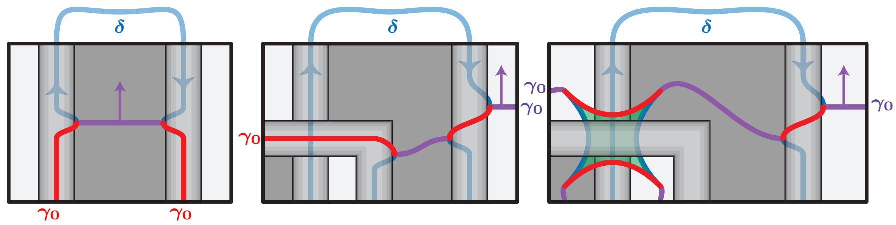


FIGURE 36. The three types of arc δ of $F \cap \text{int}(T_-)$

Proof of Lemma 3.26. Proposition 3.18 implies that the lexicographical quantity $(\mathbf{|}F\mathbf{|}_1, \mathbf{|}F\mathbf{|}_2, \mathbf{|}F\mathbf{|}_3)$ is always at least $(0,0,0)$, and so Lemma 3.24 implies that any sequence of Moves 1-7 terminates. Thus, any maximal sequence of Moves 1-9 (terminating only in 9-good position) has the form $F \to \cdots \to F_1 \to \cdots$, where F_1 is in 7-good position with $\mathbf{|}F_1\mathbf{|}_3 \geq 0$. By Lemma 3.25 (D), the remaining sequence $F_1 \to \cdots$ uses only Moves 8-9; both decrease $\mathbf{|}\cdot\mathbf{|}_3$. $\square$

8. PROOFS OF TECHNICAL LEMMAS FROM §4

In §8, set up as in §3.1, we prove Lemmas 4.2, 4.3, and 4.9.

8.1. Innermost circles in 9-good position.

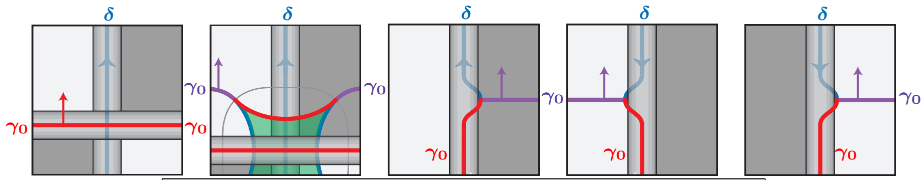
In §8.1, we adopt all setup from in §4.1, assuming in particular that F is in 9-good position with $F \cap S_+ \neq \varnothing$, and that T_+ is an innermost disk of $S_+ \backslash\backslash F$ with $\partial T_+ = \gamma_0$ and $T_- = S_- \cap (\pi^{-1} \circ \pi(T_+))$.

Proof of Lemma 4.1. For (A), if $|\gamma_0 \cap C_t^+| \geq 1$, then, as shown right in Figure 26, there would be a push-through move along a nearby arc $\alpha \subset S_{+W}$, violating Lemma 3.21. For (B), Sublemma 7.5 implies that $\omega \cap E = \varnothing$, and this implies that $\gamma \cap E = \varnothing$: otherwise, F would admit a push-through move along an arc in S_{+W}, again violating Lemma 3.21. Part (B) implies that γ_0 does not traverse the overpass at C_s; parts (C)-(D) now follow from (A), Lemma 3.4 (C), and the facts that γ_0 is innermost and D is alternating. $\square$

As we prepare to prove Lemma 4.2, note that each circle of $F \cap \text{int}(T_-)$ is disjoint from S_0 and intersects C^- only where it abuts crossing bands, hence is isotopic in $T_- \backslash\backslash S_0$ into $\partial \widehat{B}$; in particular, each such circle is innermost on S_-. Likewise, and more importantly:

Observation 8.1. *Let δ be an arc of $F \cap \text{int}(T_-)$. Then $\overline{\delta}$ is properly isotopic in $T_- \backslash\backslash S_0$ to an arc β of $T_- \cap \partial \widehat{B}$, and β is parallel through a disk $B_0 \subset \widehat{B} \cap T_-$ into γ_0; hence, δ is outermost in int(T_-).*

