THE VIRTUAL FLYPING THEOREM

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Abstract. We extend the flyping theorem to alternating links in thickened surfaces and alternating virtual links. The proof of the former result uses work of Boden–Karimi to adapt the author's geometric proof of Tait's 1898 flyping conjecture (first proved in 1993 by Menasco–Thistlethwaite), while the proof of the latter involves a new correspondence between abstract link *diagrams*, cellularly embedded link *diagrams* on closed surfaces, and equivalence classes of virtual link *diagrams*.

1. Introduction

P.G. Tait asserted in 1898 that all reduced alternating diagrams of a given prime nonsplit link in S^3 minimize crossings, have equal writhe, and are related by *flype* moves (see Figure 1) [Ta1898]. The first proofs came almost a century later, and all involved the Jones polynomial [Ka87, Mu87, Mu87ii, Th87, MT91, MT93]. In 2017, Greene gave the first *purely geometric* proof of part of the classical Tait conjectures [Gr17], and in 2020, the author gave the first purely geometric proof of Tait's flyping conjecture [Ki23].

Recently, Boden, Chrisman, Karimi, and Sikora extended much of this to alternating links in thickened surfaces. First, using generalizations of the Kauffman bracket, Boden–Karimi–Sikora proved that Tait's first two conjectures hold for alternating links in thickened surfaces [BK18, BKS19].[1] Second, Boden–Chrisman–Karimi extended the Gordon–Litherland pairing to spanning surfaces in thickened surfaces [BCK21]. Third, Boden–Karimi applied this pairing to extend Greene's characterization of classical alternating links to links L in thickened surfaces $\Sigma \times I$, proving that L bounds connected definite surfaces of opposite signs if and only if L is alternating and $(\Sigma \times I, L)$ is nonstabilized [BK22].[2]

The first main result of this paper combines and adapts several of these recent developments to prove that the flyping theorem extends to alternating links in (nonstabilized) thickened surfaces.

[1]Boden–Karimi proved Tait's first two conjectures for alternating links in thickened surfaces, with a few extra conditions [BK18], and with Sikora they extended those results to adequate links and removed the extra conditions [BKS19].

[2]See §2.1 for definitions of *stabilized*, *prime*, *weakly prime*, *fully alternating*, *cellularly embedded*, *end-essential*, *definite*, and *removably nugatory*.

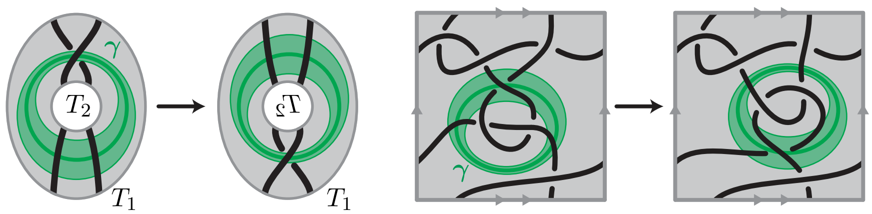

FIGURE 1. A *flype* along an annulus $A = \nu\gamma \subset \Sigma$.

Theorem 3.5. *Let $D \subset \Sigma$ be a weakly prime, fully alternating diagram of a link L in a thickened surface $\Sigma \times I$. Then any other such diagram of L is related to D by flypes on Σ.*

The approach is parallel to that in [Ki23], and indeed most of the arguments translate directly. For some, which we mark with the symbol ♊, the statements and proof hold without further comment. Appendix A lists pertinent cross-referencing information for these and other results marked with the symbol ♈. The upshot is a geometric proof of Theorem 3.5 and other generalized Tait conjectures:

Theorem 3.3 (Part of Tait's extended first conjecture [BK18, BKS19])**.** *If $D, D' \subset \Sigma$ are alternating diagrams of a link $L \subset \Sigma \times I$, neither containing removable nugatory crossings, then D and D' have the same number of crossings.*

Theorem 3.6 (Tait's extended second conjecture [BK18, BKS19])**.** *All weakly prime, fully alternating diagrams of a given link $L \subset \Sigma \times I$ have the same writhe.*

Section 4 extends Theorems 3.3, 3.5, and 3.6 to virtual links:

Theorem 4.14. *Any two weakly prime, alternating virtual diagrams*[3] *of a given virtual link $\widetilde{L}$ are related by virtual (non-classical) Reidemeister moves and classical flypes.*[4]

To prove this, we establish a new *diagrammatic* analog to the correspondence established by Kauffman, Kamada–Kamada, and Carter–Kamada–Saito between virtual links, equivalence classes of abstract links, and stable equivalence classes of links in thickened surfaces [Ka98, KK00, CKS02]; also see [Ku03]. Given a virtual link diagram V, let $[V]$ denote its equivalence class under virtual (non-classical) Reidemeister moves. We show that such classes $[V]$ correspond bijectively to abstract link diagrams and thus to cellularly embedded link diagrams on closed surfaces.

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[3] A virtual link diagram is *alternating* if its *classical crossings* alternate between over and under.

[4] A **classical flype** on a virtual link diagram appears as in Figure 1, where T_1 contains no virtual crossings.

As corollaries, we extend Theorems 3.3 and 3.6 to virtual diagrams, and we observe that connect sum is not a well-defined operation on virtual knots or links:

Theorem 4.15. *All weakly prime, alternating diagrams of a given virtual link have the same crossing number and writhe.*

Corollary 4.16. *Given any two non-classical, weakly prime, alternating virtual links* V_1 *and* V_2*, there are infinitely many distinct virtual links that decompose as a connect sum of* V_1 *and* V_2*.*

Before all this, in §2, we introduce the required background regarding links in thickened surfaces. Some of this reviews the existing literature, and some of it is new.

2. Links and spanning surfaces in thickened surfaces

Convention 2.1. Throughout, Σ is a connected, closed, orientable surface with genus $g(\Sigma) > 0$.[5] We denote the intervals $[-1, 1]$ and $[0, 1]$ by I and I_+, respectively. In $\Sigma \times I$, we identify Σ with $\Sigma \times \{0\}$ and denote $\Sigma \times \{\pm 1\} = \Sigma_\pm$. For a pair (Σ, L) or $(\Sigma \times I, L)$, L is a link in $\Sigma \times I$, and for a pair (Σ, D), D is a link diagram on Σ.

2.1. Alternating links in thickened surfaces.

A pair (Σ, L) is **stabilized** if, for some circle[6] $\gamma \subset \Sigma$, L can be isotoped so that it intersects each component of $(\Sigma \times I) \setminus (\gamma \times I)$ but not the annulus $\gamma \times I$; one can then *destabilize* the pair (Σ, L) by cutting $\Sigma \times I$ along $\gamma \times I$ and attaching two 3-dimensional 2-handles in the natural way (this may disconnect Σ); the reverse operation is called *stabilization.* Equivalently, (Σ, L) is *nonstabilized* if every diagram D of L on Σ is **cellularly embedded**, meaning that D cuts Σ into disks.

