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Please picture, in your mind's eye, a surface F with nonempty boundary $\partial F = K$ embedded in 3-space $\mathbb{R}^3$. If F and K are both compact and connected, then K is embedded circle, called a **knot**, and F is said to **span** K. (If K is disconnected, it is an embedded disjoint union of circles, called a link; for simplicity I will not mention links again until §5.) I choose to study knots, and to do so chiefly via *spanning surfaces*, in part because luck and privilege have allowed me the opportunity, and in part because I find these objects to be radically accessible on a human level and far-reaching in their mathematical significance. I describe some of this significance in §5.

In general, math research aims to discover and prove new mathematical facts; yet, proving *that* something is true does not always convey *why* it is true; to *justify* is not necessarily to *illuminate*. My research, however, emphasizes *understanding* above knowledge. This theme will recur throughout my statement, which is organized as follows:

- §1 sketches the way that linear algebra arises in my research;
- §2 describes several related problems that nicely capture the spirit of my research; one of these is a 30 (or 125) year old problem that I recently solved;
- §3 summarizes the results of my 14 completed papers;
- §4 discusses the two Master's thesis projects that I have co-advised at Wake Forest together with other possible topics for future undergraduate research; and
- §5 motivates knot theory vis a vis mathematics more broadly.

1. The starring role of linear algebra in my research

Every knot in 3-space has spanning surfaces, lots of them actually. Given a knot $K \subset \mathbb{R}^3$, take a generic projection of K to $\mathbb{R}^2$, and record over-under information at the self-intersections, or *crossings*, to get a **diagram** D of K. It is always possible to color the regions of $\mathbb{R}^2 - D$ light and dark in "checkerboard fashion," as shown left in Figure 1, so that like-shaded regions abut only at crossings. If we do this so that the unbounded region is light, then there is a spanning surface B for K that lies almost entirely in the dark regions of $\mathbb{R}^2 - D$, except near crossings, where it twists; B is called a **checkerboard surface**. If we instead view $K \subset S^3$ (where $S^3 - \{\text{point}\}$ is $\mathbb{R}^3$) and project to S^2, then there are two checkerboard surfaces B and W, one of which lies almost entirely in the dark regions of $S^2 - D$, the other in the light ones. See Figure 1, center and right, which show the same pair of checkerboard surfaces. Different diagrams of the same knot usually give different checkerboard surfaces.

There is a symmetric bilinear pairing $\langle \cdot, \cdot, \rangle : H_1(F) \times H_1(F) \to \mathbb{Z}$ on the first homology group of any spanning surface F,[1] first described by Gordon and Litherland [8].[2] It is easiest to describe when

[1] $H_1(F)$ is a free abelian group whose elements are represented by (disjoint unions of) oriented circles embedded in F; its rank, denoted $\beta_1(F)$, counts "how many holes" are in F

[2] This is my all-time favorite paper, partly because its content, but especially because they prove their results twice: first, they take a sophisticated approach, involving double-branched covers of the 4-ball (in some sense, this is the best way to understand the deep significance of their work), and then, restarting from scratch, they take a down-to-earth approach that uses only elementary tools and thus makes this profound paper remarkably accessible.

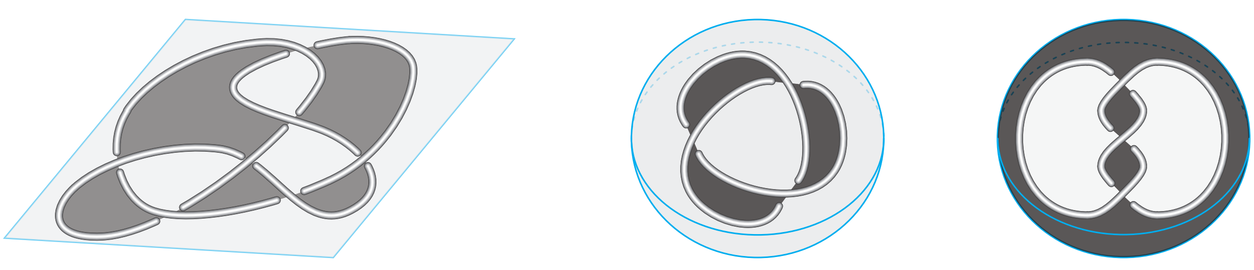

Figure 1. Checkerboard surfaces of knot diagrams

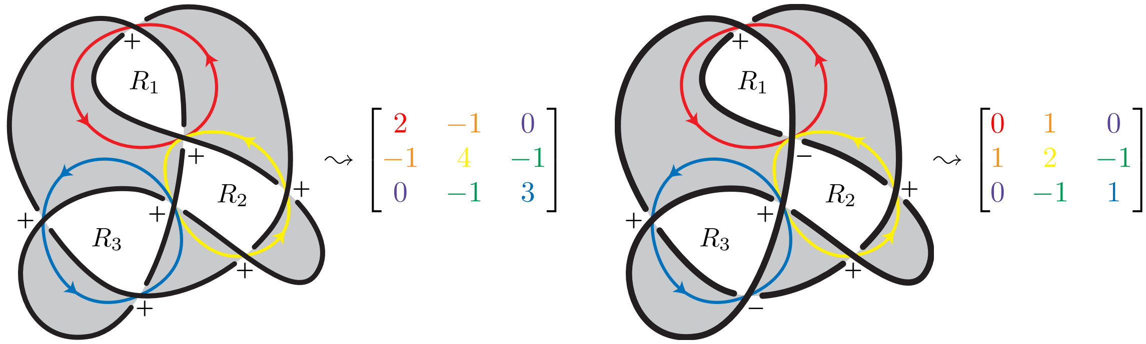

FIGURE 2. Goeritz matrices for two checkerboard surfaces

F is the dark checkerboard surface of a diagram $D \subset \mathbb{R}^2$: then $H_1(F)$ has a basis $\mathcal{A} = (a_1, \ldots, a_n)$ represented by circles α_i, each of which goes counterclockwise around exactly one bounded light region R_i, and $\langle \cdot, \cdot \rangle$ is represented, with respect to $\mathcal{A}$, by a **Goeritz matrix** $G = (g_{ij})$,[3] where each g_{ii} counts (with sign) the number of crossings incident to R_i and each g_{ij} counts (with opposite sign) the number of crossings incident to both R_i and R_j. Figure 2 shows two examples. Here are four key features of **Gordon-Litherland pairings** $\langle \cdot, \cdot \rangle$ and Goeritz matrices $G \in \mathbb{Z}^{n \times n}$:

- If $\alpha \subset F$ is a circle representing a homology class a, then the self-pairing $\langle a, a \rangle$ equals (half of) the *framing* of α in F, which measures how much F "twists" along α.
- Since $G \in \mathbb{Z}^{n \times n}$ represents a bilinear mapping, the change-of-basis formula is $G \to PGP^T$, where $P \in \mathbb{Z}^{n \times n}$ is invertible (i.e. "unimodular").
- "Attaching a crosscap" as in Figure 3 changes a Goeritz matrix like this: $G' \to \begin{bmatrix} G & \mathbf{0} \\ \mathbf{0} & \pm 1 \end{bmatrix}$.
- $|\det(G)|$ depends only on K, not on one's choice of F or of basis for $H_1(F)$.[4]

Not only does every knot K have spanning surfaces, it has some, called **Seifert surfaces**, that are orientable (i.e. 2-sided). *Seifert's algorithm* is a way of constructing such a surface using a diagram D of K; Figure 4 shows an example. Any Seifert surface F is homeomorphic to a connect sum of tori with an open disk removed: $F \cong \left(\#_{i=1}^n S^1 \times S^1\right) - (\text{open disk})$; we call n the *genus* of F and write $g(F) = n$. The **knot genus** $g(K)$ is the minimum of $g(F)$ over all Seifert surfaces F for K.

