WHAT IS A VIRTUAL LINK DIAGRAM?

THOMAS KINDRED

Abstract. We establish a new correspondence between abstract link diagrams, cellularly embedded link diagrams on closed surfaces, and equivalence classes of virtual link diagrams. This is analogous to a well-known correspondence among the links represented by these diagrams, but with a crucial subtlety.

A *virtual link diagram* is the image of an immersion $\bigsqcup S^1 \to S^2$ in which all self-intersections are transverse double-points, some of which are labeled with over-under information. These labeled points are called *classical crossings*, and the other double-points are called *virtual crossings*. Traditionally, virtual crossings are marked with a circle, as in Figure 1. A *virtual link* is an equivalence class of such diagrams under generalized Reidemeister moves ("R-moves"), as shown in Figure 1. There are seven types of such moves, the three *classical* moves and four *virtual* moves.[1]

An *abstract link diagram* (S, G) consists of a 4-valent graph G embedded in a compact orientable surface S, such that G has over-under information at each vertex, and G is a deformation retract of S. An *abstract link* is an equivalence class of such diagrams under the following equivalence relation $\sim$: $(S_1, G_1) \sim (S_2, G_2)$ if there are embeddings $\phi_i : S_i \to S$, $i = 1, 2$, into a surface S, such that $\phi_1(G_1)$ and $\phi_2(G_2)$ are related by classical R-moves on S.

Suppose Σ is a closed orientable surface and L is a link in the thickened surface. The pair (Σ, L) is *stabilized* if, for some simple closed curve $\gamma \subset \Sigma$, L can be isotoped so that it intersects each component of $(\Sigma \times I) \setminus (\gamma \times I)$ but not the annulus $\gamma \times I$; one can then *destabilize* the pair (Σ, L) by cutting $\Sigma \times I$ along $\gamma \times I$ and attaching two 3-dimensional 2-handles in the natural way; the reverse operation is called *stabilization*. Equivalently, (Σ, L) is *nonstabilized* if every diagram D of L on Σ is *cellularly embedded*, meaning that D cuts Σ into disks.

By work of Kauffman [Ka98], Kamada–Kamada [KK00], and Carter–Kamada–Saito [CKS02], there is a triple bijective correspondence between (1) virtual links, (2) abstract links, and (3) stable equivalence classes of links in thickened surfaces. The purpose of this note is

[1]The move involving two virtual crossings and one classical crossing is sometimes called a *mixed* move, but we include it as a virtual (non-classical) move.

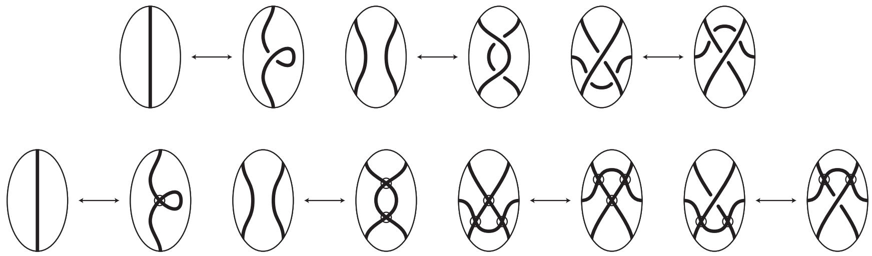

FIGURE 1. Classical (top) and virtual (bottom) Reidemeister moves, up to mirror symmetry

to establish a new correspondence between the associated *diagrams*. This enables the author to adapt and prove Tait's flyping conjecture for virtual links in [Ki23v]. It also leads to a new diagrammatic proof of the older, well-known correspondence. Nothing here will be particularly difficult, but there is a *hidden subtlety*, which is fundamental to *understanding* both correspondences and which, to the author's knowledge, has not previously been observed in the literature. See Theorem 5, Remark 6, and Example 7.

Notation 1. Given a virtual link diagram $V \subset S^2$, let $[V]$ denote the set of all virtual diagrams related to V by planar isotopy and virtual R-moves.

