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Research Statement

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Please picture, in your mind's eye, a surface F with nonempty boundary $\partial F = K$ embedded in 3-space $\mathbb{R}^3$. If F and K are both compact and connected, then K is embedded circle, called a **knot**, and F is said to **span** K. (If K is disconnected, it is an embedded disjoint union of circles, called a link; for simplicity I will not mention links again until §4.) I choose to study knots, and to do so chiefly via *spanning surfaces*, in part because luck and privilege have allowed me the opportunity, and in part because I find these objects to be radically accessible on a human level and far-reaching in their mathematical significance. I describe some of this significance in §4.

In general, math research aims to discover and prove new mathematical facts; yet, proving *that* something is true does not always convey *why* it is true; to *justify* is not necessarily to *illuminate*. My research, however, emphasizes *understanding* above knowledge. This theme will recur throughout my statement, which is organized as follows:

- §1 sketches the way that linear algebra arises in my research;
- §2 describes several related problems that nicely capture the spirit of my research; one of these is a 30 (or 125) year old problem that I recently solved;
- §3 discusses the two Master's thesis projects that I have co-advised at Wake Forest; and
- §4 motivates knot theory vis a vis mathematics more broadly.

A longer version, available on my website, contains an additional section, which summarizes the results of my 14 completed papers.

1. The starring role of linear algebra in my research

Every knot in 3-space has spanning surfaces, lots of them actually. Given a knot $K \subset \mathbb{R}^3$, take a generic projection of K to $\mathbb{R}^2$, and record over-under information at the self-intersections, or *crossings*, to get a **diagram** D of K. It is always possible to color the regions of $\mathbb{R}^2 - D$ light and dark in "checkerboard fashion," as shown left in Figure 1, so that like-shaded regions abut only at crossings. If we do this so that the unbounded region is light, then there is a spanning surface B for K that lies almost entirely in the dark regions of $\mathbb{R}^2 - D$, except near crossings, where it twists; B is called a **checkerboard surface**. If we instead view $K \subset S^3$ (where $S^3 - \{\text{point}\}$ is $\mathbb{R}^3$) and project to S^2, then there are two checkerboard surfaces B and W, one of which lies almost entirely in the dark regions of $S^2 - D$, the other in the light ones. See Figure 1, center and right, which show the same pair of checkerboard surfaces. Different diagrams of the same knot usually give different checkerboard surfaces.

There is a symmetric bilinear pairing $\langle \cdot, \cdot \rangle : H_1(F) \times H_1(F) \to \mathbb{Z}$ on the first homology group of any spanning surface F,[1] first described by Gordon and Litherland [8].[2] It is easiest to describe when

[1] $H_1(F)$ is a free abelian group whose elements are represented by (disjoint unions of) oriented circles embedded in F; its rank, denoted $\beta_1(F)$, counts "how many holes" are in F

[2] This is my all-time favorite paper, partly because its content, but especially because they prove their results twice: first, they take a sophisticated approach, involving double-branched covers of the 4-ball (in some sense, this is the best way to understand the deep significance of their work), and then, restarting from scratch, they take a down-to-earth approach that uses only elementary tools and thus makes this profound paper remarkably accessible.