Any Seifert surface F comes with a bilinear pairing $H_1(F) \times H_1(F) \to \mathbb{Z}$ called the *Seifert pairing* and is represented by a *Seifert matrix* A. This pairing carries all the information of the Gordon-Litherland pairing, as $A + A^T$ is a Goeritz matrix for F, but it has something extra: the polynomial

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[3] Given $x = \sum_{i=1}^n x_i a_i$ and $y = \sum_{i=1}^n y_i a_i$ in $H_1(F)$ and denoting $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, we have $\langle x, y \rangle = \vec{x}^T G \vec{y}$.

[4] The same is also true of the signature of G, albeit with a correction term.

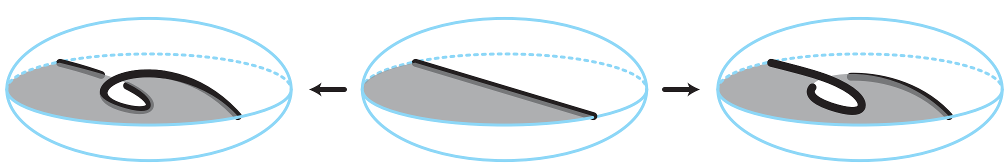

FIGURE 3. Attaching a crosscap to a spanning surface F is this local operation near ∂F.

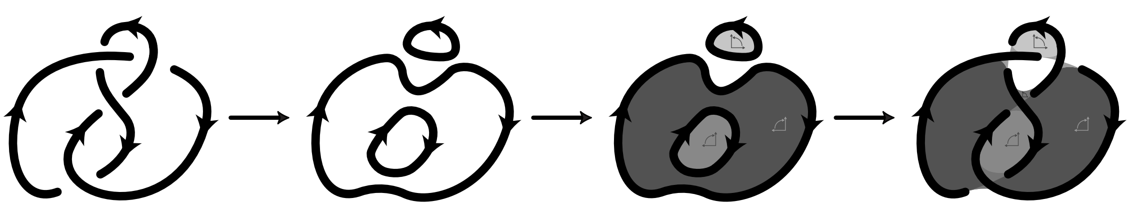

FIGURE 4. Seifert's algorithm obtains an *orientable* spanning surface from any knot diagram.

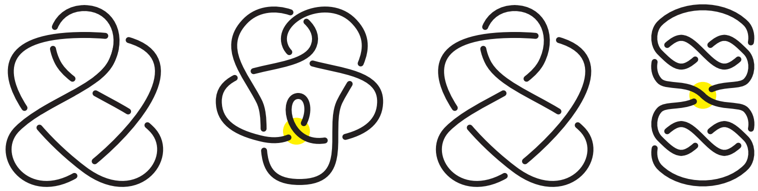

FIGURE 5. Four knot diagrams, three of them alternating, two with nugatory crossings (highlighted)

$\Delta_K(t) = \det(A - tA^T)$, called the **Alexander polynomial** of F, depends (up to degree shift) only on K. Moreover, the degree span or *breadth* $\mathrm{bth}(\Delta_K(t))$ depends only on K and provides a lower bound for $g(K)$: $\mathrm{bth}(\Delta_K(t)) \leq g(K)$. More on this polynomial toward the end of §2.

2. Classical knowledge, new understanding

Much of my research concerns **alternating** knots: a diagram $D \subset S^2$ of a knot $K \subset S^3$ is alternating if its crossings alternate between over and under, like the last three diagrams in Figure 5, and K is alternating if it has an alternating diagram; $c(D)$ denotes the number of crossings in D, and $c(K)$ is the smallest number of crossings among all diagrams of K. (We regard two knots as equivalent if they are related by a continuous deformation between their embeddings, called an *isotopy*.) In general, it is hard to determine the crossing number of an arbitrary knot, or to determine whether or not a given diagram minimizes crossings, but certainly no diagram that minimizes crossings can have a "nugatory" crossing, like the two highlighted in Figure 5. A diagram without nugatory crossings is called *reduced*, whether or not it minimizes crossings. Interestingly, if a knot K has a diagram D that is reduced *and* alternating, then D *always* minimizes crossings: $c(D) = c(K)$. Although this fact was first observed empirically by P.G. Tait in 1898 [28], it, and two related conjectures, remained unproven for almost a century:

Tait's conjectures. *Given a knot $K \subset S^3$ and two prime*[5] *alternating diagrams D and D' of K:*

(1) D and D' have the same number of crossings, which is minimal: $c(D) = c(D') = c(K)$.
(2) D and D' have the same writhe:[6] *$w(D) = w(D')$.* [7]
(3) D and D' are related by a sequence of flype moves: see Figure 6.

Tait's conjectures remained open until the discovery of the Jones polynomial in the mid-1980's [13],[8] which almost immediately led to three independent proofs of Tait's first conjecture Jones polynomial of a knot K: the degree span, or *breadth*, of $V_K(t)$, denoted $\mathrm{bth}(V_K(t))$, always provides

[5] A knot diagram D is *prime* if D has at least two crossings and, for every circle $\gamma \subset S^2$ that intersects D generically in two points, all crossings of D lie on the same side of γ; every prime diagram is reduced.

[6] Orient D arbitrarily. Then every crossing looks like ⤫ or ⤬. The *writhe* of D is $w(D) = |$⤫$| - |$⤬$|$, where bars count components. It is independent of the orientation on D.

[7] Tait's first two conjectures are true even if D and D' are not prime, as long as they are reduced.

[8] Jones described a new way of assigning a polynomial $V_K(t) \in \mathbb{Z}[t, t^{-1}]$ to any knot; different pictures of the same knot always yield the same polynomial, but different knots "usually" have different Jones polynomials.

