HEEGAARD DIAGRAMS CORRESPONDING TO TURAEV SURFACES

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Abstract. We describe a correspondence between Turaev surfaces of link diagrams on $S^2 \subset S^3$ and special Heegaard diagrams for S^3 adapted to links.

1. Introduction

To construct the Turaev surface Σ of a link diagram D on $S^2 \subset S^3$, one pushes the all-A and all-B states of D to opposite sides of S^2, connects these two states with a certain cobordism, and caps the state circles with disks. Turaev's original construction [19] streamlined Murasugi's proof [16], based on Kauffman's work [12] on the Jones polynomial [11], of Tait's longstanding conjecture on the crossing numbers of alternating links [17]. See also [18]. More recently, Turaev surfaces have provided geometric means for interpreting Khovanov and knot Floer homologies, as in [3, 5, 6, 9, 10, 14, 20].

Dasbach, Futer, Kalfagianni, Lin, and Stoltzfus showed that the Turaev surface of any connected link diagram D on $S^2 \subset S^3$ is a splitting surface for S^3 on which D forms an alternating link diagram [8]. When equipped with the type of crossing ball structure developed by Menasco [15], the projection sphere provides natural attaching circles for the two handlebodies of this splitting, completing a Heegaard diagram (Σ, α, β) for S^3. By characterizing the interplay between this Heegaard diagram and the original link diagram D, we obtain a correspondence between Turaev surfaces and particular Heegaard diagrams adapted to links. Figure 1 shows a typical example of such a diagram $(\Sigma, \alpha, \beta, D)$.

First, §2 defines Heegaard splittings and diagrams, link diagrams, crossing ball structures, and Turaev surfaces. Next, §3 constructs and describes the special, link-adapted Heegaard diagrams $(\Sigma, \alpha, \beta, D)$. Finally, §4 establishes the following correspondences:

Theorem 4.1. *There is a 1-to-1 correspondence between Turaev surfaces of connected link diagrams on $S^2 \subset S^3$ and diagrams $(\Sigma, \alpha, \beta, D)$ with the following properties:*

- *(Σ, α, β) is a Heegaard diagram for S^3, with $\alpha \pitchfork \beta$.*
- *D is an alternating link diagram on Σ which cuts Σ into disks, with $D \pitchfork \alpha$ and $D \pitchfork \beta$.*
- *$D \cap \alpha = D \cap \beta = \alpha \cap \beta$, none of these points being crossings of D.*
- *There is a checkerboard partition $\Sigma \setminus (\alpha \cup \beta) = \Sigma_\varnothing \cup \Sigma_K$, in which $\Sigma_\varnothing$ consists of disks disjoint from D, in which D cuts Σ_K into disks each of whose boundary contains at least one crossing point and at most two points of $\alpha \cap \beta$, and in which $2g(\Sigma) + |\Sigma_\varnothing| = |\alpha| + |\beta|$.*

Theorem 4.2: *There is a 1-to-1 correspondence between generalized Turaev surfaces, constructed from dual pairs of states of connected link diagrams on $S^2 \subset S^3$, and diagrams $(\Sigma, \alpha, \beta, D)$ with the properties in Theorem 4.1, except that D need not alternate on Σ.*

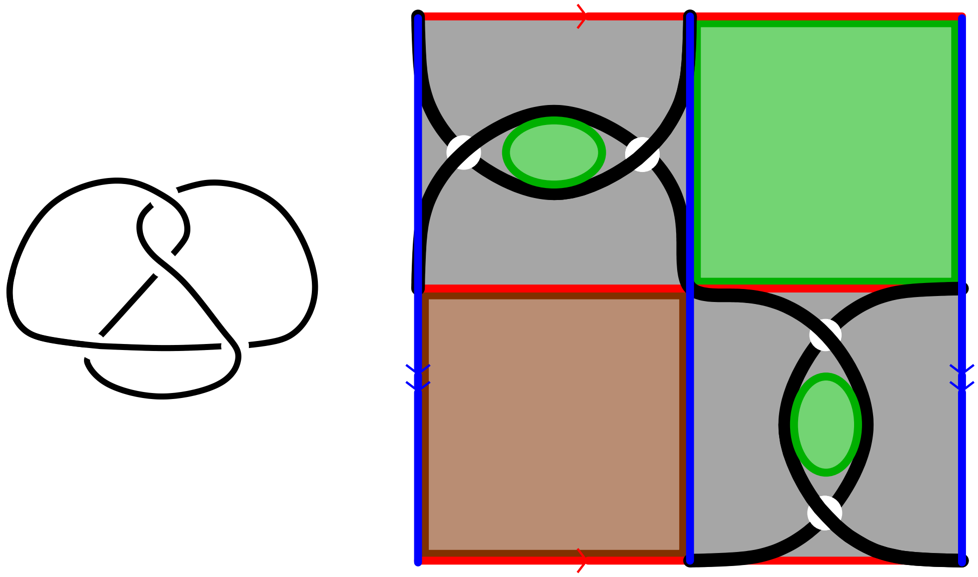

FIGURE 1. A link diagram on S^2, and the link-adapted Heegaard diagram $(\Sigma, \alpha, \beta, D)$ corresponding to its Turaev surface, the torus in Figure 7. As in all figures, the link is black; the crossing balls are white; the attaching circles comprising α and β are red and blue, respectively; and the circles and disks from the all-A state are green, while those from the all-B state are brown.