A pair (Σ, L) is **split** if L has a disconnected diagram on Σ. Note that if (Σ, L) is split then it is also stabilized (as we assume that Σ is connected). The converse is false. In fact, the number of split components is an invariant of stable equivalence classes.

Kuperberg's Theorem states that the stable equivalence class of (Σ, L) contains a unique nonstabilized representative (§4 gives a new diagrammatic proof, using Correspondence 4.3); this implies that when (Σ, L) is nonsplit, (Σ, L) is nonstabilized if and only if Σ has *minimal genus* in this stable equivalence class.

Theorem 2.2 (Theorem 1 of [Ku03])**.** *If* (Σ, L) *and* $(\Sigma' \times I, L')$ *are stably equivalent and nonstabilized, then there is a pairwise homeomorphism* $(\Sigma \times I, L) \to (\Sigma' \times I, L')$.

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[5] [Ki23b, Ki23c] also allow Σ to be disconnected with components of any genus.

[6] We use "circle" as shorthand for "simple closed curve."

If L is nonsplit and $g(\Sigma) > 0$, then $(\Sigma \times I) \setminus L$ is irreducible, as $\Sigma \times I$ is always irreducible, since its universal cover is $\mathbb{R}^2 \times \mathbb{R}$.[7] The converse of this, too, is false,[8] due to the next observation, which follows from a standard innermost circle argument:

Observation 2.3. *If $(\Sigma_i \times I) \setminus L_i$ is irreducible for $i = 1, 2$ and $\Sigma = \Sigma_1 \#_\gamma \Sigma_2$ with $L = L_1 \sqcup L_2 \subset \Sigma \times I$, where the annulus $A = \gamma \times I$ separates L_1 from L_2 in $\Sigma \times I$, then $(\Sigma \times I) \setminus L$ is irreducible.*

We call (Σ, D) *cellularly embedded* if D cuts Σ into disks and **fully alternating** if it is alternating and cellularly embedded. We will use this result of Boden–Karimi and the generalization that follows:

Fact 2.4 (Corollary 3.6 of [BK22])**.** *If (Σ, L) has a fully alternating diagram, then (Σ, L) is nonsplit and nonstabilized.*

Corollary 2.5. *Suppose (Σ, L) has an alternating diagram $D \subset \Sigma$. Then (Σ, L) is nonsplit if and only if D is connected, and (Σ, L) is nonstabilized if and only if D is cellularly embedded.*

We call (Σ, D) **prime** if any pairwise connect sum decomposition $(\Sigma, D) = (\Sigma_1, D_1)\#(\Sigma_2, D_2)$ has $(\Sigma_i, D_i) = (S^2, \bigcirc)$ for either $i = 1, 2$. Likewise, we call (Σ, L) *prime* if any every pairwise connect sum decomposition $(\Sigma, L) = (\Sigma_1, L_1)\#(\Sigma_2, L_2)$[9]is trivial: $(\Sigma_i, L_i) = (S^2, \bigcirc)$ for either $i = 1, 2$. Thus, (Σ, L) is prime if and only if, whenever $\gamma \subset \Sigma$ is a separating curve and L is isotoped to intersect the annulus $\gamma \times I$ in two points, γ bounds a disk $X \subset \Sigma$ such that L intersects $X \times I$ in a single unknotted arc. Note that if (Σ, D) is prime then D is connected, and if (Σ, L) is prime then it is nonsplit.

Following Howie-Purcell, we also call (Σ, D) **weakly prime** if, for every pairwise connect sum decomposition $(\Sigma, D) = (\Sigma, D_1)\#(S^2, D_2)$, either $D_2 = \bigcirc$ is the trivial diagram of the unknot or $(\Sigma, D_1) = (S^2, \bigcirc)$ [HP20], and we call (Σ, L) *weakly prime* if, for every pairwise connect sum decomposition $(\Sigma, L) = (\Sigma, L_1)\#(S^2, L_2)$, either $L_2 = \bigcirc$ is the unknot or $(\Sigma, L_1) = (S^2, \bigcirc)$ [HP20].[10]

As in the classical case [Me84], certain diagrammatic conditions constrain an alternating link L as one might wish:

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[7]For more detail, see Proposition 12 of [BK22]; the proof cites [CSW14].

[8]If $(\Sigma_i \times I, L_i)$ is nonsplit (implying that $\Sigma_i \times I \setminus L_i$ is irreducible) for $i = 1, 2$, then choose disks $X_i \subset \Sigma_i$ with $(X_i \times I) \cap L_i = \varnothing$ and construct the connect sum $\Sigma = (\Sigma_1 \setminus \text{int}(X_1)) \cup (\Sigma_2 \setminus \text{int}(X_2)) = \Sigma_1 \# \Sigma_2$. Let $L = L_1 \sqcup L_2 \subset \Sigma \times I$. Then (Σ, L) is split. Yet, $(\Sigma \times I) \setminus L$ is irreducible by Observation 2.3.

[9]This pairwise connect sum is sometimes called an *annular connect sum*.

[10]A third notion of primeness for D on Σ also appears in the literature: Ozawa calls (Σ, D) *strongly prime* if every circle on Σ (not necessarily separating) that intersects D in two generic points also bounds a disk in Σ which contains no crossings of D [Oz06].

Theorem 2.6 ([Oz06, BK22, Aetal19, Ki23b]). *If $D \subset \Sigma$ is a fully alternating diagram of a link $L \subset \Sigma \times I$, then L is (i) nullhomologous over $\mathbb{Z}/2$ and (ii) nonsplit; in particular, $(\Sigma \times I) \setminus L$ is irreducible if $g(\Sigma) > 0$. Moreover, (iii) if (Σ, D) is weakly prime, then (Σ, L) is weakly prime, and (iv) if (Σ, D) is prime, then (Σ, L) is prime.*

Parts (i) and (ii) were proven by Ozawa in [Oz06] and by Boden-Karimi in [BK22]. Part (iii) was proven by Adams et al in [Aetal19]. Part (iv) is one of the main results of [Ki23b], where we also give new proofs of (i)-(iii).

2.1.1. *End-essential spanning surfaces.* Part (i) of Theorem 2.6 implies that L has *spanning surfaces*: embedded, unoriented, compact surfaces $F \subset \Sigma \times I$ with $\partial F = L$; while we *do not* require F to be connected, we do require that each component of F has nonempty boundary. By deleting the interior of a regular neighborhood of L from F and $\Sigma \times I$, one may instead view F as a properly embedded surface in the link exterior $(\Sigma \times I) \setminus \mathring{\nu} L$[11][12] We take this view throughout, except in Definition 2.7, Note 19, and §2.3.1.