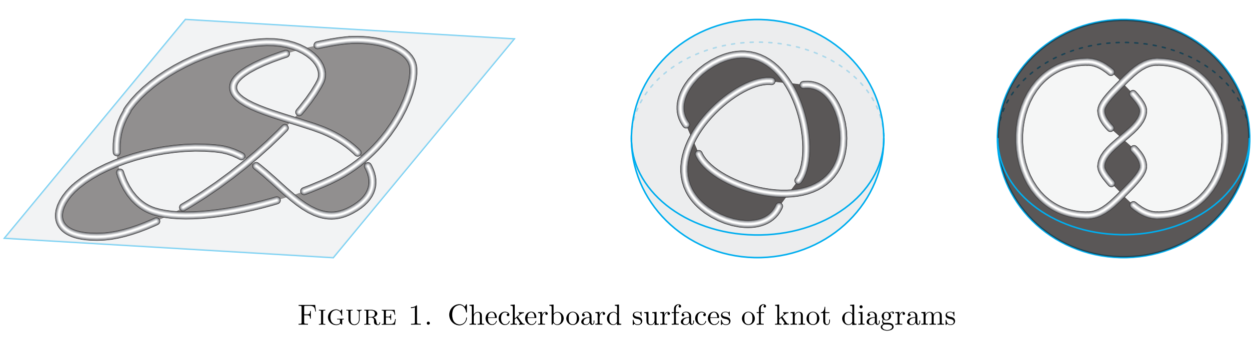

Figure 1. Checkerboard surfaces of knot diagrams

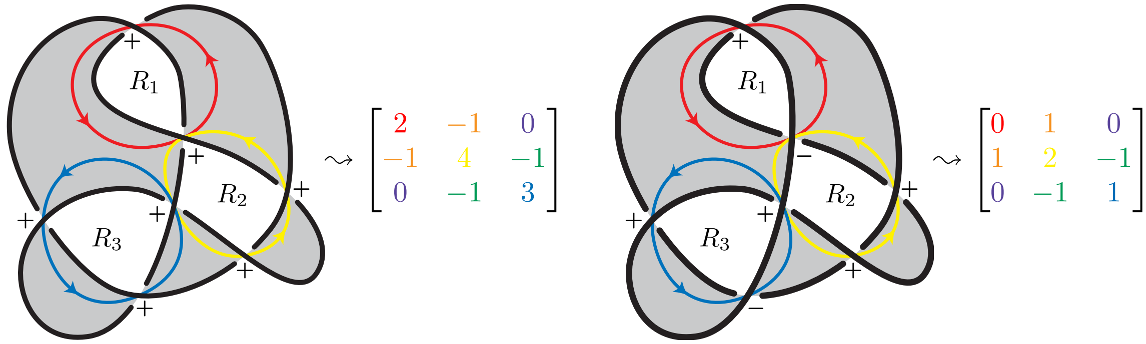

FIGURE 2. Goeritz matrices for two checkerboard surfaces

F is the dark checkerboard surface of a diagram $D \subset \mathbb{R}^2$: then $H_1(F)$ has a basis $\mathcal{A} = (a_1, \ldots, a_n)$ represented by circles α_i, each of which goes counterclockwise around exactly one bounded light region R_i, and $\langle \cdot, \cdot \rangle$ is represented, with respect to $\mathcal{A}$, by a **Goeritz matrix** $G = (g_{ij})$,[3] where each g_{ii} counts (with sign) the number of crossings incident to R_i and each g_{ij} counts (with opposite sign) the number of crossings incident to both R_i and R_j. Figure 2 shows two examples. Here are four key features of **Gordon-Litherland pairings** $\langle \cdot, \cdot \rangle$ and Goeritz matrices $G \in \mathbb{Z}^{n \times n}$:

- If $\alpha \subset F$ is a circle representing a homology class a, then the self-pairing $\langle a, a \rangle$ equals (half of) the *framing* of α in F, which measures how much F "twists" along α.
- Since $G \in \mathbb{Z}^{n \times n}$ represents a bilinear mapping, the change-of-basis formula is $G \to PGP^T$, where $P \in \mathbb{Z}^{n \times n}$ is invertible (i.e. "unimodular").
- "Attaching a crosscap" as in Figure 3 changes a Goeritz matrix like this: $G' \to \begin{bmatrix} G & \mathbf{0} \\ \mathbf{0} & \pm 1 \end{bmatrix}$.
- $|\det(G)|$ depends only on K, not on one's choice of F or of basis for $H_1(F)$.[4]

Not only does every knot K have spanning surfaces, it has some, called **Seifert surfaces**, that are orientable (i.e. 2-sided). *Seifert's algorithm* is a way of constructing such a surface using a diagram D of K; Figure 4 shows an example. Any Seifert surface F is homeomorphic to a connect sum of tori with an open disk removed: $F \cong \left(\#_{i=1}^n S^1 \times S^1\right) - (\text{open disk})$; we call n the *genus* of F and write $g(F) = n$. The **knot genus** $g(K)$ is the minimum of $g(F)$ over all Seifert surfaces F for K.