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2. Background

2.1. **Heegaard splittings and diagrams.** A *Heegaard splitting* of an orientable 3-manifold M is a decomposition of M into two handlebodies H_α and H_β with common boundary. The surface $\partial H_\alpha = \partial H_\beta = \Sigma$ is called a *splitting surface* for M. In this paper, we address only the case in which $M = S^3$.

One can describe a handlebody H by identifying on its boundary $\partial H = \Sigma$ a collection of disjoint, simple closed curves $\alpha_1, \ldots, \alpha_k$, such that each α_i bounds a disk $\hat{\alpha}_i$ in H, and such that these disks together cut H into a disjoint union of balls. The α_i are called *attaching circles* for H. Some conventions require that the $\hat{\alpha}_i$ together cut H into a single ball, hence $k = g(\Sigma)$; though not requiring this, our definition does imply that $k \geq g(\Sigma)$.

A *Heegaard diagram* (Σ, α, β) combines these ideas to blueprint a 3-manifold. The diagram consists of a splitting surface $\Sigma = \partial H_\alpha = \partial H_\beta$, together with a union $\alpha = \bigcup \alpha_i$ of attaching circles for H_α and a union $\beta = \bigcup \beta_i$ of attaching circles for H_β. If (Σ, α, β) is a Heegaard diagram for S^3, then the circles of α and β together generate $H_1(\Sigma)$. The Appendix provides an easy proof of this fact, using Seifert surfaces.

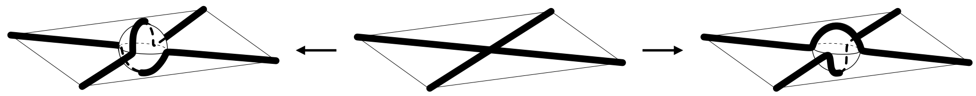

FIGURE 2. Each crossing in a link diagram is labeled in one of two ways. The label tells one how to adjust the link after inserting a crossing ball.

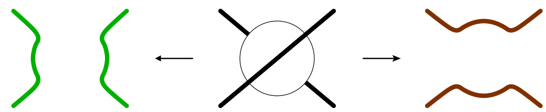

FIGURE 3. The A-smoothing (left) and B-smoothing (right) of a crossing.

2.2. **Link diagrams and crossing balls.** A *link diagram* D on a closed surface $F \subset S^3$ is the image, in general position, of an immersion of one or more circles in F; each arc at any crossing point is labeled with a direction normal to F near that point, so that under- and over-crossings have been identified. By inserting small, mutually disjoint crossing balls $C = \bigcup C_i$ centered at the crossing points of D and pushing the two intersecting arcs of each $D \cap C_i$ off F to the appropriate hemisphere of $\partial C_i \setminus F$, as in Figure 2, one obtains a configuration of a link $K \subset (F \setminus C) \cup \partial C \subset S^3$. Call this a *crossing ball configuration* of the link K corresponding to the link diagram D.

Conversely, given mutually disjoint crossing balls $C = \bigcup C_i$ centered at points on a closed surface $F \subset S^3$, and a link $K \subset (F \setminus C) \cup \partial C$ in which each crossing ball appears as in Figure 2, one may obtain a corresponding link diagram as follows. Consider a regular neighborhood of F that contains C and is parameterized by an orientation-preserving homeomorphism with $F \times [-1, 1]$ which identifies F with $F \times \{0\}$. If $\pi : F \times [-1, 1] \to F$ denotes the natural projection, the link diagram corresponding to the crossing ball configuration $K \subset (F \setminus C) \cup \partial C \subset S^3$ is the projected image $\pi(K) \subset F$ with appropriate crossing labels.

In such a crossing ball configuration, each arc of $K \cap \partial C$ lies either in $F \times [-1, 0]$ or in $F \times [0, 1]$. The former arcs are called *under-passes*, and the latter are called *over-passes*. A link diagram D is said to be *alternating* if each arc of $K \setminus C$ in a corresponding crossing ball configuration joins an under-pass with an over-pass. A link $K \subset S^3$ is alternating if it has an alternating diagram on S^2.

In particular, any Heegaard diagram (Σ, α, β) for S^3 provides an embedding of the closed surface Σ in S^3. One may therefore superimpose a link diagram D on the Heegaard diagram to obtain a new type of diagram $(\Sigma, \alpha, \beta, D)$. This new diagram describes a Heegaard splitting of S^3 in which the splitting surface contains a link diagram.

2.3. **Turaev surfaces.** Each crossing in a link diagram D on a surface F can be smoothed in two different ways, by inserting a crossing ball C_i and replacing $D \cap C_i$ with one of the two pairs of arcs of $(\partial C_i \cap F) \setminus D$ opposite to another. Figure 3 shows the two possibilities, called the *A-smoothing* and the *B-smoothing* of the crossing. Making a choice of smoothing for each crossing in the diagram produces a disjoint union of circles on F, called a *state* of the diagram D. Two states of D are *dual* if they have opposite smoothings at each crossing.