If (Σ, D) is a fully alternating diagram of (Σ, L), then it is possible to orient each disk of $\Sigma \setminus D$ so that, under the resulting boundary orientation, over- and under-strands are oriented respectively toward and away from crossings. Since Σ is orientable, these orientations determine a checkerboard coloring of $\Sigma \setminus\setminus D$,[13] i.e. a way of shading the disks of $\Sigma \setminus\setminus D$ black and white so that regions of the same shade abut only at crossings.[14] One can use this checkerboard coloring to construct *checkerboard surfaces* B and W for L, where B projects into the black regions, W projects into the white, and B and W intersect in *vertical arcs* which project to the the crossings of D. The main result of [Ki23c] is that these checkerboard surfaces satisfy several convenient properties:

Definition 2.7. Let $F \subset \Sigma \times I$ be a spanning surface for (Σ, L). Denote $M_F = (\Sigma \times I) \setminus\setminus F$, and use the natural map $h_F : M_F \to \Sigma \times I$ to denote $h_F^{-1}(L) = \widetilde{L}$, $h_F^{-1}(\Sigma_\pm) = \widetilde{\Sigma_\pm}$, and $h_F^{-1}(F) = \widetilde{F}$, so that $h_F : \widetilde{L} \to L$ and $h_F : \widetilde{\Sigma_\pm} \to \Sigma_\pm$ are homeomorphisms and $h_F : \widetilde{F} \setminus \widetilde{L} \to \text{int}(F)$ is a 2:1 covering map. Then we say that F is:

[11]Throughout, given a manifold X and a submanifold $Y \subset X$, νY denotes a *closed* regular neighborhood of Y in X.

[12]We also assume that ∂F is transverse on $\partial \nu L$ to each meridian, where a meridian is the preimage of a point in L under the bundle map $\nu L \to L$.

[13]For compact $X, Y \subset \Sigma \times I$, $X \setminus\setminus Y$ denotes the metric closure of $X \setminus Y$; see Note 7 of [Ki23] for a precise definition.

[14]Interestingly, fully alternating link diagrams on nonorientable surfaces are never **checkerboard colorable**.

(a) **incompressible** if any circle $\gamma \subset \widetilde{F} \setminus \widetilde{L}$ that bounds a disk in M_F also bounds a disk in $\widetilde{F} \setminus \widetilde{L}$.[15]
(b) **end-incompressible** if any circle $\gamma \subset \widetilde{F} \setminus \widetilde{L}$ that is parallel in M_F to $\widetilde{\Sigma_\pm}$ bounds a disk in $\widetilde{F} \setminus \widetilde{L}$.
(c) **∂-incompressible** if, for any circle $\gamma \subset \widetilde{F}$ with $|\gamma \cap \widetilde{L}| = 1$ that bounds a disk in M_F, $\gamma \setminus\setminus \widetilde{L}$ is parallel in $\widetilde{F} \setminus\setminus \widetilde{L}$ into $\widetilde{L}$.
(d) **essential** if F satisfies (a) and (c).
(e) **end-essential** if F satisfies (b) and (c).[16]

A crossing c of a diagram $D \subset \Sigma$ is **removably nugatory** if there is a disk $X \subset \Sigma$ such that $\partial X \pitchfork D = \{c\}$; in that case, one can remove c from D via a flype and a Reidemeister 1 move. No cellularly embedded, weakly prime diagram has removable nugatory crossings. Also, any diagram (Σ, D) with a removable nugatory crossing, has at least one ∂-compressible checkerboard surface. Conversely:

Theorem 2.8 (Theorem 1.1 of [Ki23c])**.** *If $D \subset \Sigma$ is a fully alternating diagram without removable nugatory crossings, then both checkerboard surfaces from D are end-essential.*

Proposition 2.9. *Suppose $F_\pm$ are definite surfaces of opposite signs spanning a link $L \subset \Sigma \times I$ and $F_+ \cap F_-$ consists only of arcs, none of which are ∂-parallel in both F_+ and F_-. If F_- (resp. F_+) is ∂-incompressible, then no arc of $F_+ \cap F_-$ is ∂-parallel in F_+ (resp. F_-).*♊︎

Proposition 2.10. *If an essential surface F spanning (Σ, L) contains an arc β which is parallel in $(\Sigma \times I) \setminus\setminus (F \cup \nu L)$ to an arc $\alpha \subset \partial\nu L \setminus\setminus \partial F$, then α is parallel in $\partial \nu L$ to ∂F.*♊︎

Observation 2.11. *Suppose B, W are the checkerboard surfaces of a fully alternating diagram $D \subset \Sigma$ of a link $L \subset \Sigma \times I$. Any properly embedded arc in W that is disjoint from B and separating in W is either ∂-parallel in W or isotopic in W to a vertical arc of $B \cap W$. Likewise with B and W reversed.*♊︎

Remark 2.12. Observation 2.11 implies in particular that no vertical arc from a weakly prime, fully alternating diagram is ∂-parallel in either checkerboard surface.♊︎

2.1.2. *Flype-related diagrams.*

Definition 2.13. If $D \subset \Sigma$ is a link diagram and $\gamma \subset \Sigma$ is an inessential circle that intersects D transversally in three points, exactly one

[15] F is incompressible if and only if F is π_1-injective, meaning that inclusion $\text{int}(F) \hookrightarrow (\Sigma \times I) \setminus L$ induces an injection of fundamental groups (for all possible choices of basepoint).

[16] Note that any end-essential surface is essential. Observe moreover that the converse is true when Σ is a 2-sphere.