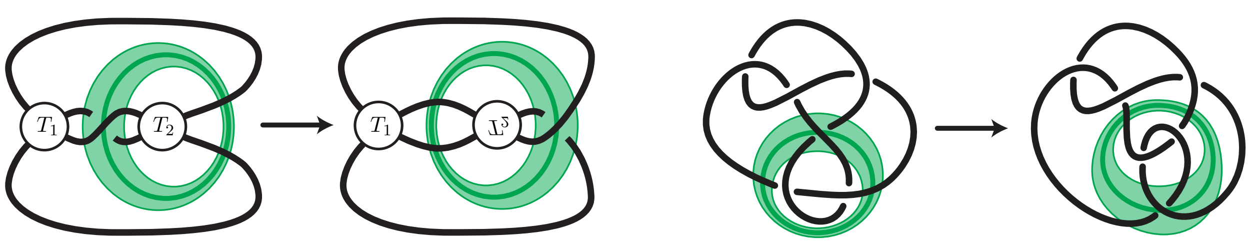

FIGURE 6. *Flype* moves

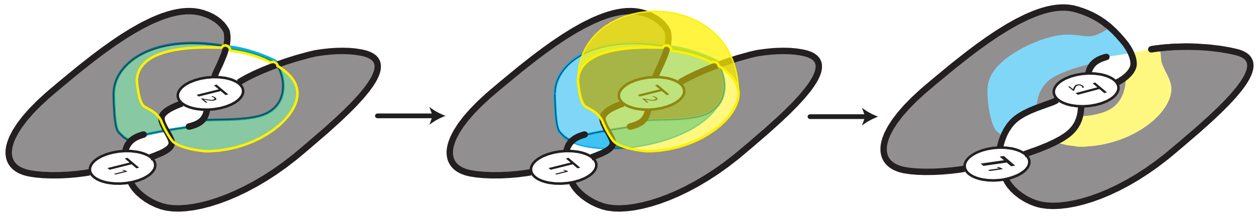

FIGURE 7. A flype move on a knot diagram corresponds to a re-plumbing move on one of its checkerboard surfaces (here, the black surface) and an isotopy of the other surface (here, the white surface).

a lower bound for crossing number, $c(K) \leq \text{bth}(V_K(t))$, and any reduced alternating diagram D of K satisfies $c(D) = \text{bth}(V_K(t))$;[9] by definition, $c(K) \leq c(D)$; therefore, $c(D) = c(K)$. QED

Within a decade, the Jones polynomial had led to proofs of all three of Tait's conjectures. Yet, these proofs were only somewhat satisfying: why did they work? What was the Jones polynomial really measuring? In their 1993 proof of Tait's flyping conjecture, Menasco and Thistlethwaite spotlighted this remaining gap in our understanding: "the question remains open as to whether there exist purely geometric proofs of this and other results that have been obtained with the help of new polynomial invariants." The aim is not just to know, but to understand. I love that the mathematical community values this.

The first partial answer came in 2017 from Greene in a paper where he answered another long-standing question, this one from Ralph Fox: "What [geometrically] is an alternating knot?" First, Greene observed that, if B and W are the checkerboard surfaces from an alternating knot diagram and G_B and G_W are their Goeritz matrices, then G_B is positive-definite and G_W is negative-definite, or vice-versa: "B and W are definite and of opposite signs."[10] Second, Greene proved that alternating knots are the only ones with a two such surfaces. In fact, he proved that if F_+ and F_- are definite spanning surfaces of opposite signs for the same knot K, then K has an alternating diagram whose checkerboard surfaces are "the same as" (are isotopic to) F_+ and F_-. Greene was then able to translate Tait's conjectures into *non-diagrammatic* statements and give the first "purely geometric" proofs of Tait's second conjecture and part of his first: any reduced alternating diagrams D and D' of the same knot satisfy $c(D) = c(D')$ and $w(D) = w(D')$ [9].[11]

Recently, I gave the first purely geometric proof of Tait's "flyping" conjecture. The first key insight (see Figure 7) was that flyping a diagram D changes one of its checkerboard surfaces by isotopy and the other, F, by a geometric operation I call *re-plumbing*, which replaces a disk $U \subset F$ (shown green, left) with another disk V (shown half yellow and half blue, center) that is disjoint from F except along its boundary $\partial V = \partial U$. Figure 8, left, shows another example of re-plumbing. This insight translates Tait's diagrammatic conjecture to a geometric statement about spanning surfaces. Then the real work begins. I would love to tell you that story; please ask me about it some time.

Around the time I proved Tait's flyping conjecture, Boden-Karimi extended Greene's insights about definite surfaces in S^3 to the context of thickened surfaces, like $S^1 \times S^1 \times [-1, 1]$ [1]. I

[9] The converse is also (nearly) true: any prime diagram D of any knot K satisfies $c(D) = \text{bth}(V_K(t))$ if and only if D is alternating.

[10] $G_B \in \mathbb{Z}^{n\times n}$ is positive-definite iff $\vec{x}^T G_B \vec{x} > 0$ for every nonzero $\vec{x} \in \mathbb{Z}^n$.

[11] It remains an open problem to give a purely geometric proof of Tait's full first conjecture, since Greene's insights are less useful regarding the possibilities of non-alternating diagrams of an alternating knot. I have some ideas....

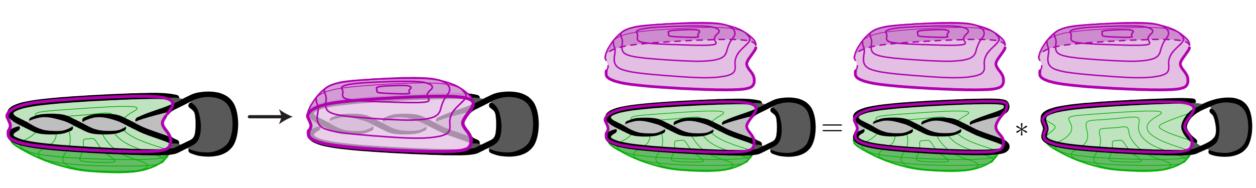

FIGURE 8. Left: A replumbing move. Right: a de-plumbing.

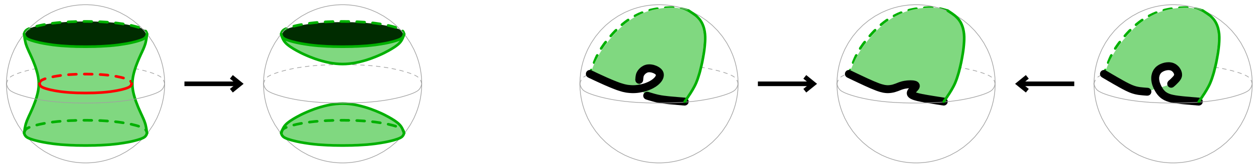

Figure 9. An spanning surface is geometrically essential if it admits none of these simplifying moves.

used their innovations to extend Tait's flyping conjecture to that context. Many of the arguments adapted directly, but a few technical points required special attention, several of which led to fundamental revelations about *virtual knots*, which correspond to knots in thickened surfaces. See §3.

Another recent project of mine regards a different seminal result in knot theory that, like Tait's flyping conjecture, was first proven in the Annals. Again, I wanted to *understand why* the result was true. Here is the background.