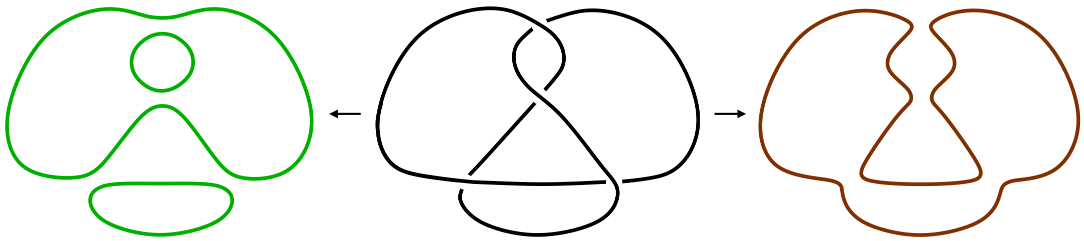

FIGURE 4. The all-A (left) and all-B (right) states for a link diagram.

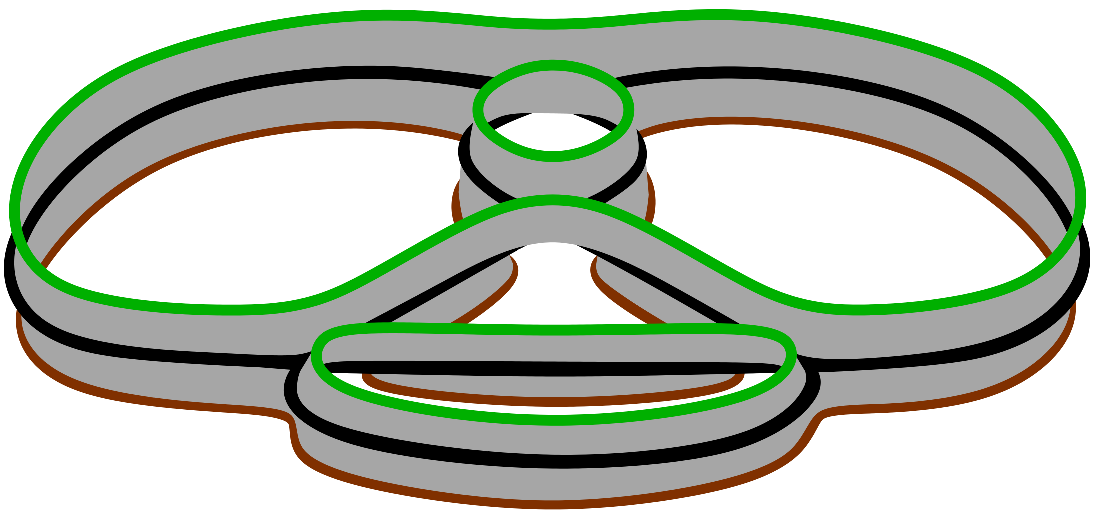

FIGURE 5. The cobordism between the all-A and all-B states from Figure 4.

Given a link diagram D on S^2, the two extreme states – the all-A and the all-B – are of particular interest, due in part to the bounds they give on the maximum and minimum degrees of the Jones polynomial. Kauffman's proof [12] that these bounds are sharp for reduced, alternating diagrams provided the impetus for Murasugi [16], Thistlethwaite [18], and Turaev [19] to prove Tait's conjecture on the crossing numbers of alternating links. Cromwell [7], Lickorish and Thistlethwaite [13] then extended these results to adequate link diagrams. Figure 4 shows the all-A and all-B states for the link diagram from Figure 1.

Following Turaev [19], one can construct a cobordism between the all-A and all-B states as follows. Parameterize a bi-collaring of S^2 as in §2.2, and push the all-A and all-B states off S^2 to $S^2 \times \{1\}$ and $S^2 \times \{-1\}$, respectively, such that each state circle sweeps out an annulus to one side of S^2. Assume that these annuli are mutually disjoint, and that they are disjoint from the crossing balls $C = \bigcup C_i$ used to construct the all-A and all-B states. Gluing together these annuli and the disks of $S^2 \cap C$ produces the cobordism between the two states. (See Figure 5.) Near each crossing, the cobordism has a saddle, as in Figure 6.

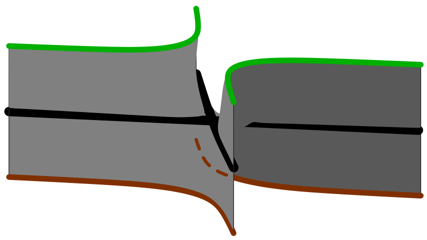

FIGURE 6. Turaev's cobordism between the all-A and all-B states has a saddle near each crossing, shown here with and without a crossing ball.