Any Seifert surface F comes with a bilinear pairing $H_1(F) \times H_1(F) \to \mathbb{Z}$ called the *Seifert pairing* and is represented by a *Seifert matrix* A. This pairing carries all the information of the Gordon-Litherland pairing, as $A + A^T$ is a Goeritz matrix for F, but it has something extra: the polynomial

[3] Given $x = \sum_{i=1}^n x_i a_i$ and $y = \sum_{i=1}^n y_i a_i$ in $H_1(F)$ and denoting $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, we have $\langle x, y \rangle = \vec{x}^T G \vec{y}$.

[4] The same is also true of the signature of G, albeit with a correction term.

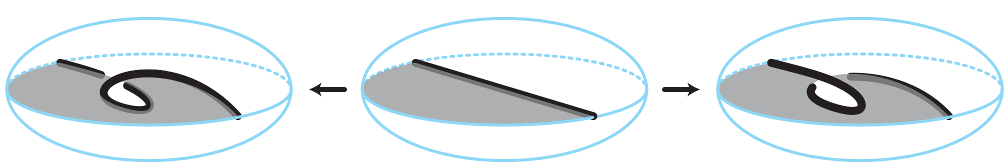

FIGURE 3. Attaching a crosscap to a spanning surface F is this local operation near ∂F.

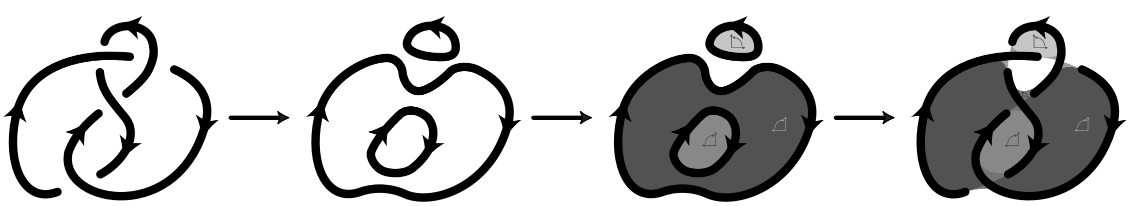

FIGURE 4. Seifert's algorithm obtains an *orientable* spanning surface from any knot diagram.

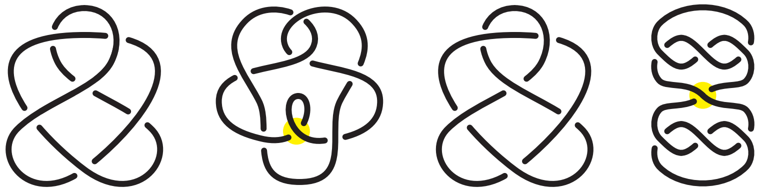

FIGURE 5. Four knot diagrams, three of them alternating, two with nugatory crossings (highlighted)

$\Delta_K(t) = \det(A - tA^T)$, called the **Alexander polynomial** of F, depends (up to degree shift) only on K. Moreover, the degree span or *breadth* $\mathrm{bth}(\Delta_K(t))$ depends only on K and provides a lower bound for $g(K)$: $\mathrm{bth}(\Delta_K(t)) \leq g(K)$. More on this polynomial toward the end of §2.

2. Classical knowledge, new understanding

Much of my research concerns **alternating** knots: a diagram $D \subset S^2$ of a knot $K \subset S^3$ is alternating if its crossings alternate between over and under, like the last three diagrams in Figure 5, and K is alternating if it has an alternating diagram; $c(D)$ denotes the number of crossings in D, and $c(K)$ is the smallest number of crossings among all diagrams of K. (We regard two knots as equivalent if they are related by a continuous deformation between their embeddings, called an *isotopy*.) In general, it is hard to determine the crossing number of an arbitrary knot, or to determine whether or not a given diagram minimizes crossings, but certainly no diagram that minimizes crossings can have a "nugatory" crossing, like the two highlighted in Figure 5. A diagram without nugatory crossings is called *reduced*, whether or not it minimizes crossings. Interestingly, if a knot K has a diagram D that is reduced *and* alternating, then D *always* minimizes crossings: $c(D) = c(K)$. Although this fact was first observed empirically by P.G. Tait in 1898 [28], it, and two related conjectures, remained unproven for almost a century:

Tait's conjectures. *Given a knot $K \subset S^3$ and two prime*[5] *alternating diagrams D and D' of K:*

(1) D and D' have the same number of crossings, which is minimal: $c(D) = c(D') = c(K)$.
(2) D and D' have the same writhe:[6] *$w(D) = w(D')$.* [7]
(3) D and D' are related by a sequence of flype moves: see Figure 6.

Tait's conjectures remained open until the discovery of the Jones polynomial in the mid-1980's [13],[8] which almost immediately led to three independent proofs of Tait's first conjecture Jones polynomial of a knot K: the degree span, or *breadth*, of $V_K(t)$, denoted $\mathrm{bth}(V_K(t))$, always provides

[5]A knot diagram D is *prime* if D has at least two crossings and, for every circle $\gamma \subset S^2$ that intersects D generically in two points, all crossings of D lie on the same side of γ; every prime diagram is reduced.

[6]Orient D arbitrarily. Then every crossing looks like ⤫ or ⤬. The *writhe* of D is $w(D) = |⤫| - |⤬|$, where bars count components. It is independent of the orientation on D.

[7]Tait's first two conjectures are true even if D and D' are not prime, as long as they are reduced.

[8]Jones described a new way of assigning a polynomial $V_K(t) \in \mathbb{Z}[t, t^{-1}]$ to any knot; different pictures of the same knot always yield the same polynomial, but different knots "usually" have different Jones polynomials.

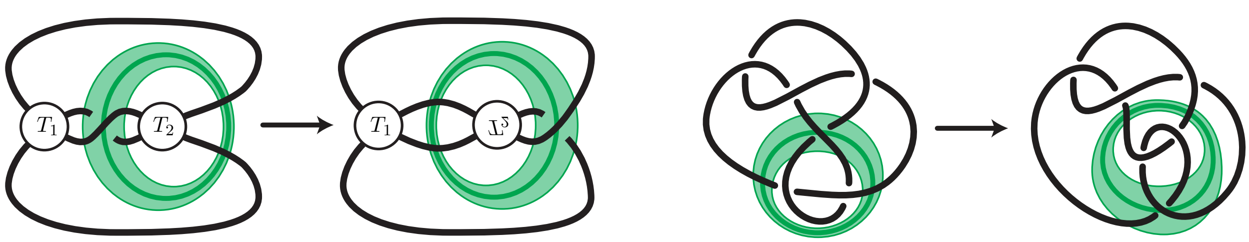

FIGURE 6. *Flype* moves

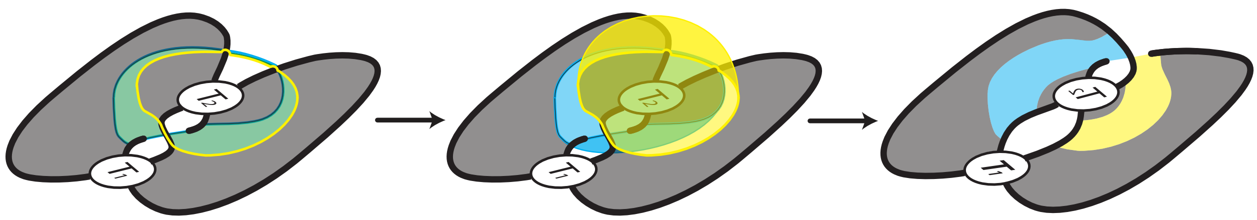

FIGURE 7. A flype move on a knot diagram corresponds to a re-plumbing move on one of its checkerboard surfaces (here, the black surface) and an isotopy of the other surface (here, the white surface).

a lower bound for crossing number, $c(K) \leq \text{bth}(V_K(t))$, and any reduced alternating diagram D of K satisfies $c(D) = \text{bth}(V_K(t))$;[9] by definition, $c(K) \leq c(D)$; therefore, $c(D) = c(K)$. QED

Within a decade, the Jones polynomial had led to proofs of all three of Tait's conjectures. Yet, these proofs were only somewhat satisfying: why did they work? What was the Jones polynomial really measuring? In their 1993 proof of Tait's flyping conjecture, Menasco and Thistlethwaite spotlighted this remaining gap in our understanding: "the question remains open as to whether there exist purely geometric proofs of this and other results that have been obtained with the help of new polynomial invariants." The aim is not just to know, but to understand. I love that the mathematical community values this.