Our main result is the following correspondence:

Theorem 2. *The following define a triple bijective correspondence between (1) equivalence classes* $[V]$ *of virtual link diagrams, (2) abstract link diagrams (up to pairwise homeomorphism* $(S, G) \to (S', G')$ *that respects crossing information), and (3) cellularly embedded link diagrams on closed surfaces (up to pairwise homeomorphism* $(\Sigma, D) \to (\Sigma', D')$ *that respects crossing information):*

$1 \to 2$: *Given an equivalence class* $[V]$ *of virtual link diagrams, choose a representative diagram* $V \subset S^2$*, and construct an abstract link diagram as follows. First, take a regular neighborhood* νV *of* V *in* S^2*. Second, embed* S^2 *in* S^3 *and modify* νV *near each virtual crossing of* V *as shown in Figure 2. Third, view the resulting pair abstractly, forgetting the embedding in* S^3*.*

$2 \to 3$: *Given an abstract link diagram* (S, G)*, cap off each component of* ∂S *with a disk to obtain a cellularly embedded link diagram on a closed surface.*

To confirm the bijectivity of this correspondence, we will use *lassos*, which are introduced and more fully explored in [Ki23b]:

Definition 3. A *lasso* for a link diagram D on a closed surface Σ is a disk $X \subset \Sigma$ that intersects D generically and contains all crossings

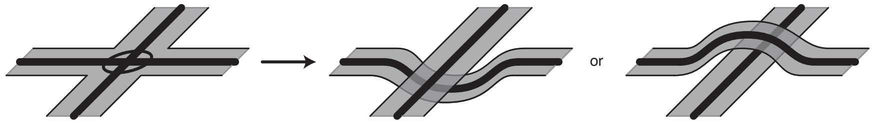

FIGURE 2. Converting the neighborhood of a virtual link diagram to an abstract link diagram

of D. A *lasso* for a virtual link diagram $V \subset S^2$ is a disk $X \subset S^2$ that intersects D generically and contains all classical crossings of D but no virtual crossings; call $V \cap X$ and $V \setminus\setminus X$ respectively the *classical* and *virtual* parts of V (with respect to X).

Proof of Theorem 2. We first describe the reverse constructions (3) $\to$ (2) and (2) $\to$ (1). The former is easy: given a link diagram D on a closed surface Σ, take a regular neighborhood νD of D in Σ; the pair $(\nu D, D)$ is the associated abstract link diagram.

For (2) $\to$ (1), consider an abstract link diagram (S, G). View $S^3 = (S^2 \times \mathbb{R}) \cup \{\pm\infty\}$, denote $\widehat{S^3} = S^3 \setminus \{\pm\infty\}$, and denote projection $\pi : \widehat{S^3} \to S^2$. Choose any embedding $\phi : S \to \widehat{S^3}$ such that $\pi|_{\phi(S)}$ has no critical points and all self-intersections in $\pi \circ \phi(G)$ are transverse double-points with neighborhoods as suggested in Figure 2. Now take the 4-valent graph $\pi \circ \phi(G) \subset S^2$, and, for each crossing point c of G, label the double-point $\pi \circ \phi(c)$ with the matching over-under information. (Thus, the double points of V coming from the crossings of D comprise the classical crossings of V, and the remaining double-points comprise the virtual crossings.) See Figure 3.

It remains to justify that this triple correspondence is indeed bijective. For (2) $\leftrightarrow$ (3), this is immediate.

For (1) $\leftrightarrow$ (2), note that in the construction (1) $\to$ (2), the pairwise homeomorphism type of the resulting abstract link diagram is well-defined and is unchanged by virtual R-moves on V. It thus remains only to show that the choice of embedding $\phi : S \to \widehat{S^3}$ described above for (2) $\to$ (1) does not affect the equivalence class $[V]$ of the resulting virtual diagram V.