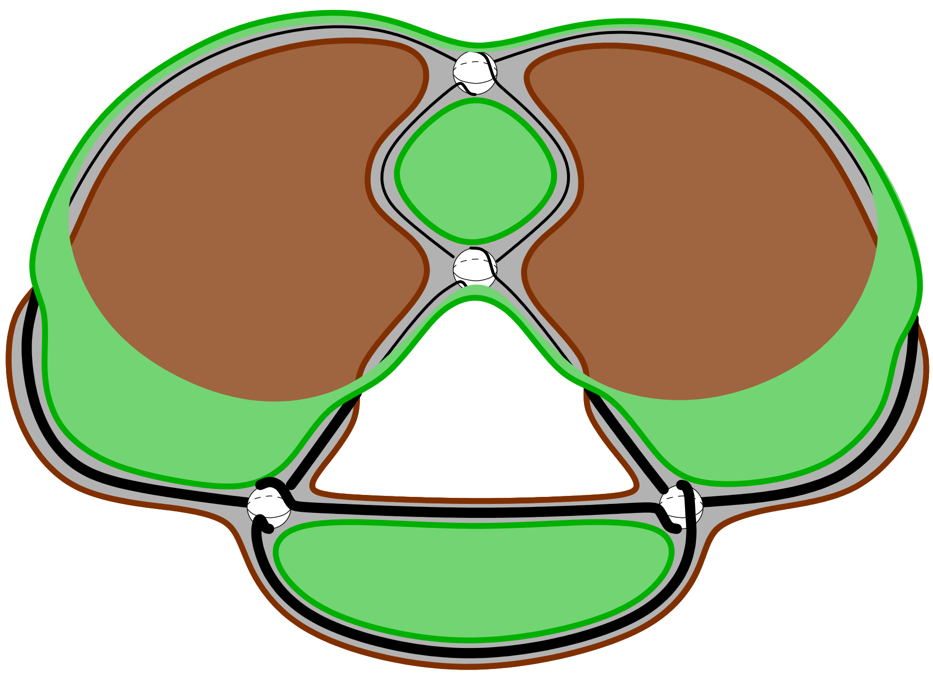

FIGURE 7. This torus is the Turaev surface of the link diagram in Figures 1 and 4, seen from the ambient space. To provide a window to the far side of the surface, one of the three disks of the all-A state is only partly shown.

Having constructed the cobordism, one caps the all-A and all-B states with mutually disjoint disks to form a closed surface Σ, called the *Turaev surface* of the original link diagram D on S^2. Since Σ contains a neighborhood of S^2 around each crossing point, the crossing information of D on S^2 translates to crossing information on the Turaev surface. Thus, D forms a link diagram on Σ. A crossing ball configuration corresponding to this link diagram is $K \subset (\Sigma \setminus C) \cup \partial C$, with under- and over-passes defined as in §2.2.

Observe that D cuts Σ into disks, each of which contains exactly one state disk, and that $S^2 \cap \Sigma = S^2 \cap (C \cup K) = \Sigma \cap (C \cup K)$. Note also that if D is alternating on S^2, then Σ is a sphere which can be isotoped to S^2 while fixing D. Figure 7 shows a less trivial example.

The construction of the Turaev surface generalizes to any pair of states s and $\tilde{s}$ dual to one another. By pushing s and $\tilde{s}$ to opposite sides of S^2 to sweep out annuli on opposite sides of S^2, gluing in disks near the crossings to obtain a cobordism between s and $\tilde{s}$, and capping off with disks, one obtains a closed surface Σ on which D forms a link diagram [1, 19]. Call this surface Σ the *generalized Turaev surface* of the dual states s and $\tilde{s}$.

3. Construction of Heegaard diagrams for Turaev surfaces

Given a connected link diagram D on $S^2 \subset S^3$ and its Turaev surface Σ, this section constructs a link-adapted Heegaard diagram $(\Sigma, \alpha, \beta, D)$. Theorem 3.4 then characterizes this diagram, providing one direction of the correspondence to come in Theorem 4.1.

Let $K \subset (S^2 \setminus C) \cup \partial C$ be a crossing ball structure corresponding to D, and let H_α and H_β be the two components of $S^3 \setminus \Sigma$. Define $\hat{\alpha} := (S^2 \setminus (C \cup K)) \cap H_\alpha$ and $\hat{\beta} := (S^2 \setminus (C \cup K)) \cap H_\beta$ to be the two checkerboard classes of $S^2 \setminus (C \cup K)$, with $\alpha := \partial\hat{\alpha}$ and $\beta := \partial\hat{\beta}$. From this setup, three modifications will complete the construction of the diagram $(\Sigma, \alpha, \beta, D)$. During these changes, Σ, D, S^2, C, and K will remain fixed.

First, perturb α and β through the cobordism as follows, carrying along the disks of $\hat{\alpha}$ and $\hat{\beta}$. Let $X = \{x_1, \ldots, x_n\}$ consist of one point on each arc of $K \setminus C$ which joins two under-passes on S^2, and let $Y = \{y_1, \ldots, y_n\}$ consist of one point on each arc of $K \setminus C$ which joins two over-passes on S^2. Each arc of $\alpha \setminus (X \cup Y)$ runs along a circle from either the all-A state or the all-B state. Isotope α through the cobordism so as to push arcs of the former type to $S^2 \times (0, 1)$ and arcs of the latter type to $S^2 \times (-1, 0)$, giving $\alpha \cap C = \varnothing$ and $\alpha \cap D = X \cup Y$. Next, isotope β in the same manner, after which α and β will both be disjoint from C, while α, β, and D will be pairwise transverse and will intersect exclusively at triple points: $\alpha \cap \beta = \alpha \cap D = \beta \cap D = X \cup Y$.

To further simplify the picture, push the state circles through the cobordism to align with $\alpha \cup \beta$, so that each state disk becomes a component of $\Sigma \setminus (\alpha \cup \beta)$. This causes the neighborhood of each arc of $K \setminus C$ to appear as in Figure 8, possibly with red and blue reversed. Note that the state disks' interiors remain disjoint from D, in fact from S^2.

To complete the construction, remove any attaching circles that are disjoint from D. Also remove the corresponding disks of $\hat{\alpha}$ and $\hat{\beta}$, and let α, β, $\hat{\alpha}$ and $\hat{\beta}$ retain their names. Because each removed circle lies in some disk of $\Sigma \setminus D$, each removed disk is parallel to Σ.