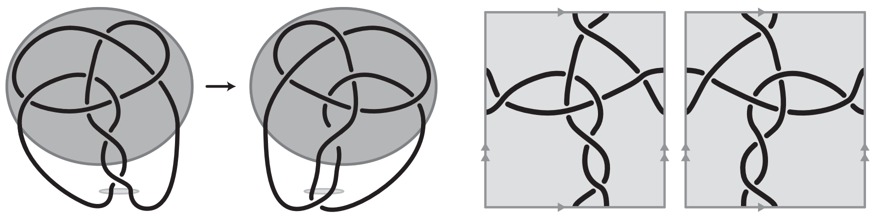

FIGURE 2. Left: an *entire flype* of a diagram of the knot 8_{17}. Right: Corollary 3.7 will imply that these links are non-isotopic; see Example 3.8.

of them a crossing point, c, then we call the circle γ a **flyping circle** for D. Up to mirror symmetry, D and γ appear as shown far left in Figure 1 (D intersects the disk component of $\Sigma \setminus \mathring{\nu}\gamma$ in a tangle T_2 and intersects the other component in a "higher-genus tangle" T_1), so one can **flype** D along γ as shown: this move fixes T_1, switches which pair of strands cross within $\nu\gamma$, and changes T_2 by reflecting the underlying projection and reversing all crossing information.ϒ

Observation 2.14. *If $D \to D'$ is a flype, then D and D' represent the same link L and have the same number of crossings. If D is oriented then D and D' have the same writhe.*[17] *If D is fully alternating (resp. weakly prime), then so is D'.*ϒ

Remark 2.15. In the classical setting, the tangle T_1 in Figure 1 might contain no crossings, in which case the flype has the effect of changing D to its mirror image and then reversing all crossings; one may think of this move as leaving D unchanged and viewing it from the opposite side of Σ (in [Ki23], we call such a flype an *entire flype*). By contrast (by an euler characteristic argument), no cellularly embedded, checkerboard colorable diagram on a surface of positive genus does. Thus, while, as in [Ki23], we regard two diagrams $D, D' \subset \Sigma$ as *equivalent* iff they are related by planar isotopy and possibly an entire flype, the latter possibility will be vacuous.

2.2. Definite surfaces.

2.2.1. *Linking numbers and slopes.* We adopt the notion of *generalized linking numbers* which was first defined for arbitrary 3-manifolds with nonempty boundary in [CT07] and applied in the context of thickened surfaces in [BCK21, BK22]. The generalized linking number of disjoint multicurves[18] $\alpha, \beta \subset \Sigma \times I$ is

$$\mathrm{lk}_\Sigma(\alpha, \beta) = |\nwarrow\!\!\!\nearrow| - |\nearrow\!\!\!\nwarrow|. \tag{2.1}$$

[17] The **writhe** of D is $w_D = |\times| - |\times|$.

[18] We call a disjoint union of embedded, *oriented* circles a **multicurve**.

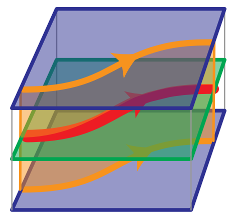

FIGURE 3. A multicurve $\gamma \subset F$ and $\widetilde{\gamma} \subset \widetilde{F}$: $[\widetilde{\gamma}] = \tau[\gamma]$.

This linking pairing, taken *relative to* Σ_+, is *asymmetric*: denoting intersection number on Σ by $\cdot_\Sigma$ and projection $p_\Sigma : \Sigma \times I \to \Sigma$,

$$\text{lk}_\Sigma(\alpha, \beta) - \text{lk}_\Sigma(\beta, \alpha) = p_\Sigma(\alpha) \cdot_\Sigma p_\Sigma(\beta).$$

If F spans a link $L = \bigsqcup_i L_i \subset \Sigma \times I$ and each $\widehat{L_i}$ is a co-oriented pushoff of L_i in F, then we call $s(F) = \sum_i \text{lk}(L_i, \widehat{L_i})$ the **slope** of F.

2.2.2. *The Gordon–Litherland pairing.* Given a surface F spanning a link $L \subset \Sigma \times I$, take νF in the link exterior $(\Sigma \times I) \backslash \mathring{\nu} L$ with projection $p : \nu F \to F$, such that $p^{-1}(\partial F) = \nu F \cap \partial \nu L$, and denote the *frontier* $\widetilde{F} = \partial \nu F \setminus\setminus \partial \nu L$ and *transfer map* $\tau : H_1(F) \to H_1(\widetilde{F})$ (see Figure 3). Following Boden–Chrisman–Karimi, the (generalized) *Gordon–Litherland pairing* (relative to Σ_+) is the symmetric bilinear mapping $\langle \cdot, \cdot \rangle_F : H_1(F) \times H_1(F) \to \mathbb{Z}$ given by [GL78, BCK21]:

$$\langle a, b \rangle_F = \frac{1}{2} \left(\text{lk}_\Sigma(\tau a, b) + \text{lk}_\Sigma(\tau b, a) \right).$$

Given a multicurve $\gamma \subset F$ representing $g \in H_1(F)$, we denote $\langle g, g \rangle_F = \lvert g \rvert_F$ and call $\frac{1}{2} \lvert g \rvert_F$ the *framing* of γ in F. Given a basis $\mathcal{B} = (a_1, \dots, a_n)$ for $H_1(F)$, the *Goeritz matrix* $G = (x_{ij}) \in \mathbb{Z}^{n \times n}$, $x_{ij} = \langle a_i, a_j \rangle_F$, represents $\langle \cdot, \cdot \rangle_F$ with respect to $\mathcal{B}$. Denoting the signature of G by $\sigma(F)$, the quantity

$$\sigma_F(L) = \sigma(F) - \frac{1}{2} s(F), \tag{2.2}$$

depends only on the S^* equivalence class of F; whenever (Σ, L) is nonsplit with diagram (Σ, D) there are exactly two such classes, each represented by a checkerboard surface of D [BCK21].[19]

[19] S^* equivalence is generated by attaching and deleting tubes and crosscaps [GL78] and thus respects relative homology classes. The checkerboard surfaces F and F' of D satisfy $[F] + [F'] = [\Sigma]$ in $H_2(\Sigma \times I, L; \mathbb{Z}/2)$, so $[F] \neq [F']$; hence, F and F' are not S^* equivalent. For the converse, following the classical approach of Yasuhara [Ya14], put an arbitrary spanning surface in disk-band form, attach tubes to make it a checkerboard surface for some diagram, and then perform Reidemester moves (requiring more tubing and crosscapping moves).

2.2.3. *Definiteness characterizes alternating links.* A spanning surface F is *positive-* (resp. *negative-*) *definite* if $\langle \alpha, \alpha \rangle_F > 0$ (resp. $\langle \alpha, \alpha \rangle_F < 0$) for all nonzero $\alpha \in H_1(F)$ [Gr17].[20][21]

Adapting work of Greene from the classical setting [Gr17], Boden–Karimi characterized nonstabilized alternating links in (and diagrams on) thickened surfaces in terms of definite surfaces:

Fact 2.16 (Proposition 3.8 of [BK22])**.** *A cellularly embedded, checkerboard colorable link diagram $D \subset \Sigma$ is alternating if and only if its checkerboard surfaces are definite and of opposite signs.*

Theorem 2.17 (Theorem 4.8 of [BK22])**.** *Suppose (Σ, L) is nonstabilized.*[22] *Then L is alternating if and only if it has connected*[23] *spanning surfaces of opposite signs.*

The proof in [BK22] of Theorem 2.17 shows moreover that if L has connected spanning surfaces of opposite signs, then there is a closed surface S in $\Sigma \times I$ on which L has a fully alternating diagram whose checkerboard surfaces are isotopic to the given surfaces; further, if (L, Σ) is nonstabilized, then S is isotopic to Σ. Formally:

Corollary 2.18. *If (Σ, L) is nonstabilized and B and W are connected spanning surfaces of opposite signs spanning L, then L has a fully alternating diagram on Σ whose checkerboard surfaces are isotopic to B and W.*

Convention 2.19. The checkerboard surfaces B and W of any fully alternating diagram are labeled such that B is positive-definite and W is negative-definite. Likewise for checkerboard surfaces B' and W' (resp. B_i and W_i) from such a diagram D' (resp. D_i).