Choose a spanning tree T for G, and take a regular neighborhood U of G in S. Now take two embeddings $\phi_i : S \to \widehat{S^3}$, $i = 1, 2$, as described for (2) $\to$ (1). Construct ϕ_1 such that $\pi \circ \phi_1(U)$ contains all classical crossings of the virtual diagram $V_1 = \pi \circ \phi_1(G)$ and no classical crossings[2]. Then the disk $\pi \circ \phi_2(U) \subset S^2$ is a lasso for the virtual diagram $V_2 = \pi \circ \phi_2(G)$, and $\pi \circ \phi_2$ restricts to a pairwise homeomorphism of $(U, G \cap U)$. Let ϕ_2 be arbitrary, and denote the virtual diagram $V_2 = \pi \circ \phi_2(G)$. Also denote $\pi \circ \phi_i = f_i$ and $f_i(U) = U_i$ for $i = 1, 2$.

[2]That is, construct ϕ_1 so that $\pi^{-1} \circ \pi \circ \phi_1(U) \cap \phi_1(G) = \phi_1(G \cap U)$.

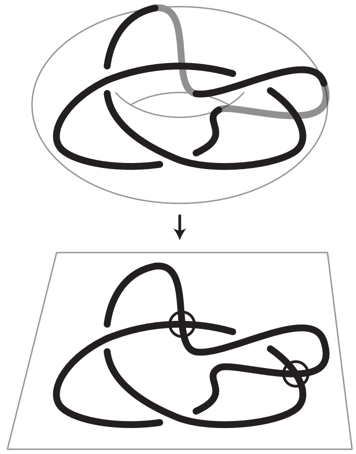

Figure 3. A link diagram on the torus and a corresponding virtual diagram

It will suffice to show that V_1 and V_2 are related by virtual R-moves. If $U_2 \cap f_2(G \setminus U) = \varnothing$, then U_2 is a lasso for V_2, and the classical parts of V_2 and V_1 match, as do the combinatorics of the virtual parts, and so V_2 and V_1 are related by virtual R-moves.

Assume instead that $U_2 \cap f_2(G \setminus U) \neq \varnothing$. Consider any (smooth) arc α of $U_2 \cap f_2(G \setminus U)$ in U_2. Note that α contains no classical crossings. Denote one of the disks of $U_2 \setminus\setminus f_2(G \setminus U)$ by U_2'. Then $\partial U_2' = \alpha \cup \beta$ for some arc $\beta \subset \partial U_2 \setminus\setminus f_2(G \setminus U)$. Since α contains no classical crossings, there is a *virtual pass move*, as shown in Figure 4, which takes α through U_2' past β; such a move can be performed via a sequence of *virtual* R-moves, since α contains no virtual crossings.

Perform such virtual pass moves successively on the arcs of $U_2 \cap f_2(G \setminus U)$ until $U_2 \cap f_2(G \setminus U) = \varnothing$. Then U_2 is a lasso for the resulting diagram, whose classical part matches that of V_1; the combinatorics of the virtual parts of these diagrams also match, and so these diagrams are related by virtual R-moves, as therefore are V_1 and V_2. $\square$

This gives a new diagrammatic perspective on a well-known correspondence:

Theorem 4 ([Ka98, KK00, CKS02])**.** *There is a triple bijective correspondence between (1) virtual links, (2) abstract links, and (3) stable equivalence classes of links in thickened surfaces. Namely, choose any representative diagram and apply the diagrammatic correspondence from Theorem 2.*

There is an important caveat in correspondences described in Theorems 2 and 4 which is worth noting explicitly. Namely, recall the

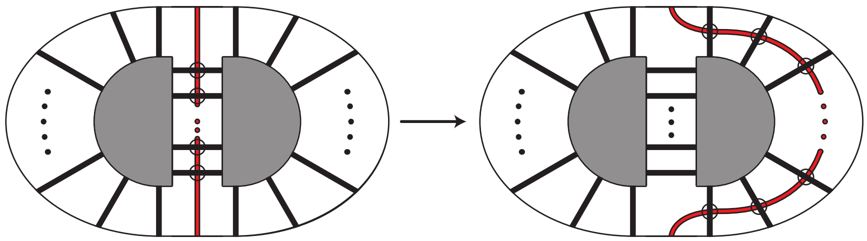

FIGURE 4. A virtual pass move; the red arc on the left may also contain virtual crossings with itself.