Lemma 3.1 (DFKLS [8]). *The Turaev surface Σ of any connected link diagram D on $S^2 \subset S^3$ is a splitting surface for S^3.*

Proof. Observe that $S^2 \cup C$ cuts S^3 into two balls, which Σ cuts into smaller balls. Also, $S^3 \setminus (S^2 \cup C \cup \Sigma) = (H_\alpha \setminus (S^2 \cup C)) \cup (H_\beta \setminus (S^2 \cup C))$, where H_α and H_β are the two components of $S^3 \setminus \Sigma$. Hence, $H_\alpha \setminus C$ and $H_\beta \setminus C$ are handlebodies, as are H_α and H_β. □

The proof of Lemma 3.1 implies that (Σ, α, β) was a Heegaard diagram for S^3 when α and β were first defined. The fact that each removed disk of $\hat{\alpha}$ and of $\hat{\beta}$ was parallel to Σ implies that (Σ, α, β) is a Heegaard diagram for S^3 in the finished construction as well.

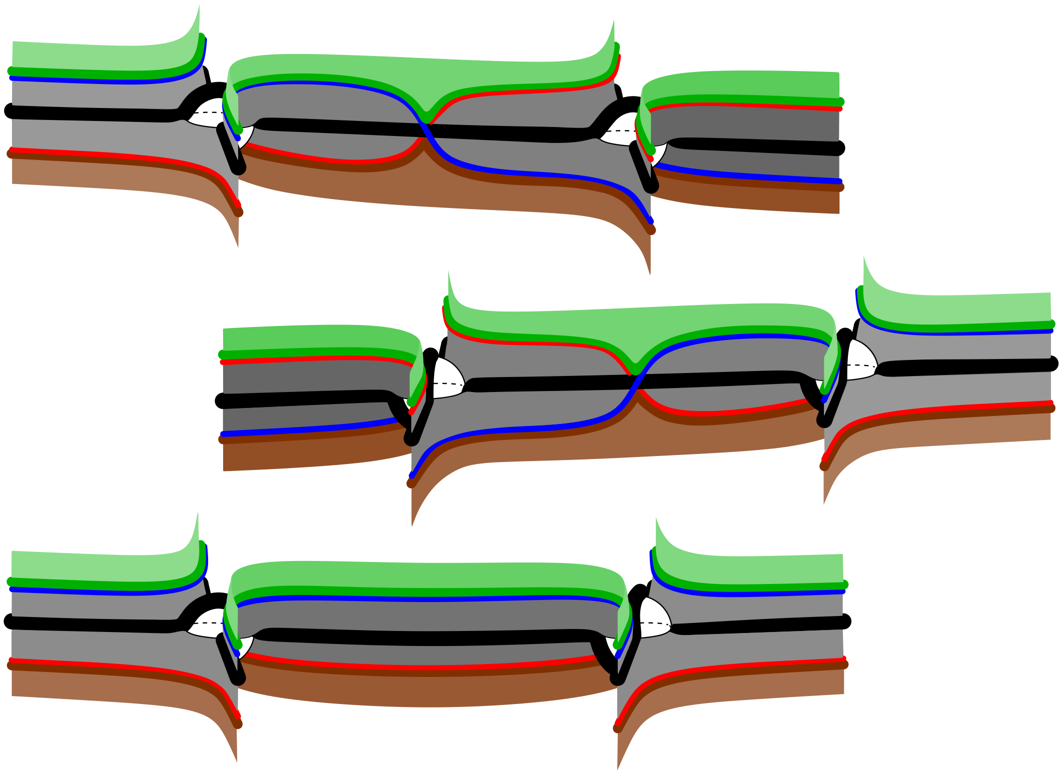

FIGURE 8. Up to reversing red and blue, these are the possible configurations of the Turaev surface Σ between two adjacent crossings, shown at the stage of the construction in which the boundary of each state disk lies in $\alpha \cup \beta$.

Lemma 3.2 (DFKLS [8])**.** *Any connected link diagram D on $S^2 \subset S^3$ forms an alternating link diagram on its Turaev surface Σ.*

Proof. Recall from §2.3 that D forms a link diagram on Σ. On S^2, each arc κ of $K \setminus C$ joins either two over-passes, two under-passes, or one of each. Figure 8 shows the three possible configurations of Σ near κ, prior to the removal of attaching circles, up to reversal of α and β. In all three cases, the two arcs of $K \cap \partial C$ incident to κ lie to opposite sides of Σ, so that one is an over-pass on Σ and the other is an under-pass on Σ. □

One defines the *Turaev genus* $g_T(K)$ of a link $K \subset S^3$ to be the minimum genus among the Turaev surfaces of all diagrams of K on S^2. The resulting invariant, surveyed in [4], measures how far a link is from being alternating. See also [2]. In particular, Turaev genus provides the crux of Turaev's proof of Tait's conjecture:

Corollary 3.3 (Turaev [19], DFKLS [8])**.** *A link K is alternating if and only if $g_T(K) = 0$.*