Lemma 2.20 (c.f. [BK22] Lemma 3.7)**.** *The checkerboard surfaces B and W of any fully alternating diagram of a link (Σ, L) satisfy*[24]

$$\sigma_B(L) - \sigma_W(L) = 2g(\Sigma).$$

Moreover, much of Boden–Karimi's proof of Theorem 2.17 goes through even if the spanning surfaces of opposite signs for L are disconnected or if (Σ, L) is stabilized, or both. In particular, if L has spanning surfaces (not necessarily connected) of opposite signs, then there is a closed surface S (not necessarily connected) in $\Sigma \times I$

[20] F is positive-definite iff $\sigma(F) = \beta_1(F)$ or equivalently iff each multicurve in F either has positive framing in F or bounds an orientable subsurface of F.

[21] When $|\partial F| \leq 2$, every primitive $g \in H_1(F)$ is represented by an oriented circle, but this is not true in general: e.g. take F to be an oriented pair of pants and g the sum of two boundary components, one with the boundary orientation.

[22] Recall that this implies that $L \subset \Sigma \times I$ is a nonsplit link.

[23] Spanning surfaces are assumed to be connected throughout [BK22].

[24] For an arbitrary diagram on Σ, $|\sigma_W(L) - \sigma_B(L)| \leq 2g(\Sigma)$.

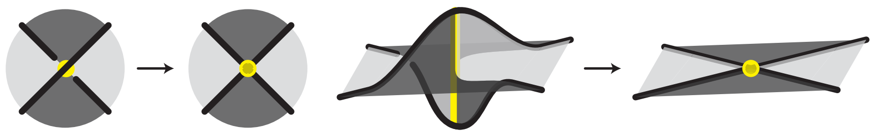

FIGURE 4. Collapsing $S \cup T$ along a standard arc

on which L has a fully alternating diagram D whose checkerboard surfaces are isotopic to the given surfaces; further, each component of S either is parallel to Σ or is a 2-sphere. In particular:

Fact 2.21. *If $F_\pm$ are definite surfaces of opposite signs spanning a link $L \subset \Sigma \times I$, then for some (possibly empty) disjoint union of 2-spheres $\Sigma' \subset (\Sigma \times I) \setminus \Sigma$, L has a fully alternating diagram $D \subset \Sigma \cup \Sigma'$ whose checkerboard surfaces are isotopic to $F_\pm$. Thus:*

(A) F_+ and F_- have the same number of connected components, and this equals the number of split components of L.
(B) L has at most one non-local component.

2.2.4. *Intersections between definite surfaces.* Let F and F' be spanning surfaces for (Σ, L) with $F \pitchfork F'$. Orient L arbitrarily, and orient ∂F and $\partial F'$ so that each is homologous in νL to L. Given an arc α of $F \cap F'$, take $\nu \partial \alpha$ in $\partial \nu L$. Following Howie [Ho18], we call α **standard** if $i(\partial F, \partial F')_{\nu\partial\alpha} = \pm 2$ and *non-standard* if $i(\partial F, \partial F')_{\nu\partial\alpha} = 0$.

$$s(F) - s(F') = i(\partial F, \partial F')_{\partial \nu L} = \sum_{\text{arcs } \alpha \text{ of } F \cap F'} i(\partial F, \partial F')_{\nu \partial \alpha} \tag{2.3}$$

Procedure 2.22. Let (Σ, L) be non-stabilized with connected spanning surfaces S, T such that $S \cap T$ consists entirely of standard arcs and $|S \cap T| = \beta_1(S) + \beta_1(T) + 2g(\Sigma)$. Then extending S, T through νL so that $\partial S = L = \partial T$ and collapsing $S \cup T$ along each arc of $\text{int}(S) \cap \text{int}(T)$ gives a closed surface Q isotopic to Σ[25] on which L collapses to a connected 4-valent graph; recovering crossing information gives a connected link diagram $D \subset Q$ for which S and T *are* checkerboard surfaces. The initial configuration of S and T, up to isotopy of $S \cup T$ in $(\Sigma \times I) \setminus \mathring{\nu} L$, uniquely determines D up to isotopy. See Figure 4.Υ

Proposition 2.23. *If (Σ, L) is local and has positive- and negative-definite connected spanning surfaces F_+ and F_-, then*

$$s(F_+) - s(F_-) = 2\left(\beta_1(F_+) + \beta_1(F_-)\right).$$

Proof. Because L is local, the surfaces F_+ and F_- are S^*-equivalent, so $\sigma_{F_+}(L) = \sigma_{F_-}(L)$, and the result follows from (2.2). □

[25] Connectedness and $|S \cap T| = \beta_1(S) + \beta_1(T)$ imply that $g(Q) = g(\Sigma)$. This and the assumption that (Σ, L) is non-stabilized imply that Q is isotopic to Σ.

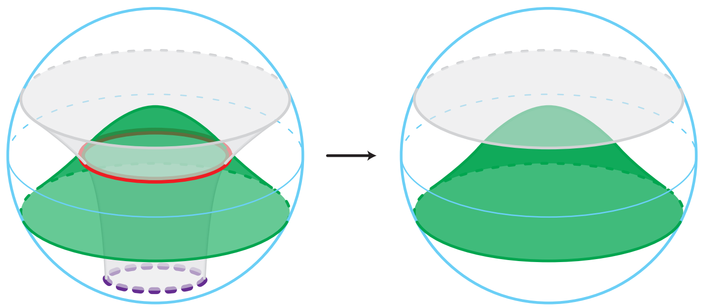

FIGURE 5. Removing a circle γ of intersection between positive- and negative-definite surfaces F_+ and F_-. The dashed purple circle bounds a disk in F_+.