Proposition 8.2. *Every arc δ of $F \cap \text{int}(T_-)$ has one of the three types of local neighborhoods shown in Figure 36.*



location	description
$\delta_- \in C^-$	$\pi(\delta_-) = \pi(w)$ for $w \in \gamma_0 \cap \partial F$
$\delta_- \in C^-$	$\pi(z) = \pi(w)$ for $w \in \gamma_0$ on saddle
$\delta_- \in \partial B$	$i(\partial F, \partial B)_{\nu\delta_-} = 1$
~~$\delta_- \in \partial W$~~	~~$i(\partial F, \partial W)_{\nu\delta_-} = -1$~~
$\delta_+ \in \partial B$	$i(\partial F, \partial B)_{\nu\delta_+} = -1$
$\delta_+ \in \partial W$	$i(\partial F, \partial W)_{\nu\delta_+} = 1$

FIGURE 37. The possible types of endpoints of an arc δ of $F \cap \text{int}(T_-)$.

`figures/BiffBCase1.{ps,eps,pdf}` not found (or no BBox)
`figures/BiffBCase2.{ps,eps,pdf}` not found (or no BBox)
`figures/BiffBCase3.{ps,eps,pdf}` not found (or no BBox)

FIGURE 38. δ cannot have exactly one endpoint on ∂B.

Proof. Orient δ so that the disk B_0 described in Observation 8.1 lies to the right of δ, when viewed from H_+. Denote the initial and terminal points of δ by δ_- and δ_+. Definition 3.2 (a) gives $\delta_- \notin \partial W$, so there are three possibilities for δ_- and two for δ_+; see Figure 37.

Comparing Figures 36 and 37, it now suffices to prove that $\delta_- \in \partial B$ if and only if $\delta_+ \subset \partial B$. Suppose otherwise. There are three cases to consider. These appear above the dashed lines in Figure 38; in each case, we must have the full configuration shown in the figure, or else F would admit Move 7 or 8 (along an arc α shown in the figure). Hence, in each case, F admits a push-through move along an arc $\omega \subset S_{-W}$, contradicting Lemma 3.21.[67] □

Proposition 8.3. *If F is in 7-good position and an arc α of $\partial F \cap S_+$ lies on a single edge, then α has one endpoint on $\partial\widehat{B}$ and one on $\partial\widehat{W}$.*

Proof. If both endpoints of α were in $\partial\widehat{W}$, then one of these endpoints would be negative, violating Definition 3.2 (a). If both endpoints of α were in $\partial\widehat{B}$, then F would admit either Move 3 or Move 7. □

[67]To check that these moves satisfy Definition 3.10 (e), we also use Lemma 3.19 (left in Figure 38), Definition 3.2 (a) and the assumption that D is reduced (center), and Sublemma 7.5 (right).

Proof of Lemma 4.2. Given a prism P_i, consider the endpoint x_i of ω_i that lies in P_i. If $x_i \in \partial C$, then P_i is of type I, by Lemma 4.1 and Proposition 8.2. Otherwise, let λ_1 denote the arc of $\gamma_0 \cap \partial\nu L$ incident to x_i. If λ_1 traverses an overpass, then P_i is of type II, due to Proposition 8.2. Otherwise, by Proposition 8.2, λ_1 is incident to a non-standard arc β of $\gamma \cap \widehat{B}$, which is incident to a second arc λ_2 of $\gamma \cap \partial\nu L$ as shown left in Figure 36. This arc λ_2 must traverse an overpass, due to Proposition 8.3, alternatingness, and Definition 3.2 (a), so Proposition 8.2 implies that P_i is of type III. $\square$

8.2. **Properties of Move 10.** Observation 8.1 implies:

Observation 8.4. *For each disk X of $F \cap H_- \cap Y_1$, $|\partial X \cap \partial Y_1| \leq 1$.*

Proposition 8.5. *If $F \to F' = (F \setminus\setminus U) \cup V$ is a Move 10 along γ_0, then the arcs of $\gamma_0 \cap S_0$ abut mutually disjoint disks of $F \cap H_-$, each of which contains at most one arc of $F \cap H_- \cap \partial Y_2$.*