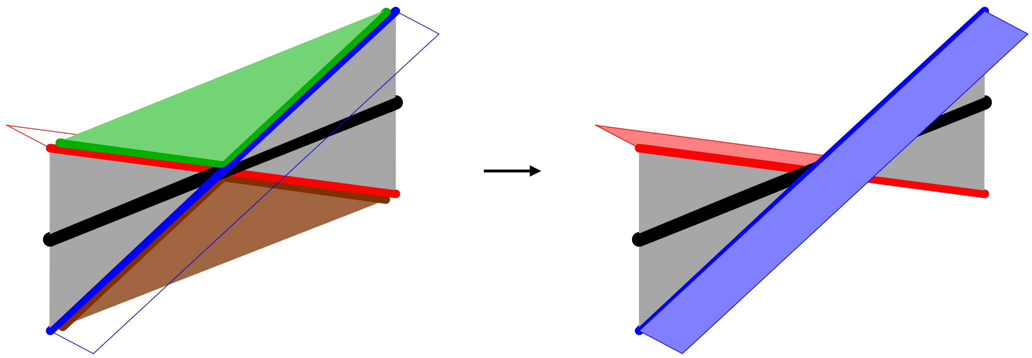

FIGURE 9. Given a diagram $(\Sigma, \alpha, \beta, D)$ with the properties in Theorems 3.4, 4.1, or 4.2, removing the disks of $\Sigma_\varnothing$ from Σ and gluing in the disks of $\hat{\alpha}$ and $\hat{\beta}$ produces a sphere on which D forms a link diagram. Near each point of $\alpha \cap \beta$, this surgery appears as shown, up to mirroring.

Theorem 3.4. *From the Turaev surface Σ of a connected link diagram D on $S^2 \subset S^3$, the construction in this section produces a diagram $(\Sigma, \alpha, \beta, D)$ with the following properties:*

- *(Σ, α, β) is a Heegaard diagram for S^3, with $\alpha \pitchfork \beta$.*
- *D is an alternating link diagram on Σ which cuts Σ into disks, with $D \pitchfork \alpha$ and $D \pitchfork \beta$.*
- *$D \cap \alpha = D \cap \beta = \alpha \cap \beta$, none of these points being crossings of D.*
- *There is a checkerboard partition $\Sigma \setminus (\alpha \cup \beta) = \Sigma_\varnothing \cup \Sigma_K$, in which $\Sigma_\varnothing$ consists of disks disjoint from D, in which D cuts Σ_K into disks each of whose boundary contains at least one crossing point and at most two points of $\alpha \cap \beta$, and in which $2g(\Sigma) + |\Sigma_\varnothing| = |\alpha| + |\beta|$.*

Proof. We have already established the first three properties. Let $\Sigma_\varnothing$ consist of the interiors of all adjusted state disks whose boundary contains at least one point of $\alpha \cap \beta$, i.e. those whose boundary still lies in $\alpha \cup \beta$ after the removal of the attaching circles disjoint from D. These state disks are disjoint from D and constitute a checkerboard class of $\Sigma \setminus (\alpha \cup \beta)$. See Figure 9.

Let Σ_K denote the other checkerboard class of $\Sigma \setminus (\alpha \cup \beta)$. Each component of $\Sigma_K \setminus D$ is also a component of $(\Sigma \setminus D) \setminus (\alpha \cup \beta)$, and each attaching circle intersects D; therefore, D cuts Σ_K into disks. Further, each arc of $K \setminus C$ contains at most one point of $\alpha \cap \beta$, and each arc of $(\alpha \cup \beta) \setminus D$ is parallel through Σ to D; consequently, the boundary of each disk of $\Sigma_K \setminus D$ contains at least one crossing point and at most one arc of $(\alpha \cup \beta) \setminus D$, hence at most two points of $\alpha \cap \beta$.

Finally, to see that $2g(\Sigma) + |\Sigma_\varnothing| = |\alpha| + |\beta|$, consider Euler characteristic in light of the observation that removing the disks of $\Sigma_\varnothing$ from Σ and gluing in the disks of $\hat{\alpha}$ and $\hat{\beta}$ yields a sphere isotopic to S^2. Near each point of $\alpha \cap \beta$, this surgery appears as in Figure 9. □

4. Correspondence between Heegaard diagrams and Turaev surfaces

4.1. **Main correspondence.** From the Turaev surface of a connected link diagram on $S^2 \subset S^3$, we have constructed a link-adapted Heegaard diagram $(\Sigma, \alpha, \beta, D)$ with several properties. We will now see that any such diagram corresponds to the Turaev surface of some link diagram on the sphere.

Theorem 4.1. *There is a 1-to-1 correspondence between Turaev surfaces of connected link diagrams on $S^2 \subset S^3$ and diagrams $(\Sigma, \alpha, \beta, D)$ with the following properties:*

- *(Σ, α, β) is a Heegaard diagram for S^3, with $\alpha \pitchfork \beta$.*
- *D is an alternating link diagram on Σ which cuts Σ into disks, with $D \pitchfork \alpha$ and $D \pitchfork \beta$.*
- *$D \cap \alpha = D \cap \beta = \alpha \cap \beta$, none of these points being crossings of D.*
- *There is a checkerboard partition $\Sigma \setminus (\alpha \cup \beta) = \Sigma_\varnothing \cup \Sigma_K$, in which $\Sigma_\varnothing$ consists of disks disjoint from D, in which D cuts Σ_K into disks each of whose boundary contains at least one crossing point and at most two points of $\alpha \cap \beta$, and in which $2g(\Sigma) + |\Sigma_\varnothing| = |\alpha| + |\beta|$.*

Proof. Theorem 3.4 provides one direction of this correspondence. It remains to prove the converse.