The first partial answer came in 2017 from Greene in a paper where he answered another long-standing question, this one from Ralph Fox: "What [geometrically] is an alternating knot?" First, Greene observed that, if B and W are the checkerboard surfaces from an alternating knot diagram and G_B and G_W are their Goeritz matrices, then G_B is positive-definite and G_W is negative-definite, or vice-versa: "B and W are definite and of opposite signs."[10] Second, Greene proved that alternating knots are the only ones with a two such surfaces. In fact, he proved that if F_+ and F_- are definite spanning surfaces of opposite signs for the same knot K, then K has an alternating diagram whose checkerboard surfaces are "the same as" (are isotopic to) F_+ and F_-. Greene was then able to translate Tait's conjectures into *non-diagrammatic* statements and give the first "purely geometric" proofs of Tait's second conjecture and part of his first: any reduced alternating diagrams D and D' of the same knot satisfy $c(D) = c(D')$ and $w(D) = w(D')$ [9].[11]

Recently, I gave the first purely geometric proof of Tait's "flyping" conjecture. The first key insight (see Figure 7) was that flyping a diagram D changes one of its checkerboard surfaces by isotopy and the other, F, by a geometric operation I call *re-plumbing*, which replaces a disk $U \subset F$ (shown green, left) with another disk V (shown half yellow and half blue, center) that is disjoint from F except along its boundary $\partial V = \partial U$. Figure 8, left, shows another example of re-plumbing. This insight translates Tait's diagrammatic conjecture to a geometric statement about spanning surfaces. Then the real work begins. I would love to tell you that story; please ask me about it some time.

Around the time I proved Tait's flyping conjecture, Boden-Karimi extended Greene's insights about definite surfaces in S^3 to the context of thickened surfaces, like $S^1 \times S^1 \times [-1, 1]$ [1]. I

[9]The converse is also (nearly) true: any prime diagram D of any knot K satisfies $c(D) = \text{bth}(V_K(t))$ if and only if D is alternating.

[10]$G_B \in \mathbb{Z}^{n \times n}$ is positive-definite iff $\vec{x}^T G_B \vec{x} > 0$ for every nonzero $\vec{x} \in \mathbb{Z}^n$.

[11]It remains an open problem to give a purely geometric proof of Tait's full first conjecture, since Greene's insights are less useful regarding the possibilities of non-alternating diagrams of an alternating knot. I have some ideas....

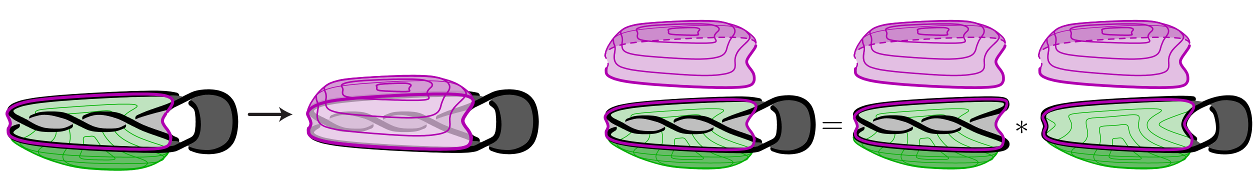

FIGURE 8. Left: A replumbing move. Right: a de-plumbing.

used their innovations to extend Tait's flyping conjecture to that context. Many of the arguments adapted directly, but a few technical points required special attention, several of which led to fundamental revelations about *virtual knots*, which correspond to knots in thickened surfaces [K10, K11, K13, K12].