requirement in part (2) → (1) of the proof of Theorem 2 that $\pi|_{\phi(S)}$ must have no critical points. This ensures that $\phi(S)$ has no "back side." If one wishes to construct a virtual link diagram directly from diagram D on a closed surface Σ, however, this is too much to require, any embedding of Σ will have a front and a back: given an embedding $\phi : \Sigma \to \widehat{S^3}$, a regular point of $\pi|_{\phi(\Sigma)}$ lies on the *front* or *back* of $\phi(\Sigma)$ depending on whether an *even* or *odd* number of points of $\phi(\Sigma)$ lie directly above it. The salient point is that one must choose an embedding ϕ of Σ under which all *crossings* of D lie on the front of $\phi(\Sigma)$:

Theorem 5. *There is a bijective correspondence between (1) equivalence classes $[V]$ of virtual diagrams and (2) cellularly embedded link diagrams (Σ, D) up to pairwise homeomorphism. Namely:*

(1) → (2) *Given $[V]$, choose a representative $V \subset S^2$, take a regular neighborhood νV of V in S^2, modify νV near each virtual crossing of V as shown in Figure 2, and (abstractly) cap off each boundary component of the resulting surface with a disk.*

(2) → (1) *Given (Σ, D), choose any embedding $\phi : \Sigma \to \widehat{S^3}$ such that (i) for each crossing point $c \in D$, $\phi(c)$ lies on the front of Σ and (ii) all self-intersections in $\pi \circ \phi(G)$ are transverse double-points. Then let $V = \pi \circ \phi(D)$, with over-under information matching D.*

Remark 6. The requirement in Theorem 5 that all crossings lie on the front of $\phi(\Sigma)$ is necessary; otherwise, different embeddings $\Sigma \to \widehat{S^3}$ may yield distinct virtual links. See Example 7.

Example 7. Let D be a minimal diagram of the RH trefoil on a 2-sphere Σ, and embed Σ in $\widehat{S^3}$ such that the critical locus of $\pi|_{\Sigma}$ is a simple closed curve and D lies entirely on the front of Σ. The corresponding virtual diagram V is also a minimal diagram of the classical RH trefoil. Now isotope D on Σ as shown in Figure 5, so that a crossing passes across the critical circle of $\pi|_{\Sigma}$. Denote the resulting diagram on Σ by D'. The virtual diagram $V' = \pi \circ \phi(D')$

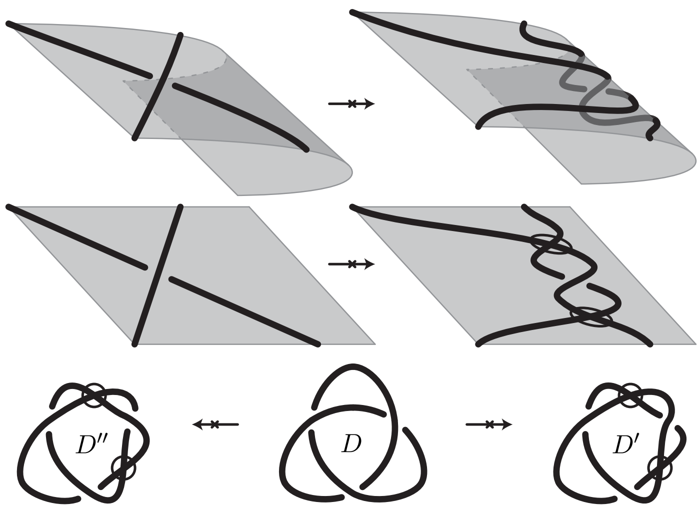

FIGURE 5. Given $D \subset \Sigma$, one obtains $V \subset S^2$ by embedding Σ in S^3 and projecting, but all crossings of D must remain on the front of Σ.

represents the virtual knot 3.5, which is distinct from the classical RH trefoil [Ka98], even though D and D' are isotopic on Σ.