Assume that the diagram $(\Sigma, \alpha, \beta, D)$ is as described. Remove the disks of $\Sigma_\varnothing$ from Σ and glue in the disks of $\hat{\alpha}$ and $\hat{\beta}$ to obtain a closed surface. (See Figure 9.) Because D is connected and $2g(\Sigma) + |\Sigma_\varnothing| = |\alpha| + |\beta|$, this surface is a sphere – call it S^2. Moreover, D, being disjoint from $\Sigma_\varnothing$ and having its crossing points in Σ_K, forms a link diagram on S^2. We claim, up to isotopy, that Σ is the Turaev surface of the link diagram D on S^2.

The property that D cuts Σ_K into disks implies that D intersects each attaching circle, cutting α and β into arcs. Because the boundary of each disk of $\Sigma_K \setminus D$ contains at most two points of $\alpha \cap \beta$, each of these arcs is parallel through one of these disks to D. The property that the boundary of each disk of $\Sigma_K \setminus D$ contains at least one crossing point then implies that there is at most one point of $\alpha \cap \beta$ on D between any two adjacent crossings.

The link diagram D cuts S^2 into disks admitting a checkerboard partition. Because S^2 appears near each point of $\alpha \cap \beta$ as in Figure 9, one of the checkerboard classes contains $\hat{\alpha}$, and the other contains $\hat{\beta}$. Yet, some disks of $S^2 \setminus D$ may be entirely contained in Σ_K, hence disjoint from α and β. Construct an attaching circle in the interior of each such disk, and incorporate it into either α or β according to the checkerboard pattern, letting α and β retain their names. Span each new circle of α by a new disk of $\hat{\alpha}$ on the same side of Σ as the other disks of $\hat{\alpha}$, and similarly span each new circle of β by a new disk of $\hat{\beta}$.

The components of $\Sigma \setminus (\alpha \cup \beta)$ still admit a checkerboard partition, $\Sigma \setminus (\alpha \cup \beta) = \Sigma_\varnothing \cup \Sigma_K$, in which $\Sigma_\varnothing$ consists of disks disjoint from D, though D no longer need cut Σ_K into disks. The preceding modification of α, β, $\hat{\alpha}$, and $\hat{\beta}$ corresponds to an isotopy of S^2, which again may be obtained from Σ by removing the disks of $\Sigma_\varnothing$ and gluing in the disks of $\hat{\alpha}$ and $\hat{\beta}$.

Let $K \subset (\Sigma \setminus C) \cup \partial C$ be a crossing ball configuration corresponding to the link diagram D on Σ, with $C \cap \alpha = \varnothing = C \cap \beta$. Note that $K \subset (S^2 \setminus C) \cup \partial C$ is also a crossing ball configuration corresponding to the link diagram D on S^2.

Currently Σ and S^2 are non-transverse, even away from C, as both Σ and S^2 contain Σ_K. Rectify this by perturbing S^2 as follows, fixing Σ, α, β, $\hat{\alpha}$, $\hat{\beta}$, D, $\Sigma_\varnothing$, Σ_K, C, and K in the process. (We initially constructed S^2 by gluing together $\hat{\alpha}$, $\hat{\beta}$, and Σ_K, but now we are pushing S^2 off of them.) Each disk of $S^2 \setminus (C \cup K)$ currently contains a disk of either $\hat{\alpha}$ or $\hat{\beta}$; push the disk of $S^2 \setminus (C \cup K)$ off Σ in the corresponding direction, while fixing its boundary, which lies in $\Sigma \cap (K \cup \partial C)$. This isotopy makes S^2 disjoint from α and β, except at the points of $\alpha \cap \beta$. In fact, this isotopy gives $S^2 \cap \Sigma = S^2 \cap (C \cup K) = \Sigma \cap (C \cup K)$, as was the case in §2.3. (Recall Figure 6.)

Because D is alternating on Σ, the disks of $\Sigma \setminus (C \cup K)$ admit a checkerboard partition – the boundaries of the disks in the two classes are the all-A and all-B state circles for the link diagram D on Σ. Further, each of these state circles on Σ encloses precisely one disk of $\Sigma_\varnothing$. Color green each disk of $\Sigma_\varnothing$ enclosed by a circle from the all-A state, and color brown each disk of $\Sigma_\varnothing$ enclosed by a circle from the all-B state. Near each arc of $K \setminus C$, Σ now appears as in Figure 8 (possibly with red and blue reversed). As a final adjustment, slightly perturb the green and brown disks so that they become disjoint from α, β, and D.

Removing the green and brown disks from Σ leaves a cobordism between their boundaries. Cutting this cobordism along $S^2 \cap (K \cup \partial C)$ yields the disks of $S^2 \setminus C$, together with annuli lying to either side of S^2, through which the boundaries of the green and brown disks are respectively parallel to the all-A and all-B states of the link diagram D on S^2. As claimed, Σ is therefore the Turaev surface of the link diagram D on S^2. $\square$

4.2. **Generalization to arbitrary dual states.** As noted at the end of §2.3, the construction of the Turaev surface from the all-A and all-B states of a link diagram D on S^2 generalizes to any pair of states of D which are dual to one another, having opposite smoothings at each crossing. The correspondence developed in §3 and §4.1 between link-adapted Heegaard diagrams $(\Sigma, \alpha, \beta, D)$ and Turaev surfaces extends to these generalized Turaev surfaces, the only difference being that D no longer need alternate on Σ.