Proposition 2.24 (c.f. Propositions 2.12 and 2.22 of [Ki23]). *If (Σ, L) is non-stabilized and has positive- and negative-definite connected spanning surfaces F_+ and F_-, then*

$$s(F_+) - s(F_-) = 2\beta_1(F_+) + 2\beta_1(F_-) + 4g(\Sigma).$$

Further, if $F_+ \cap F_-$ is comprised of arcs α with $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} = +2$:

(A) $|F_+ \cap F_-| = \beta_1(F_+) + \beta_1(F_-) + 2g(\Sigma)$,
(B) $F_\pm$ yield an alternating diagram D via Procedure 2.22, and
(C) if F_+ and F_- are ∂-incompressible, then D has no removable nugatory crossings.

Proof. Isotope $F_\pm$ so that each component α of $F_+ \cap F_-$ is an arc with $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} = +2$. Now

$$|F_+ \cap F_-| = \frac{1}{2}|\partial F_+ \cap \partial F_-| = \frac{1}{2}\left(s(F_+) - s(F_-)\right),$$

which equals $\beta_1(F_+) + \beta_1(F_-) + 2g(\Sigma)$ by (2.2) and Lemma 2.20. Therefore, the pair $F_\pm$ determines a connected diagram D of L via Procedure 2.22. The checkerboard surfaces of D are $F_\pm$, so D is alternating by Fact 2.16. Part (C) follows easily. ☐

Fact 2.25 (c.f. Fact 2.23 of [Ki23], Lemma 3.4 of [Gr17]). *If $F_+ \pitchfork F_-$ are definite surfaces of opposite signs spanning a link $L \subset \Sigma \times I$, then any circle $\gamma \subset F_+ \cap F_-$ bounds disks in both F_+ and F_-.*

Procedure 2.26. Suppose $F_+ \pitchfork F_-$ are definite surfaces of opposite signs spanning a link $L \subset \Sigma \times I$. Fixing F_-, isotope F_+ via the following hierarchy of moves:[26]

(1) If $F_+ \cap F_-$ contains circles, then (using Fact 2.25) choose an innermost one γ in F_+; γ bounds disks $X_\pm \subset F_\pm$. Using the

[26] That is, perform (1) whenever possible, perform (2) whenever possible unless (1) is possible, and perform (3) whenever possible unless (1) or (2) is possible.

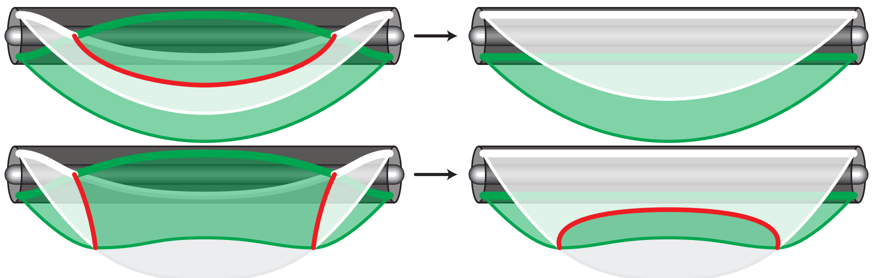

FIGURE 6. Removing adjacent points of $\partial F_+ \cap \partial F_-$ of opposite sign

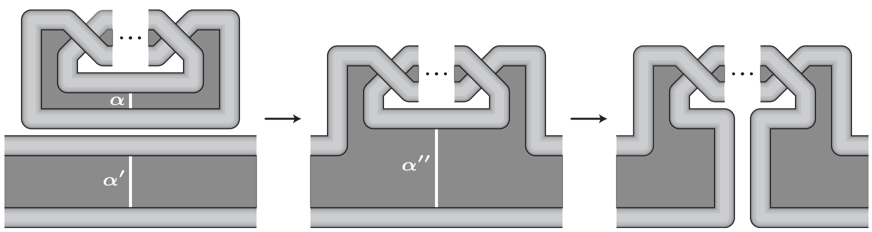

FIGURE 7. Adding positive twists to a spanning surface

irreducibility of $(\Sigma \times I) \setminus L$, isotope X_+ past X_- as shown in Figure 5. Meanwhile, fix F_+ away from X_+.

(2) If any arc α of $F_+ \cap F_-$ is parallel in $F_- \backslash\backslash F_+$ to ∂F_- *and* in $F_+ \backslash\backslash F_-$ to ∂F_+, then remove α as shown in Figure 6, top.

(3) If arcs $\alpha_+ \subset \partial F_+ \backslash\backslash \partial F_-$ and $\alpha_- \subset \partial F_- \backslash\backslash \partial F_+$ are parallel in $\partial \nu L$, then push α_+ past α_- as in Figure 6, bottom.Υ

We also recall:

Fact 2.27. *If α is a system of disjoint properly embedded arcs in a definite surface F, then $F \setminus \mathring{\nu}\alpha$ is definite.*Υ

Fact 2.28. *If F' is obtained by adding positive twists to a positive-definite surface F as in Figure 7, then F' is positive-definite.*Υ[27]

Fact 2.29. *If $F_\pm$ are definite surfaces of opposite signs spanning (Σ, L) and α is a non-standard arc of $F_+ \cap F_-$, then denoting $F'_+ = F_+ \setminus \mathring{\nu}\alpha$, $L' = \partial F'_+$, and $F'_- = F_- \setminus \mathring{\nu}\alpha$, the following are equivalent:*

(I) α is separating on F_+;
(II) α is separating on F_-;
(III) L' has one more split component than L.Υ

The next two facts differ notably from their classical analogs:

[27] Likewise for adding negative twists to a negative-definite surface.

Fact 2.30 (c.f. Proposition 6.6 of [Ki23]). *Let F be a positive-definite surface spanning a weakly prime alternating link L, and let K be the kernel of the map $H_1(F) \to H_1(\Sigma \times I)$ induced by inclusion $F \hookrightarrow \Sigma \times I$. Then F is end-essential if and only if every nonzero $a \in K$ satisfies $\langle a, a \rangle_F \geq 2$.*[28]

Proof. Take an end-essential negative-definite spanning surface W for L with $W \pitchfork F$, and let D be an alternating diagram of L associated to F, W (via Procedure 2.26 and then 2.22). If D is weakly prime, then both conditions are satisfied, the first by Theorem 2.8 and the second by an argument analogous to the proof of Lemma 4 of [Ki23a]. Conversely, if D admits a removable nugatory crossing c, then neither condition holds, because W is end-essential. □

Proposition 2.31 (c.f. Proposition 6.7 of [Ki23]). *Let F be a positive-definite surface spanning a weakly prime alternating link L, and let $\alpha \subset F$ be a properly embedded arc such that $F' = F \setminus \mathring{\nu}\alpha$ spans a weakly prime alternating link L'. If F is end-essential, then F' is also end-essential.*