In general, a Seifert surface resulting from Seifert's algorithm *need not* have minimal genus, but when Seifert's algorithm is applied to an *alternating* diagram, the resulting surface F *always* has minimal genus: $g(F) = g(K)$. Gabai gave a short, elegant, purely geometric, proof of this fact in 1986 [6]. The original proofs, however, due independently to Crowell and Murasugi in 1958-59 [3, 22], had something extra, regarding the Alexander polynomial $\Delta_K(t)$. Recall from §2 that every knot K satisfies $\text{bth}(\Delta_K(t)) \leq g(K)$. Crowell and Murasugi proved that when F is obtained via Seifert's algorithm from an alternating diagram, we have $g(F) = \frac{1}{2}\text{bth}(\Delta_K(t))$. Since $g(K) \leq g(F)$ by definition, it follows that $g(F) = g(K) = \text{bth}(\Delta_K(t)) \leq g(K)$. When I taught a topics course at UNL in 2021, I wanted to understand, and then share, *why* this was true. This led me to discover a new, short, extremely satisfying proof of the Crowell-Murasugi theorem [K8]. It involves all my favorite characters: checkerboard surfaces, plumbing operations, and linear algebra. Here is a sketch:

First, to prove that $g(F) = g(K) = \frac{1}{2}\text{bth}(\Delta_K(t))$, I showed that it suffices to prove that the Seifert matrix A from F is invertible. Second, I showed that this is true if F happens to be a checkerboard surface (from an alternating diagram). Third, the surfaces obtained via Seifert's algorithm from alternating diagrams are always plumbings of alternating checkerboard surfaces, so it suffices to prove that if F is a plumbing of surfaces F_1 and F_2 which have invertible Seifert matrices A_1 and A_2, then F also has an invertible Seifert matrix. Finally, I showed that, indeed, F has a Seifert matrix of the form $A = \begin{bmatrix} A_1 & 0 \\ B & A_2 \end{bmatrix}$; applying the pigeonhole principle to the formula $\det[a_{ij}]_{i=1}^n = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)}$ thus confirms that $\det(A) = \det(A_1) \cdot \det(A_2) \neq 0$.

3. Papers

Here are synopses of my fourteen completed papers, in reverse chronological order:

3.1. The essence of a spanning surface [K14].

A spanning surface F for a knot $K \subset S^3$ is *geometrically essential* if it admits neither of the simplifying moves shown in Figure 9, and is π_1-*essential* if F is neither  nor  and the inclusion map of the interior of F into the knot complement $S^3 - K$ induces an injective map on fundamental groups. Essential surfaces are fundamental to our understanding of the topology and geometry of 3-manifolds, but the two notions of "essential" were rarely mentioned in the 20th century, largely because they are equivalent for *orientable* spanning surfaces. For example, Gabai proved in 1983 that any plumbing of essential Seifert surfaces is essential, no qualifier needed [5]. In 2011, however, when Ozawa extended Gabai's result to all spanning surfaces, regardless of orientability, he needed to specify that his surfaces were π_1-essential [24]. The question remained open whether plumbing likewise respects the property of being *geometrically* essential. My first main result answers this question in the negative via the example in Figure 10. The only hard part of the proof is showing that the surface on the left is indeed geometrically essential; I develop a new technique to do this. The second main result extends Ozawa's result in another direction. I introduce a quantity that measures "how essential" a spanning surface F is. I call it the *essence* of F, denoted $\text{ess}(F)$; F is π_1-essential if and only if

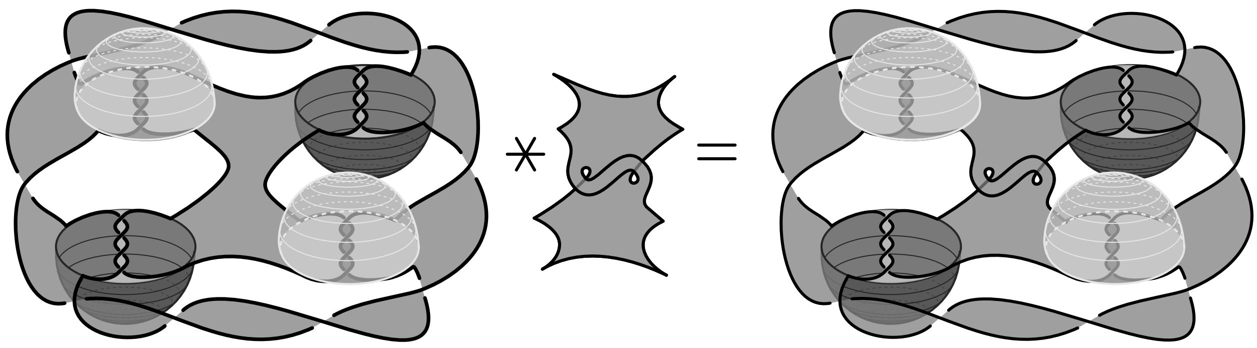

FIGURE 10. Plumbing does not always respect the property of being geometrically essential.

$\mathrm{ess}(F) \geq 2$. I extend Ozawa's theorem by proving that if F is a plumbing of π_1-essential surfaces F_1 and F_2, then $\mathrm{ess}(F) \geq \min_{i=1,2} \mathrm{ess}(F_i)$.

3.2. **The virtual flyping theorem** [K13]**.** As described in §2, I extended Tait's flyping conjecture to knots in thickened surfaces. In the process, I encountered some interesting subtleties, leading to three spin-off projects.

3.3. **What is a virtual link diagram?** [K12]**.** Under a long-known correspondence, any knot K in a thickened surface $\Sigma \times [-1, 1]$ corresponds to a unique *virtual knot* [15, 14, 2]. Figure 11 shows the rough idea:[12] take a diagram D of K on Σ, embed Σ in S^3, and project to S^2 to get a *virtual knot diagram* V. Each crossing from D gives a "classical" crossing in V, but there are often other self-intersections in V, called virtual crossings, which are instead marked with a circle. Presumably, for this correspondence to make sense, different embeddings of Σ must all give the same virtual knot.[13] Yet, I found simple examples where this did not seem to be the case: the three virtual diagrams shown bottom-right in Figure 11 all represent different virtual links. What was I misunderstanding?

It turns out that there was a hidden subtlety: one must embed Σ in S^3 so that all crossings of D lie on its *front* of Σ. The early papers about virtual knots had not needed to mention this explicitly, because they never went straight from a diagram $D \subset \Sigma$ to a virtual diagram V; instead, they used an intermediate step, called an abstract link diagram. Everything they did was correct, but this subtlety remained hidden. And so it remained, to my knowledge, for 20 years (which featured roughly 300 published papers about virtual knots and links), until now.

[12] I am suppressing a detail here about "stable equivalence."

[13] A virtual knot is defined to be an equivalence class of virtual diagrams; the equivalence relation is generated by seven diagrammatic moves.