Interestingly, the virtual knot 3.5 has the same Jones polynomial as the RH trefoil, but the two can be distinguished using the involutory quandle, also called the fundamental quandle. Indeed, by Lemma 5 of [Ka98], the virtual knot 3.5 has the same involutory quandle as the unknot, which is distinct from that of the RH trefoil, since the former is trivial and the latter is not [Jo82].

We note too that the diagram D'' in Figure 5 represents the virtual knot 3.7, which has trivial Jones polynomial.

We close by noting a consequence of Theorem 2 and Kuperberg's Theorem. Recall:

Theorem 8 (Theorem 1 of [Ku03])**.** *If (Σ, L) and (Σ', L') are stably equivalent and nonstabilized, then there is a pairwise homeomorphism $(\Sigma \times I, L) \to (\Sigma' \times I, L')$.*

Given a virtual diagram V of, say, a nonsplit virtual link, one may define the *genus* $g(V)$ to be the genus $g(\Sigma)$, where (Σ, D) corresponds to $[V]$ under Theorem 2. Note that $g([V])$ is well-defined. Say that V has *minimal genus* if $g(V) \leq g(V')$ for all diagrams V' of the same virtual link. In general, given diagrams V and V' of the same link, it is possible that *all* sequences $V = V_0 \to \cdots \to V_n = V'$ have some V_i with $g(V_i) \gg \max\{g(V), g(V')\}$. A priori, this seems plausible even when V and V' have minimal genus, but:

Corollary 9. *All minimal genus diagrams of a nonsplit virtual link are related by minimal-genus-preserving generalized R-moves.*

Proof. Let V and V' be two such virtual diagrams, let (Σ, D) and (Σ', D') be the diagrams corresponding to $[V]$, $[V']$ under Theorem 2, and let $L \subset \Sigma \times I$ and $L' \subset \Sigma' \times I$ be the links they represent. The minimal genus condition ensures that L and L' are nonstabilized, so by Kuperberg's Theorem there is a pairwise homeomorphism $(\Sigma \times I, L) \to (\Sigma' \times I, L')$. Therefore, there is a sequence $D = D_0 \to \cdots \to D_n$ of R-Moves on Σ such that there is a pairwise homeomorphism $(\Sigma, D_n) \to (\Sigma', D')$. Applying Theorem 2 to the sequence $(\Sigma, D_0) \to \cdots \to (\Sigma, D_n)$ gives a sequence $[V] = [V_0] \to \cdots \to [V_n] = [V']$ where each $g([V_i])$ is minimal and each pair $[V_i], [V_{i+1}]$ have representatives that differ by a single classical R-move; refining this sequence with virtual R-moves gives the desired sequence of generalized R-moves taking V to V'. □

References

[CKS02] J.S. Carter, S. Kamada, M. Saito, *Stable equivalence of knots on surfaces and virtual knot cobordisms*, J. Knot Theory Ramifications, 11 (2002), no. 3, 311–322.

[Jo82] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra 23 (1982), no. 1, 37-65.

[KK00] N. Kamada, S. Kamada, *Abstract link diagrams and virtual knots*, J. Knot Theory Ramifications 9 (2000), no. 1, 93-106.

[Ka98] L.H. Kauffman, Virtual knot theory, European J. Combin. 20 (1999), no. 7, 663-690.

[Ki23b] T. Kindred, *Primeness of alternating virtual links*, arXiv:2210.03225.

[Ki23v] T. Kindred, *The virtual flyping theorem*, arXiv:

[Ku03] G. Kuperberg, *What is a virtual link?*, Alg. Geom. Topol. 3 (2003), 587–591.

Department of Mathematics & Statistics, Wake Forest University, Winston-Salem North Carolina, 27109

Email address: `thomas.kindred@wfu.edu`

URL: `www.thomaskindred.com`