Proof. Letting K and K' denote the kernels of the maps $H_1(F) \to H_1(\Sigma \times I)$ and $H_1(F') \to H_1(\Sigma \times I)$ induced by inclusion, Fact 2.30 tells us that every nonzero $c \in K$ satisfies $\langle c, c \rangle_F \geq 2$, and Fact 2.27 implies that F' is positive-definite. Therefore every nonzero $c \in K'$ satisfies $\langle c, c \rangle_F \geq 2$, and so Fact 2.30 implies that F' is end-essential. □

Proposition 2.32. *As a result of Procedure 2.26, $F_+ \cap F_-$ consists only of standard positive arcs.*♊[29]

Proposition 2.33. *If $F_\pm$ are definite surfaces of opposite signs spanning a link $L \subset \Sigma \times I$ and α is an arc of $F_+ \cap F_-$ that is ∂-parallel in both F_+ and F_-, then α is non-standard.*♊

Lemma 2.34 (c.f. Lemma 2.30 of [Ki23]). *Suppose $F_\pm$ are positive- and negative-definite surfaces spanning a non-stabilized link $L \subset \Sigma \times I$, and α is an arc of $F_+ \pitchfork F_-$. Then:*

- *(A) $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} \neq -2$.*
- *(B) If α is nonseparating on F_-, then $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} = 2$.*
- *(C) In particular, if L is weakly prime, both $F_\pm$ are essential, and α is not ∂-parallel in both $F_\pm$, then $i(\partial F_+, \partial F_-)_{\nu\partial\alpha} = 2$.*

Proof. The argument is largely the same as in [Ki23]. For (A) and (B), we just describe the differences: if (Σ, L') is nonstabilized, then replacing $\beta_1(F_+) + \beta_1(F_-)$ with $\beta_1(F_+) + \beta_1(F_-) + 2g(\Sigma)$ in (6.1) and

[28] Definiteness implies that F is end-incompressible.

[29] Note that Procedure 2.26 always terminates because each move decreases $|F_+ \cap F_-| + |\partial F_+ \cap \partial F_-|$.

(6.2) of [Ki23] contradicts Proposition 2.24 (A); if (Σ, L') is stabilized, then Fact 2.21 (A) (and, for (B), the assumption that α is non-separating on F_-) implies that L' is local, so Proposition 2.23 gives:

$$\begin{aligned} -2 &= (s(F_+) - s(F_-)) - \big(s(F'_+) - s(F'_-)\big) \\ -2 &= 2(\beta_1(F_+) + \beta_1(F_-) + 2g(\Sigma)) - 2(\beta_1(F'_+) + \beta_1(F'_-)) \quad (2.4) \\ -1 &= g(\Sigma). \end{aligned}$$

We prove (C) by contradiction. Apply Procedure 2.26 $F_+ = F_0 \to F_1 \to \cdots \to F_t$ until it terminates, and consider the last move (3) $F_s \to F_{s+1}$ in the sequence, which involves two arcs α_1, α_2 of $F_s \cap F_-$ and one arc α of $F_{s+1} \cap F_-$; perturb α_1 in F_- so that it is disjoint from F_s. Parts (A) and (B) imply without loss of generality that α_1 is non-standard, so $F_- \setminus \nu\alpha_1$ and $F_s \setminus \nu\alpha_1$ are definite surfaces of opposite sign spanning the same link L'. Observe that, for all $i = s+1, \dots, t$ (c.f. (6.3) of [Ki23]), and each arc α' of $F_- \setminus\setminus F_i$ that separates F_-, either α' is ∂-parallel in F_- or $\partial(F_- \setminus \nu\alpha')$ is split with no local components. The latter "possibility" uses the assumption that L is weakly prime; it also contradicts Fact 2.21 (B). Therefore, α_1 is ∂-parallel in F_-, which contradicts the hierarchy of the moves in Procedure 2.26. □

Using Lemma 2.34, the same reasoning as in [Ki23] leads to:

Theorem 2.35. *Suppose (Σ, D) and (Σ, D') are weakly prime, fully alternating diagrams of (Σ, L) with checkerboard surfaces B, W and B', W'. Then D and D' are equivalent if and only if B and B' are isotopic in $(\Sigma \times I) \setminus \mathring{\nu}L$, as are W and W'.* ⌑

Corollary 2.36. *There is a bijective correspondence between equivalence classes of weakly prime, fully alternating link diagrams on Σ and pairs of isotopy classes of essential definite surfaces of opposite signs spanning the same weakly prime, nonstabilized link in $\Sigma \times I$.* ⌑[30]

2.3. **Plumbing.** A *plumbing cap* for a surface F spanning (Σ, L) is an embedded disk $V \subset (\Sigma \times I) \setminus \mathring{\nu}L$ with $V \cap (F \cup \partial\nu L) = \partial V$ where:

- ∂V bounds a disk $\widehat{U} \subset F \cup \nu L$,
- $\widehat{U} \cap F$ is a disk U called the *shadow* of V, and
- denoting the components of $(\Sigma \times I) \setminus\setminus (\widehat{U} \cup V)$ by Y_1, Y_2, neither subsurface $F_i = F \cap Y_i$ is a disk.

If the first two properties hold but the third fails, we call V a *fake plumbing cap* for F.[31] If V is a plumbing cap for F with shadow U,

[30] Example 2.37 of [Ki23] shows that Theorem 2.35 and Corollary 2.36 become false if one removes "weakly prime" or "fully alternating."

[31] The decomposition $F = F_1 \cup F_2$ is a *de-plumbing* of F along U and V, denoted $F = F_1 * F_2$. The reverse operation, in which one obtains F by gluing F_1 and F_2 along U, is called *generalized plumbing* or *Murasugi sum.*

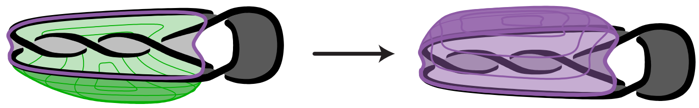

FIGURE 8. Re-plumbing a spanning surface replaces a plumbing shadow with its cap.

then the operation $F \to (F \setminus U) \cup V$ is called **re-plumbing**. See Figure 8. The same operation along a fake plumbing cap, a "fake re-plumbing," is an isotopy move. Two spanning surfaces are *plumb-related* if they are related by re-plumbing and isotopy moves.