Another recent project of mine regards a different seminal result in knot theory that, like Tait's flyping conjecture, was first proven in the Annals. Again, I wanted to *understand why* the result was true. Here is the background.

In general, a Seifert surface resulting from Seifert's algorithm *need not* have minimal genus, but when Seifert's algorithm is applied to an *alternating* diagram, the resulting surface F *always* has minimal genus: $g(F) = g(K)$. Gabai gave a short, elegant, purely geometric, proof of this fact in 1986 [6]. The original proofs, however, due independently to Crowell and Murasugi in 1958-59 [3, 22], had something extra, regarding the Alexander polynomial $\Delta_K(t)$. Recall from §2 that every knot K satisfies $\mathrm{bth}(\Delta_K(t)) \leq g(K)$. Crowell and Murasugi proved that when F is obtained via Seifert's algorithm from an alternating diagram, we have $g(F) = \frac{1}{2}\mathrm{bth}(\Delta_K(t))$. Since $g(K) \leq g(F)$ by definition, it follows that $g(F) = g(K) = \mathrm{bth}(\Delta_K(t)) \leq g(K)$. When I taught a topics course at UNL in 2021, I wanted to understand, and then share, *why* this was true. This led me to discover a new, short, extremely satisfying proof of the Crowell-Murasugi theorem [K8]. It involves all my favorite characters: checkerboard surfaces, plumbing operations, and linear algebra. Here is a sketch:

First, to prove that $g(F) = g(K) = \frac{1}{2}\mathrm{bth}(\Delta_K(t))$, I showed that it suffices to prove that the Seifert matrix A from F is invertible. Second, I showed that this is true if F happens to be a checkerboard surface (from an alternating diagram). Third, the surfaces obtained via Seifert's algorithm from alternating diagrams are always plumbings of alternating checkerboard surfaces, so it suffices to prove that if F is a plumbing of surfaces F_1 and F_2 which have invertible Seifert matrices A_1 and A_2, then F also has an invertible Seifert matrix. Finally, I showed that, indeed, F has a Seifert matrix of the form $A = \begin{bmatrix} A_1 & 0 \\ B & A_2 \end{bmatrix}$; applying the pigeonhole principle to the formula $\det[a_{ij}]_{i=1}^n = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)}$ thus confirms that $\det(A) = \det(A_1) \cdot \det(A_2) \neq 0$.

3. Advising

I proved in [K6] that any two checkerboard surfaces F_1 and F_2 (from any two diagrams) for a given knot K are related by a sequence of "crosscapping moves" (Figure 3), which change a Goeritz matrix like this, $G \leftrightarrow \begin{bmatrix} G & \mathbf{0} \\ \mathbf{0} & \pm 1 \end{bmatrix}$; also, change of basis changes a Goeritz matrix like this: $G \leftrightarrow PGP^T$ and $G \leftrightarrow \begin{bmatrix} G & \mathbf{0} \\ \mathbf{0} & \pm 1 \end{bmatrix}$. This raises the following practical question: given F_1 and F_2, how to find such a sequence of moves between them? Maybe, I thought, a good approach would be to forget the surfaces altogether and just focus on the linear algebra: if G_1 and G_2 are the Goeritz matrices for F_1 and F_2, then any sequence of crosscapping moves between F_1 and F_2 gives a sequence of moves like $G \leftrightarrow PGP^T$ and $G \leftrightarrow \begin{bmatrix} G & \mathbf{0} \\ \mathbf{0} & \pm 1 \end{bmatrix}$ from G_1 to G_2. Last spring, I posed this question to a student, John, in my topology topics course as we walked together from our classroom back to the math building. The question quickly grew into a Master's thesis project, which I now co-advise with Hugh Howards and Frank Moore. In particular, John discovered that not every move like $\begin{bmatrix} G & \mathbf{0} \\ \mathbf{0} & \pm 1 \end{bmatrix} \to G$ can be realized geometrically. We called this phenomenon a "fake unkinking move." Stepping a little further into linear algebra and away from topology, John wondered whether every positive-definite integer matrix is "naively kink equivalent" to a negative-definite integer matrix. (Were it not for fake unkinking moves, there would be no chance of this.) A couple months later, John answered this question in the negative. I would love to tell you more about this and the other half of John's Master's thesis project (also about Goeritz matrices, but from a completely different perspective).