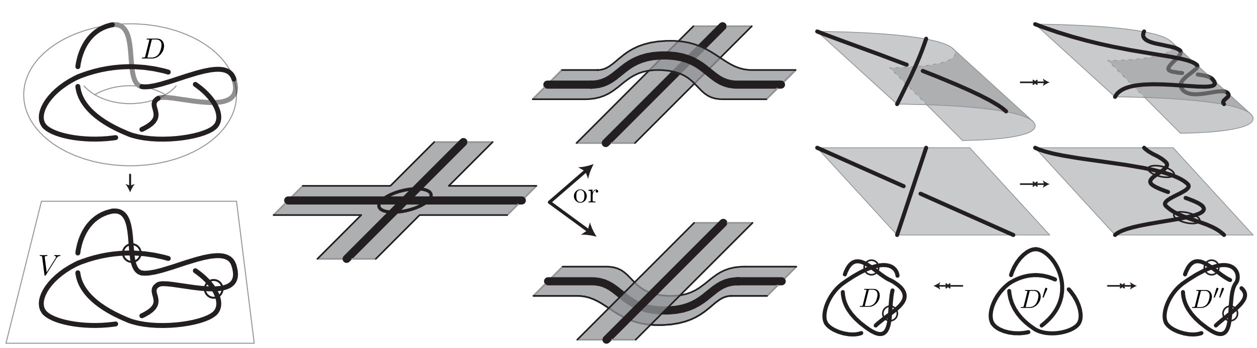

FIGURE 11. Left: A knot diagram D on a torus and a corresponding virtual diagram V. Center: To recover D from V, change each virtual crossing in V like this and (abstractly) cap off the resulting surface. Right: the caveat.

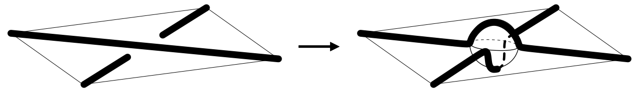

FIGURE 12. Menasco's crossing ball structures

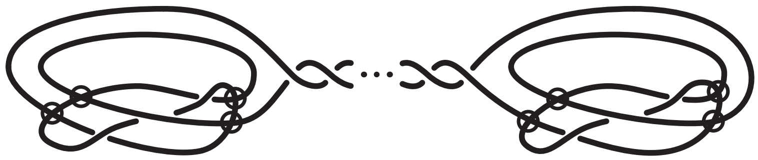

FIGURE 13. How to construct infinitely many distinct connect sums of the virtual knots shown left and right

3.4. **End-essential spanning surfaces for links in thickened surfaces** [K11]**.** If $D \subset S^2$ is a reduced, alternating diagram of a knot $K \subset S^3$, then one can use D to embed K near S^2: near each crossing of D, insert a tiny bubble C and push the two strands inside C to opposite hemispheres of C according to the over-under information. See Figure 12. Then, given a spanning surface F for K, if F is (geometrically and π_1) essential, one can continuously deform F (while fixing its boundary) so that it intersects "nicely" with the projection sphere S^2 and all the inserted bubbles C. In particular, one can insist that S^2 and these bubbles cut F into disks; if not, then, because it is essential, F admits a certain simplifying move.

Once F is set up this way, all sorts of combinatorial arguments are possible. This technique, introduced by Menasco [17], is vital to several of my papers [K1, K4, K9, K13, K14]. In the context of a thickened surface, however, being essential is not quite enough to insist that F gets cut into disks in this way. In this paper, I introduced a slightly strengthened notion of essential surfaces, adapted to this context, and proved that if F is "end-essential," then the projection surface and crossing bubbles always cut it into disks, as needed. I also proved that, just as both checkerboard surfaces from any reduced alternating diagram on S^2 are essential in S^3, both checkerboard surfaces from any reduced alternating on a closed surface Σ diagram are end-essential in $\Sigma \times I$.

3.5. **Primeness of alternating virtual links** [K10]**.** A knot $K \subset S^3$ or knot diagram $D \subset S^2$ is **prime** if it cannot be written as a trivial connect sum. For virtual knots and their diagrams, however, things are somewhat more complicated. In fact, I realized that there were two common, but different notions of primeness for virtual knots, and likewise for their diagrams. In addition to articulating this distinction, I proved that, given an alternating virtual diagram, one can tell by inspection whether or not the link it represents is prime in either sense. This distinction is particularly important for the virtual flyping theorem which holds for alternating virtual link diagrams that are "weakly prime", whether or not they are "prime" (in the stronger sense). As a corollary, it follows that, given any two non-classical alternating virtual knots, there are infinitely many distinct ways to take their connect sum; see Figure 13.

3.6. **A geometric proof of the flyping theorem** [K9]**.** As described in §2, I gave the first purely geometric proof of Tait's flyping conjecture, resolving an open question of 30 (or 125) years [18, 20].

3.7. **A simple proof of the Crowell-Murasugi theorem** [K8]**.** As described in §2, I gave a short, elementary proof of the theorem, due to Crowell-Murasugi, that any surface F obtained by applying Seifert's algorithm to an alternating diagram of a knot K satisfies $g(F) = g(K) = \frac{1}{2}\text{bth}(\Delta_K(t))$.

3.8. **Efficient multisections of odd-dimensional tori** [K7]**.** This is my one paper, so far, that has nothing to do with knots. Its main result is a construction regarding piecewise-linear, or PL, manifolds, but most of the hard work in the paper is combinatorial. In arbitrary odd dimension $n = 2\ell - 1$, I described a symmetric, efficient **multisection** of the n-torus $T^n = S^1 \times \cdots \times S^1$ (n copies): this is a decomposition into ℓ simple pieces X_i, $i \in \mathbb{Z}/\ell\mathbb{Z}$ with the following intersection property: for each $I \subset \mathbb{Z}_\ell$, the dimension and complexity of $X_I = \bigcap_{i \in I} X_i$ depend only on $|I|$

(namely, $\dim(X_I) = n+1-|I|$, and X_I can be built from a ball of that dimension d by gluing on "h-handles" $B^h \times B^{d-h}$ along $S^{h-1} \times B^{d-h}$ for $1 \leq h \leq |I|$).[14] In particular, each X_i is an n-dimensional 1-handlebody, homeomorphic to a "thickened up" wedge of some number, g, of circles;[15] g is called the *genus* of X_i.

Rubinstein and Tillmann had introduced multisections shortly before this [27], as a natural generalization of an important construction in 3-, and more recently 4-dimensions [7]. They proved that every PL manifold of arbitrary (finite) dimension has a multisection. Yet, their construction tends to produce handlebodies of very large genus, even for simple manifolds. By contrast, the multisections that I constructed for T^n were *efficient* in the sense that each X_i has genus n, which I proved is minimal. Each multisection is also *symmetric* with respect to both the permutation action of S_n on the indices and the $\mathbb{Z}_\ell$ translation action along the main diagonal. I also constructed a related trisection of T^4, lifted all symmetric multisections of tori to certain cubulated manifolds, and obtained two combinatorial identities as corollaries. For example, for any $n = 2\ell - 1$, we have:

$$\ell^{n-1} = \sum_{i_0=2}^{n} \binom{n}{i_0} \sum_{i_2=4-i_0}^{n-i_0} \binom{n-i_0}{i_1} \sum_{i_3=6-i_0-i_1}^{n-i_0-i_1} \binom{n-i_0-i_1}{i_2} \cdots \sum_{i_{\ell-2}=2\ell-2-\sum_{j=0}^{\ell-3} i_j}^{n-\sum_{j=0}^{\ell-3} i_j} \binom{n-\sum_{j=0}^{\ell-3} i_j}{i_{\ell-1}},$$

which also equals the number of spanning trees of the complete bipartite graph $K_{\ell,\ell}$ [23].