Proof. Suppose instead that distinct arcs α_1, α_2 of $\gamma_0 \cap S_0$ abut the same disk X of $F \cap H_-$. Choose points $x_i \in \alpha_i$. By Observation 8.4 and Lemma 4.2, we may construct a properly embedded arc $\alpha_- \subset X$ for which $\pi(\alpha_-) \cap \pi(T_+) = \partial\alpha_- = \{x_i, x_j\}$. Also construct a properly embedded arc $\alpha_+ \subset F_{\gamma_0}$ with $\partial\alpha_+ = \{x_i, x_j\}$. Then the circle $\alpha_+ \cup \alpha_- \subset F$ is 0-framed but not nullhomologous, contrary to definiteness. The last part then follows, using Lemma 4.2. $\square$

Proof of Lemma 4.3. Adopt the notation preceding the definition of Move 10, so that $F' = (F \setminus\setminus U) \cup V$, and recall Figure 27. Applying Lemma 3.23 to F, Lemma 4.2 implies that arcs comprise $F' \cap S_0$ and that no disk of $W \setminus\setminus (F' \cup v)$ is a bigon.

We check that F' satisfies conditions (a) and (h) of Definition 3.2, as (b)-(g) are then straightforward. For (a), if $F' \cap W$ contains circles, then each one bounds a disk in W by Fact 2.23, and an innermost one γ bounds a disk W_0 in W disjoint from F'; W_0 must intersect v, or else γ would be a circle of $F' \cap S_0$; yet, an outermost disk W_1 of $W_0 \setminus\setminus v$ is a bigon of $W \setminus\setminus (F' \cup v)$. Thus, $F' \cap W$ contains no circles. To complete the proof of (a), note that each point x of $\partial F' \cap \partial W$ either is an endpoint of an arc of $F \cap W$ or lies in P, and in either case is positive: $i(\partial F', \partial W)_{\nu x} = +1$ (see Figure 27).

For (h), each component of $F' \cap H_+$ is also a component of $F \cap H_+$, hence a disk. Likewise, each component of $F' \cap C$ is either a component of $F \cap C$ or a crossing band. Regarding $F'_- = F' \cap H_-$, each component of $F'_- \cap Y_1 \setminus V$ is also a component of $F \cap H_- \cap Y_1$, hence a disk, and likewise for $F'_- \cap Y_2$. Observation 8.4 and the last part of Proposition 8.5 further imply that each of these disks abuts ∂P in at most one arc. It thus suffices to observe in Figure 27 that each component of $F' \cap P$ is a disk. $\square$

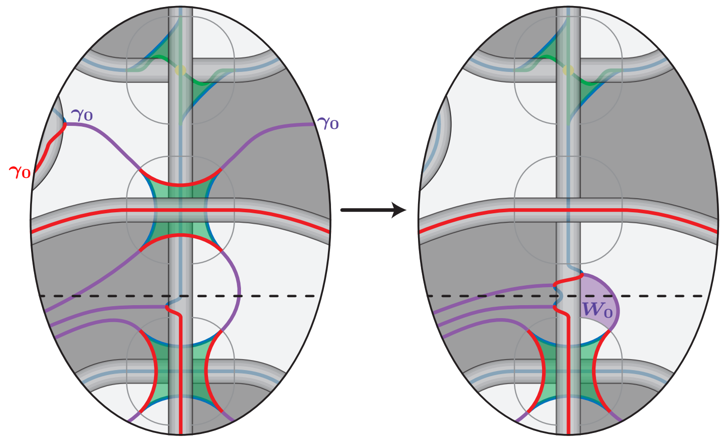


FIGURE 39. A triangle W_0 arising via Move 10

Proposition 8.6. *If $F_0 \to F_1$ is a Move 10 and $F_1 \to F_2$ is a sequence of Moves 1-9 leaving F_2 in 10-good position, then the isotopy $F_1 \to F_2$ restricts to to an isotopy $F_1 \cap W \setminus v_{F_1} \to v \setminus v_{F_1}$ in $W \setminus v_{F_1}$.*

Proof. By Lemma 4.3, F_1 is in fair position. Now apply Lemma 3.25 (B); note that $v_{F_2} = v$, by 10-good position. □

Proof of Lemma 4.9. By Lemma 3.23, no disk X of $W \setminus\setminus (F \cup v)$ satisfies $|\partial X \cap v| = 1 = |\partial X \cap F|$, so any disks W_0 of $W \setminus\setminus (F' \cup v)$ with $|\partial W_0 \cap v| = 1 = |\partial W_0 \cap F'|$ are triangles that arise near type I prisms as shown in Figure 39. Thus, using Proposition 8.6, Lemma 2.26 implies that $F_1 \cap W = v_{F_1}$. This confirms (A). Lemma 4.8 (B) thus implies that F_1 is in 9-good position; hence, by hypothesis, F_1 is in 10-good position, giving (B): $F \cap S_+ = \gamma_0$.