The previous two years, Hugh Howards and I advised a Master's thesis about Khovanov homology. In general, this is not the first topic I would recommend for a Master's or undergraduate thesis, but Emma's best friend growing up was the daughter of a mathematician specializing in Khovanov homology, and she had a burning desire to know what this was all about, so we went for it! We spent the first year learning about Khovanov homology works; then we got to a research question. Emma explored examples and ended up discovering something surprising. The longer version of my research statement has details.

I have been incredibly fortunate to have five excellent, and very different, research advisors, and my experiences with them inform a variety of different modes in which I can interact with research advisees. Sometimes it makes sense to give advisees more structure. Sometimes, the best thing to do is let them take the lead, but stay actively involved. Sometimes, it is best to really step back and just listen until they ask a question. Sometimes, advisees will benefit most from a complete change in the topic of conversation, either to another area of math or something else entirely. My research advisors remain among the most important influences in my development as a mathematician and a human being. The advisor–advisee relationship enables a unique depth of intellectual connection. It is a privilege now to be on both sides of these relationships.

4. Why study knots and links?

Perhaps the best reason to study knots and links (especially via spanning surfaces) is the accessibility of these objects and open questions about them. This is thanks in part to my undergraduate advisor, Colin Adams, and his undergraduate-level text, *The Knot Book*, which conveys the essential nature of knot theory without assuming any knowledge of algebraic, differential, or even point-set topology.

In addition to being accessible and fun, knot theory is worth studying for deep mathematical reasons:

(1) Every closed[12] 3-manifold can be obtained by integral Dehn surgery on S^3, so framed links in S^3 carry the information of all 3-manifolds.[13]

(2) Every closed smooth 4-manifold can be described by a Kirby diagram, as can any smooth 4-manifold with no 3-handles, so framed links in connect sums of $S^1 \times S^2$ carry the information of all such 4-manifolds. Smooth 4-manifolds are perhaps the most active topic of study in low-dimensional topology today, in part because several fundamental problems remain unsolved.[14]

(3) The Jones polynomial was originally discovered via Von Neumann algebras and later reinterpreted in terms of Feynmann path integrals (among other ways). This connection between knot theory and the mathematics of quantum mechanics largely remains mysterious.

(4) "Almost every" knot complement admits a hyperbolic geometric structure (with constant sectional curvature -1). Moreover, such a hyperbolic structure is always unique; in particular, almost every knot has a well-defined hyperbolic volume. Empirical evidence strongly suggests that this volume equals a certain limit from the Jones polynomial and its offshoots. Yet, it remains an open question whether this equality always holds, let alone *why*. Moreover, because 3-dimensional hyperbolic geometry arises naturally in physics as the geometry of any timelike slice of spacetime, there seems to be a deep connection, via knot theory, between the mathematics of quantum mechanics and the math of relativity. Could a future revelation in knot theory unlock the mathematics suited to describe a unified model of the four fundamental forces?

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[12] That is, compact, connected, orientable, and without boundary.

[13] Given a 3-manifold M, there is a link $L \subset S^3$ such that one can transform S^3 into M (up to homeomorphism) by removing a regular neighborhood νL (each component is a solid torus following a component of L) and "gluing it back in (another certain way)." This gluing is described by decorating each component of L with an integer, called a *framing*.

[14] Out of all categories of manifolds and all dimensions less than 60, the Poincare conjecture remains open only in the smooth category in dimension 4.[21] The smooth Schonflies problem is also open in dimension 4: Does every smooth 3-sphere in 4-space bound a 4-ball? The slice-ribbon conjecture is also open: Given a smooth, properly embedded disk $X \subset B^4$, is there always a diffeomorphism $f : B^4 \to B^4$ so that $f(X)$ has no local maxima under radial projection (using polar coordinates) $p : B^4 \to [0, 1]$, $(r, \theta) \mapsto r$?

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