3.9. **Nonorientable spanning surfaces for knots** [K6]**.** In this short, mainly expository, chapter, I also proved two new theorems, extending the well-known fact that all Seifert surfaces for a knot are related by attaching and deleting tubes and the related fact, proven by Yasuhara, that all spanning surfaces for a given knot (or link) are related by attaching and deleting tubes and crosscaps [30]. (These are the moves shown in Figure 9 and their inverses.) One new result was that, given a knot K, all checkerboard surfaces from all diagrams of K are related by attaching and deleting crosscaps, no tubing moves needed. More on this in §4

3.10. **Crosscap numbers of alternating knots via unknotting splices** [K5]**.** The crosscap number of a knot K is $cc(K) = \min\{\beta_1(F) :\ F \text{ is 1-sided and spans } K\}$.[16] While an earlier paper I wrote with Colin Adams (see §3.14) had determined the crosscap number of every alternating knot in theory, the problem remained open as to how best to compute these in practice. In 2018, Ito and Takimura defined a new diagrammatic knot invariant $u^-(K)$ which I later called the **splice-unknotting number** and proved that every knot satisfies $cc(K) \leq u^-(K)$. For alternating knots K, I used tangle decompositions to prove the reverse inequality, giving $cc(K) = u^-(K)$ (Ito-Takimura independently proved this as well [11, 12]). Then I learned Python and wrote two different programs, each of which used this new result to compute crosscap numbers of all alternating knots through 13 crossings. See §4 for more detail on the computational aspect of this project.

3.11. **Alternating links have representativity 2.** [K4]**.** Gromov defined the **distortion** $\delta(\gamma)$ of a rectifiable curve $\gamma \subset \mathbb{R}^3$ by considering how many times farther apart two points are along the curve than in space:

$$\delta(\gamma) = \sup_{p,q\in\gamma} \frac{d_\gamma(p,q)}{d_{\mathbb{R}^3}(p,q)}$$

Gromov asked whether knots have arbitrarily large distortion [10]. To show that they do, Pardon established a lower bound for distortion, $160\delta(L) \geq r(L)$, in terms of what is now called the **representativity** of L [26]:

$$r(L) = \max_{F\in\mathcal{F}_L} \min_{X\in\mathcal{X}_F} |\partial X \cap L|.$$

Here, $\mathcal{F}_L$ is the set of positive genus closed surfaces containing L, and $\mathcal{X}_F$ is the set of compressing disks for F. Ozawa computed the representativity of certain pretzel links and all torus and 2-bridge links, and conjectured that alternating links have representativity 2 [25]. I used Menasco's crossing ball structures to confirm Ozawa's conjecture:

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[14]Notation: $B^h = \{\vec{x} \in \mathbb{R}^h :\ |\vec{x}| \leq 1\}$ and $S^{h-1} = \partial B^h = \{\vec{x} \in \mathbb{R}^h :\ |\vec{x}| = 1\}$.

[15]A wedge of g circles is the space obtained from a disjoint union of g circles by choosing one point on each circle and identifying the chosen points to a single point.

[16]Recall that $\beta_1(F)$ denotes the rank of $H_1(F)$ and counts "how many holes" are in F.

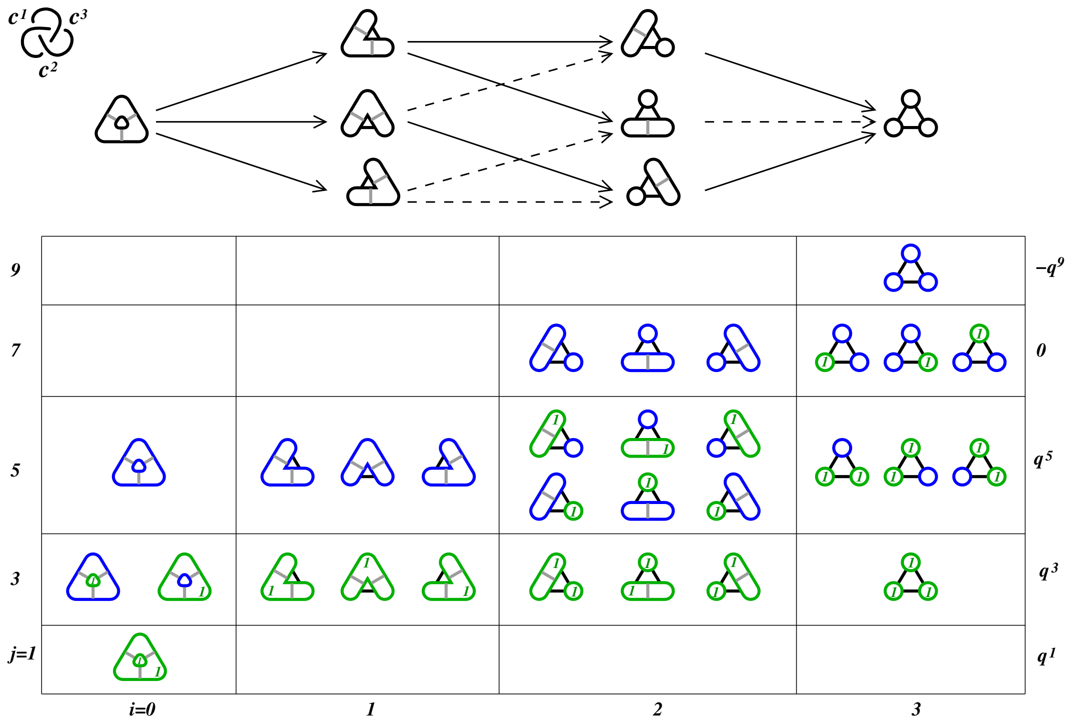

FIGURE 14. The Jones polynomial of this knot K is $V_K(q) = q + q^3 + q^5 - q^9$.