Therefore (c.f. Observation 8.1), in each prism P_i, the points labeled y_i, z_i in Figure 27 lie on the boundary of the same disk of $F \cap H_-$. This nearly contradicts Proposition 8.5; the only possibility is that there is only one prism, i.e. $|\gamma_0 \cap \widehat{W}| = 1$. The prism cannot be of type I by (A), nor of type (B) because D is prime, so it is of type III. Hence, γ_0 is a flyping circle. □

REFERENCES

[AK13] C. Adams, T. Kindred, *A classification of spanning surfaces for alternating links*, Alg. Geom. Topol. 13 (2013), no. 5, 2967-3007.

[Au56] R. J. Aumann, *Asphericity of alternating knots*, Ann. of Math. (2) 64 (1956), 374-392.

[GL78] C. McA. Gordon, R.A. Litherland, *On the signature of a link*, Invent. Math. 47 (1978), no. 1, 53-69.

[Gr17] J. Greene, *Alternating links and definite surfaces*, with an appendix by A. Juhasz, M Lackenby, Duke Math. J. 166 (2017), no. 11, 2133-2151.

[Ho17] J. Howie, *A characterisation of alternating knot exteriors*, Geom. Topol. 21 (2017), no. 4, 2353-2371.

[Ho15] J. Howie, Surface-alternating knots and links, Ph.D. thesis, University of Melbourne (2015).

[Ki23a] T. Kindred, *A simple proof of the Crowell-Murasugi theorem*, to appear in Alg. Geom. Topol.

[Ki23b] T. Kindred, *The virtual flyping theorem*, preprint.

[Ki23b] T. Kindred, *End-essential spanning surfaces for links in thickened surfaces*, preprint.

[Ka87] L.H. Kauffman, *State models and the Jones polynomial*, Topology 26 (1987), no. 3, 395-407.

[Li97] W.B.R. Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, 175. Springer-Verlag, New York, 1997. x+201 pp.

[Me84] W. Menasco, *Closed incompressible surfaces in alternating knot and link complements*, Topology 23 (1984), no. 1, 37-44.

[MT91] W. Menasco, M. Thistlethwaite, *The Tait flyping conjecture*, Bull. Amer. Math. Soc. (N.S.) 25 (1991), no. 2, 403-412.

[MT93] W. Menasco, M. Thistlethwaite, *The classification of alternating links*, Ann. of Math. (2) 138 (1993), no. 1, 113-171.

[Mu87] K. Murasugi, *Jones polynomials and classical conjectures in knot theory*, Topology 26 (1987), no. 2, 187-194.

[M87ii] K. Murasugi, *Jones polynomials and classical conjectures in knot theory II*, Math. Proc. Cambridge Philos. Soc. 102 (1987), no. 2, 317-318.

[Oz06] M. Ozawa, *Nontriviality of generalized alternating knots*, J. Knot Theory Ramifications 15 (2006), no. 3, 351-360.

[T1898] P.G. Tait, *On Knots I, II, and III*, Scientific papers 1 (1898), 273-347.

[Th87] M.B. Thistlethwaite, *A spanning tree expansion of the Jones polynomial*, Topology 26 (1987), no. 3, 297-309.

[T88a] M.B. Thistlethwaite, *On the Kauffman polynomial of an adequate link*, Invent. Math. 93 (1988), no. 2, 285-296.

[T88b] M.B. Thistlethwaite, *Kauffman's polynomial and alternating links*, Topology 27 (1988), no. 3, 311-318.

[Th91] M.B. Thistlethwaite, *On the algebraic part of an alternating link*, Pacific J. Math. 151 (1991), no. 2, 317-333.

[Tu87] V.G. Turaev, *A simple proof of the Murasugi and Kauffman theorems on alternating links*, Enseign. Math. (2) 33 (1987), no. 3-4, 203-225.

Department of Mathematics, Wake Forest University, Winston-Salem, North Carolina 27109, USA

E-mail address: `kindret@wfu.edu`

URL: `www.thomaskindred.com`