Theorem 4.2. *There is a 1-to-1 correspondence between generalized Turaev surfaces of connected link diagrams on $S^2 \subset S^3$, and diagrams $(\Sigma, \alpha, \beta, D)$ with the following properties:*

- *(Σ, α, β) is a Heegaard diagram for S^3, with $\alpha \pitchfork \beta$.*
- *D is a link diagram on Σ which cuts Σ into disks, with $D \pitchfork \alpha$ and $D \pitchfork \beta$.*
- *$D \cap \alpha = D \cap \beta = \alpha \cap \beta$, none of these points being crossings of D.*
- *There is a checkerboard partition $\Sigma \setminus (\alpha \cup \beta) = \Sigma_\varnothing \cup \Sigma_K$, in which $\Sigma_\varnothing$ consists of disks disjoint from D, in which D cuts Σ_K into disks each of whose boundary contains at least one crossing point and at most two points of $\alpha \cap \beta$, and in which $2g(\Sigma) + |\Sigma_\varnothing| = |\alpha| + |\beta|$.*

Proof. Given the generalized Turaev surface Σ for dual states s and $\tilde{s}$ of a connected link diagram D on $S^2 \subset S^3$, reverse some collection of the crossings of D to obtain a new link diagram D' for which s and $\tilde{s}$ are the all-A and all-B states. Construct the corresponding diagram $(\Sigma, \alpha, \beta, D')$ as in §3. Finally, switch back the reversed crossings of D' to obtain the required diagram $(\Sigma, \alpha, \beta, D)$.

Conversely, suppose that $(\Sigma, \alpha, \beta, D)$ is as described. The proof of Theorem 4.1 extends almost verbatim. The only concern, as D need not alternate on Σ, is whether or not the disks of $\Sigma \setminus D$ admit a checkerboard partition; it suffices to show that they do.

The condition that $D \cap \Sigma_\varnothing = \varnothing$ implies that one endpoint of each arc of $(\alpha \cup \beta) \setminus D$ appears as in Figure 9, and the other appears as the mirror image. Thus, each attaching circle intersects D in an even number of points. The fact that the attaching circles generate $H_1(\Sigma)$ then implies that any simple closed curve on Σ in general position with respect to D must intersect D in an even number of points, and hence that the disks of $\Sigma \setminus D$ admit a checkerboard partition. □

4.3. **Conclusion.** Up to isotopy, each link diagram on $S^2 \subset S^3$ has a unique Turaev surface. Theorem 4.1 thus establishes – via Turaev surfaces – a 1-to-1 correspondence between link diagrams on $S^2 \subset S^3$ and alternating, link-adapted Heegaard diagrams $(\Sigma, \alpha, \beta, D)$.

Similarly, Theorem 4.2 establishes – via generalized Turaev surfaces constructed from dual states – a 2-to-1 correspondence between states of link diagrams on $S^2 \subset S^3$ and link-adapted Heegaard diagrams $(\Sigma, \alpha, \beta, D)$ for S^3 in which D need not alternate on Σ.

5. Appendix

Let (Σ, α, β) be a Heegaard diagram for S^3, and let $\gamma \subset \Sigma$ be an oriented, simple closed curve. The following construction yields an expression for $[\gamma] \in H_1(\Sigma)$ in terms of the homology classes of the oriented attaching circles, proving that the latter generate $H_1(\Sigma)$.

Because $H_1(S^3)$ is trivial, γ bounds a Seifert surface $S \subset S^3$, on which γ induces an orientation. Fixing γ, isotope S so that its interior intersects Σ transversally – along simple closed curves and along arcs with endpoints on γ.

Given a component $S_{\alpha,i}$ of $S \cap H_\alpha$, one may obtain an expression for $[\partial S_{\alpha,i}] \in H_1(\Sigma)$ in terms of the $[\alpha_j]$ by surgering $S_{\alpha,i}$ along successive outermost disks of $\hat{\alpha} \setminus S_{\alpha,i}$ until $\partial S_{\alpha,i}$ lies entirely in the punctured sphere $\Sigma \setminus \alpha$, at which point the expression is evident. An analogous procedure expresses the homology class of each component of $S \cap H_\beta$ in terms of the $[\beta_j]$. Summing over all components of $S \setminus \Sigma$ gives the desired expression for $[\gamma] \in H_1(\Sigma)$:

$$[\gamma] = [\partial S] = \sum_{\substack{\text{Components} \\ S_{\alpha,i} \text{ of } S \cap H_\alpha}} [\partial S_{\alpha,i}] + \sum_{\substack{\text{Components} \\ S_{\beta,i} \text{ of } S \cap H_\beta}} [\partial S_{\beta,i}] = \sum_{i,j} a_{i,j}[\alpha_j] + \sum_{i,j} b_{i,j}[\beta_j]$$

Conversely, if (Σ, α, β) is a Heegaard diagram for a 3-manifold M with nontrivial first homology, then the oriented attaching circles do not generate $H_1(\Sigma)$, since inclusion $\Sigma \hookrightarrow M$ induces a surjective map $H_1(\Sigma) \to H_1(M)$, whose kernel contains all the $[\alpha_j]$ and $[\beta_j]$.

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