Theorem 3.1. *If $L \subset S^3$ is an alternating link and F is a closed surface that contains L (without crossings), then F has a compressing disk whose boundary intersects F in at most two points.*

3.12. **Plumbing essential states in Khovanov homology** [K3]**.** Earlier, I mentioned the Jones polynomial, but so far I have not said anything about what it is, or how it is computed. Here is one perspective. Given a diagram D of a knot K, "smooth" each crossing in one of two ways, $\xleftarrow{A}$ $\xrightarrow{B}$. The resulting diagram x is called a (Kauffman) **state** of D and consists of *state circles* joined by A- and B-labeled arcs, one from each crossing. "Enhance" x by assigning each state circle a binary label: $\bigcirc^1 \xleftarrow{1} \bigcirc \xrightarrow{0} \bigcirc$. Then use the number of A- versus B-smoothings leading to each state and the number of blue versus green circles in each enhancement to assign each "enhanced state" X a bigrading (i_X, j_X), and use this bigrading to arrange the enhanced states in a grid; Figure 14 shows an example.[17] For each pair (i, j), let $C_{i,j}(D)$ be the free $\mathbb{Z}$-module generated by those enhanced states X with $(i_X, j_X) = (i, j)$. Then the Jones polynomial of K is

$$V_K(q) = \sum_j q^j \sum_i (-1)^i \mathrm{rank}(C_{i,j}(D)).^{18}$$

Amazingly, $V_K(q)$ does not depend on D. Even more amazingly, Khovanov described "(co)-boundary maps" $d : C_{i,j}(D) \to C_{i+1,j}(D)$ such that the resulting (co)homology groups also are independent of D [16, 29]. All of this works over any commutative ring R with $1 \neq 0$; I will focus on $R = \mathbb{Z}$ and $R = \mathbb{Z}/2\mathbb{Z}$.

When I first learned about Khovanov homology, I worked a few simple examples, all coming from alternating diagrams, and noticed that many of the nonzero Khovanov homology classes were represented by states that described essential surfaces (in a sense that I will explain in §3.14). In some cases, these essential surfaces were checkerboard surfaces, and in others they were obtained by plumbing checkerboard surfaces. Moreover, any "diagrammatic plumbing" of alternating checkerboard states gives what is called a **homogeneously adequate state**, and these always describe essential surfaces [24]. I proved that such states *always* correspond to nonzero Khovanov homology classes, at least over $\mathbb{Z}/2\mathbb{Z}$:

[17] The other consideration in the bigrading is the writhe of D; see Note 6.

[18] The version of the Jones polynomial described in §2 is obtained from this version by dividing by $q + q^{-1}$ and making a change of variable $q^2 = t$.

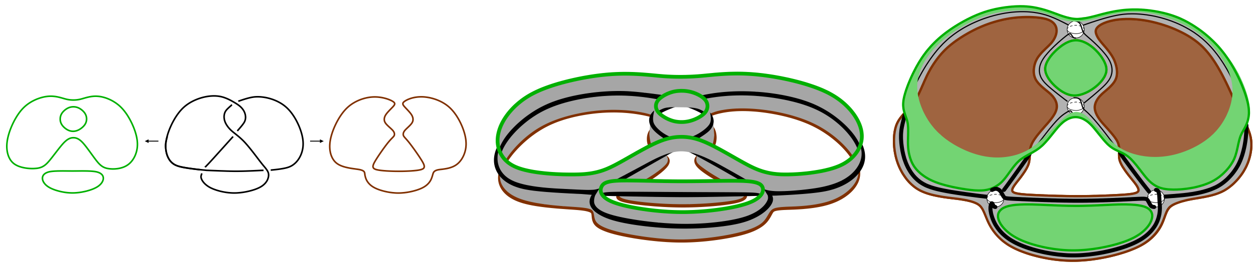

FIGURE 15. An example of how to construct a Turaev surface

Theorem 3.2. *If x is a homogeneously adequate state, then x gives nonzero Khovanov homology classes over $\mathbb{Z}/2\mathbb{Z}$ in two gradings. If a certain graph G_{x_A} is bipartite*[19]*, then this is also true with integer coefficients.*

After proving the theorem, I wanted to better understand the role of plumbing in Khovanov homology, at least in this class of examples. Had this gluing operation led me to a naive conjecture, which just turned out to be true, or was plumbing structurally important? I sought to express the nonzero classes from my theorem in terms of plumbing. Developing plumbing into an operation in Khovanov homology required a new rule of "trumps," (to indicate when the label on one circle overrides the label on another) with which plumbing behaves roughly like interior multiplication composed with an exterior product,

$$d(X * Y) = dX \;{}_{\diamond}\!* \, Y + (-1)^{|)(|_x} X *_{\diamond} dY,$$

leading to an alternate, inductive proof of the main theorem. (My first, simpler proof was constructive.) More recently, I co-advised a Master's thesis about Khovanov homology; see §4.

3.13. **Heegaard diagrams corresponding to Turaev surfaces (with Cody Armond and Nathan Druivenga)** [K2]**.** One builds a **Turaev surface** from a link diagram D on $S^2 \subset S^3$ by pushing the all-A and all-B states of D (see Figure 15, left) to opposite sides of S^2, joining them with a cobordism whose saddle points occur precisely at the crossings of D (center), and capping off each state circle with a disk (right). Figure 15 shows an example. Dasbach, Futer, Kalfagianni, Lee, and Stolzfus showed that the resulting surface Σ is a Heegaard surface for S^3, on which D forms an alternating diagram [4]. Our main theorem proved a converse to this fact, providing a correspondence between Turaev surfaces and certain link-adapted Heegaard diagrams [K2].

3.14. **A classification of spanning surfaces for alternating links (with Colin Adams)** [K1]**.** My first research project began in 2005, at Williams' REU, SMALL, where Colin Adams and I generalized Seifert's algorithm in order to construct many spanning surfaces from a given knot diagram. At the time, the construction was new; the surfaces we constructed have since become known as **state surfaces**. We continued the project in an independent study, at SMALL 2006, and in my senior thesis.

Using Menasco's crossing ball technique, we classified spanning surfaces for alternating knots (and links) up to homeomorphism type and boundary slope. That is, we solved the *geography* problem in the alternating case: given a knot K, the problem is to list all pairs $(s(F), \beta_1(F))$ realized by spanning surfaces F for K. The geography problem remains open for most classes of knots and is a fertile area for future research.

As an immediate corollary, our classification gives the **crosscap number** and **overall genus**[20] for all alternating links; previously, these were known only for 2-bridge links and certain pretzel knots. We also gave the first proof that all spanning surfaces for alternating links are connected and the first example of a knot whose overall genus can be realized by a 1-sided surface or by 2-sided surface. We explicitly computed the crosscap numbers of all alternating knots through 10 crossings and described an algorithm for computing the overall genus of an arbitrary alternating link.

[19]This corresponds to the A-smoothed part of x being a Seifert state.

[20]The crosscap number of K is $cc(K) = \min\{\beta_1(F) : F \text{ is 1-sided and spans } K\}$, and the **overall genus** is $\Gamma(K) = \min\{\beta_1(F) : F \text{ spans } K\}$.

4. Advising

Recall from §§1 and 3.9 that any two checkerboard surfaces F_1 and F_2 (from any two diagrams) for a given knot K are related by a sequence of "crosscapping moves" (Figure 3), which change a Goeritz matrix like this, $G \leftrightarrow \begin{bmatrix} G & \mathbf{0} \\ \mathbf{0} & \pm 1 \end{bmatrix}$; also, change of basis changes a Goeritz matrix like this: $G \leftrightarrow PGP^T$ and $G \leftrightarrow \begin{bmatrix} G & \mathbf{0} \\ \mathbf{0} & \pm 1 \end{bmatrix}$. This raises the following practical question: given F_1 and F_2, how to find such a sequence of moves between them? Maybe, I thought, a good approach would be to forget the surfaces altogether and just focus on the linear algebra: if G_1 and G_2 are the Goeritz matrices for F_1 and F_2, then any sequence of crosscapping moves between F_1 and F_2 gives a sequence of moves like $G \leftrightarrow PGP^T$ and $G \leftrightarrow \begin{bmatrix} G & \mathbf{0} \\ \mathbf{0} & \pm 1 \end{bmatrix}$ from G_1 to G_2. Last spring, I posed this question to a student, John, in my topology topics course as we walked together from our classroom back to the math building. The question quickly grew into a Master's thesis project, which I co-advise with Hugh Howards and Frank Moore. In particular, we soon discovered that not every move like $\begin{bmatrix} G & \mathbf{0} \\ \mathbf{0} & \pm 1 \end{bmatrix} \to G$ can be realized geometrically. We called this phenomenon a "fake unkinking move." Stepping a little farther into linear algebra and further from topology, John wondered whether every positive-definite integer matrix is "naively kink equivalent" to a negative-definite integer matrix. (Were it not for fake unkinking moves, there would be no chance of this.) A couple months later, John answered this question in the negative. I would love to tell you more about this project, and the other half of John's Master's thesis project (also about Goeritz matrices, but from a completely different perspective). Please ask me about these some time.

The previous two years, Hugh Howards and I advised a Master's thesis about Khovanov homology. In general, for a Master's or undergraduate thesis, I would recommend almost any other area of my research interest above this topic above Khovanov homology. But I do have a couple specific projects in mind, and so I included the topic in a list of four or five possibility when Hugh and I first met with our new advisee, Emma. It turns out that Emma's best friend growing up was the daughter of a mathematician specializing in Khovanov homology, and she had a burning desire to know what this was all about. So we went for it! We spent the first year learning about Khovanov homology works; then we got to a research question. Recall from §3.12 that some Khovanov homology classes come entirely from single states. The question was what happens when we change the underlying diagram, say by adding or removing a crossing via a simple kink (called an R1 move). I suspected that such a simple move would preserve all the important features of Khovanov homology classes and their representatives. Surprisingly, though, Emma discovered that sometimes adding a crossing via an R1 move could make things *nicer*. For example, in a 3-crossing diagram of the trefoil, there is a Khovanov homology class that doesn't have a "nice" representative (coming from *a single* state), but after certain R1 moves, this homology class has a nice representative.

In general, knot theory is famously well-suited to undergraduate research. I have several more ideas of projects, which I would love to tell you about. Some involve constructing knots and surfaces from line segments and polygons. Some are computational, using a way of representing any n-crossing knot diagram as a $2n$-tuple of integers, described by the first knot theorist, Gauss: pick a basepoint and orient the knot; then walk along the knot, recording each new crossing with the next unused integer and each repeated crossing with the corresponding integer.[21] See Figure 4.

I have been incredibly fortunate to have five excellent, and very different, research advisors, and my experiences with them inform a variety of different modes in which I can interact with research advisees. Sometimes it makes sense to give advisees more structure. Sometimes, the best thing to do is let them take the lead, but stay actively involved. Sometimes, it is best to really step back and just listen until they ask a question. Sometimes, advisees will benefit most from a complete change in the topic of conversation, either to another area of math or something else entirely. My research advisors remain among the most important influences in my development as a mathematician and a human being. The advisor–advisee relationship enables a unique depth of intellectual connection. It is a privilege now to be on both sides of these relationships.

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[21] In general, one uses the over/under information at the crossings to attache signs to these integers; when the diagram **alternates** between over- and under-crossings, this is unnecessary, subject to the convention that the first crossing is an over-crossing.

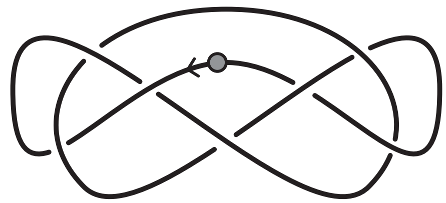

Figure 16. An alternating knot diagram. Using the indicated basepoint and orientation, the Gauss code is $(1, 2, 3, 1, 4, 5, 6, 3, 2, 4, 7, 6, 5, 7)$.

5. Why study knots and links?

Perhaps the best reason to study knots and links (especially via spanning surfaces) is the accessibility of these objects and open questions about them. This is thanks in part to my undergraduate advisor, Colin Adams, and his undergraduate-level text, *The Knot Book*, which conveys the essential nature of knot theory without assuming any knowledge of algebraic, differential, or even point-set topology.

In addition to being accessible and fun, knot theory is worth studying for deep mathematical reasons:

(1) Every closed[22] 3-manifold can be obtained by integral Dehn surgery on S^3, so framed links in S^3 carry the information of all 3-manifolds.[23]

(2) Every closed smooth 4-manifold can be described by a Kirby diagram, as can any smooth 4-manifold with no 3-handles, so framed links in connect sums of $S^1 \times S^2$ carry the information of all such 4-manifolds. Smooth 4-manifolds are perhaps the most active topic of study in low-dimensional topology today, in part because several fundamental problems remain unsolved.[24]

(3) The Jones polynomial was originally discovered via Von Neumann algebras and later reinterpreted in terms of Feynmann path integrals (among other ways). This connection between knot theory and the mathematics of quantum mechanics largely remains mysterious.

(4) "Almost every" knot complement admits a hyperbolic geometric structure (with constant sectional curvature -1). Moreover, such a hyperbolic structure is always unique; in particular, almost every knot has a well-defined hyperbolic volume. Empirical evidence strongly suggests that this volume equals a certain limit from the Jones polynomial and its offshoots. Yet, it remains an open question whether this equality always holds, let alone *why*. Moreover, because 3-dimensional hyperbolic geometry arises naturally in physics as the geometry of any timelike slice of spacetime, there seems to be a deep connection, via knot theory, between the mathematics of quantum mechanics and the math of relativity. Could a future revelation in knot theory unlock the mathematics suited to describe a unified model of the four fundamental forces?

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[22] That is, compact, connected, orientable, and without boundary.

[23] Given a 3-manifold M, there is a link $L \subset S^3$ such that one can transform S^3 into M (up to homeomorphism) by removing a regular neighborhood νL (each component is a solid torus following a component of L) and "gluing it back in (another certain way)." This gluing is described by decorating each component of L with an integer, called a *framing*.

[24] Out of all categories of manifolds and all dimensions less than 60, the Poincare conjecture remains open only in the smooth category in dimension 4.[21] The smooth Schonflies problem is also open in dimension 4: Does every smooth 3-sphere in 4-space bound a 4-ball? The slice-ribbon conjecture is also open: Given a smooth, properly embedded disk $X \subset B^4$, is there always a diffeomorphism $f : B^4 \to B^4$ so that $f(X)$ has no local maxima under radial projection (using polar coordinates) $p : B^4 \to [0, 1]$, $(r, \theta) \mapsto r$?

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