2.3.1. *The 4-dimensional perspective.*

Proposition 2.37 (c.f. Proposition 2.36 of [Ki23])**.** *Given surfaces F_1, F_2 spanning (Σ, L), let F_i' be properly embedded surfaces in $\Sigma \times I \times I_+$ obtained by perturbing int(F_i), while fixing $\partial F_1 = L = \partial F_2$. If $F_1 \setminus \mathring{\nu} L$ and $F_2 \setminus \mathring{\nu} L$ are plumb-related, then:*

(A) F_1' and F_2' are related by an ambient isotopy of $\Sigma \times I \times I_+$ which fixes $\Sigma \times I \supset L$;
(B) there is an isomorphism $\phi : H_1(F_1) \to H_1(F_2)$ satisfying $\langle \alpha, \beta \rangle_{F_1} = \langle \phi(\alpha), \phi(\beta) \rangle_{F_2}$ for all $\alpha, \beta \in H_1(F_1)$;
(C) if F_1 is definite, then F_2 is definite of the same sign;
(D) in particular, if F_1 is a checkerboard surface from an alternating diagram of L on Σ, then so is F_2;
(E) F_1 and F_2 are S^* equivalent, and thus $\sigma_{F_1}(L) = \sigma_{F_2}(L)$.

Proof. Part (A) is the same as in [Ki23]. For (B), construct the desired isomorphism $\phi : H_1(F_1) \to H_1(F_2)$ as follows. Given $a \in H_1(F_1)$, take a multicurve $\alpha \subset F_i$ representing a, replace each arc of $\alpha \cap U$ with an arc in V (with the same initial and terminal points), and denote the resulting multicurve by α'; set $\phi(a) = [\alpha']$. This immediately gives (C) and (D), and (E) now follows from the observation that $[F_1] + [F_2] = 0 \in H_2(\Sigma \times I, L; \mathbb{Z}/2)$, since the union of any plumbing cap and its shadow is nullhomologous. □

Next, we extend Theorem 3 of [GL78] to the context of thickened surfaces. Let F be a spanning surface of a link $L \subset \Sigma \times I$. Isotope F so that $F \subset (\Sigma \setminus \mathring{\nu} x) \times I$ for some point $x \in \Sigma$.[32] Let F' be a properly embedded surface in $(\Sigma \setminus \mathring{\nu} x) \times I \times I_+$ obtained by perturbing the interior of F while fixing ∂F. One can construct the double-branched cover $M_{\widehat{F}}$ of $(\Sigma \setminus \mathring{\nu} x) \times I \times I_+$ along F' by cutting $\Sigma \times I \times I_+$ along the trace of this isotopy, taking two copies, and gluing. Yet, these two copies are homeomorphic to $\Sigma \times I \times I_+$, and the gluing region corresponds to a regular neighborhood N of F in $\Sigma \times I$.

[32] To see that this is always possible, consider isotoping F into disk-band form.

Therefore, one may instead construct $M_{\widehat{F}}$ as follows. Let $\iota : N \to N$ be involution given by reflection in the fiber, take two copies Σ^4_1 and Σ^4_2 of $(\Sigma \setminus \mathring{\nu} x) \times I \times I_+$, and define

$$M_{\widehat{F}} = \left(\Sigma^4_1 \cup \Sigma^4_2\right) / \left(y \in N \subset \partial\Sigma^4_1 \sim \iota(y) \in N \subset \partial\Sigma^4_2\right).$$

Consider the Mayer-Vietoris sequence for $M_{\widehat{F}}$:

$$0 = H_2(\Sigma^4_1) \oplus H_2(\Sigma^4_2) \to H_2(M_{\widehat{F}}) \xrightarrow{\varphi} H_1(N) \xrightarrow{\psi} H_1(\Sigma^4_1) \oplus H_1(\Sigma^4_2) \to \cdots$$

If $g(\Sigma) = 0$, as in [GL78], then both Σ^4_i are 4-balls, so φ is an isomorphism; Gordon–Litherland then use the inverse map to compare the intersection form $\cdot$ on $M_{\widehat{F}}$ with their pairing $\mathcal{G}_F$ on F. After restricting appropriately, the same ideas work here:

Theorem 2.38 (c.f. Theorem 3 of [GL78])**.** *With the setup above, let $i_* : H_1(F) \to H_1(N)$ be the isomorphism induced by inclusion, and denote $K = i_*^{-1}(\ker(\psi))$. Then there is an isomorphism $S : (K, \mathcal{G}_F) \to \left(H_2(M_{\widehat{F}}), \cdot\right)$.*

Proof. Consider the following map $S : K \to H_2(M_{\widehat{F}})$. Given $A \in K$, choose a multicurve $\alpha \subset F$ with $[\alpha] = A$. Then α bounds properly embedded oriented surfaces $s_i \subset \Sigma^4_i$ for $i = 1, 2$. Define $S(A) = [s_1] - [s_2] \in H_2(M_{\widehat{F}})$.

To see that this is the required isomorphism $(K, \mathcal{G}_F) \to \left(H_2(M_{\widehat{F}}), \cdot\right)$, let $A, B \in K$, represented respectively by multicurves $\alpha, \beta \subset F$. Then α and $\widetilde{\beta}$ are disjoint multicurves in N with $[\widetilde{\alpha}] = 2A$, $[\widetilde{\beta}] = 2B$, $\iota(\alpha) = \alpha$, and $\iota(\widetilde{\beta}) = \widetilde{\beta}$. Hence:

$$\begin{aligned} S(A) \cdot S(B) &= \frac{1}{4}\left(S([\widetilde{\alpha}]) \cdot S([\beta]) + S([\widetilde{\beta}]) \cdot S([\alpha])\right) \\ &= \frac{1}{4}\left(\mathrm{lk}_\Sigma(\widetilde{\alpha}, \beta) + \mathrm{lk}_\Sigma(\iota\widetilde{\alpha}, \iota\beta) + \mathrm{lk}_\Sigma(\widetilde{\beta}, \alpha) + \mathrm{lk}_\Sigma(\iota\widetilde{\beta}, \iota\alpha)\right) \\ &= \frac{1}{2}\left(\mathrm{lk}_\Sigma(\widetilde{\alpha}, \beta) + \mathrm{lk}_\Sigma(\widetilde{\beta}, \alpha)\right) \\ &= \mathcal{G}_F(A, B). \end{aligned}$$

□

2.3.2. *Flyping caps.* Let $D \subset \Sigma$ be a weakly prime, fully alternating link diagram with checkerboard surfaces B, W. Say that a plumbing cap V for B is a **flyping cap** if V appears as in Figure 10, left-center. There is then a corresponding flype move as shown in Figures 10 and 9. Namely, denoting the shadow of V by U, the flype move proceeds along an annular neighborhood of a circle $\gamma \subset \Sigma$ comprised of the arc $V \cap W$ together with an arc in $U \cup \nu L$. (The resulting link diagram might be equivalent to D.) More formally:

Proposition 2.39 (c.f. Proposition 2.37 of [Ki23])**.** *Let V be an flyping cap for B, $D \to D'$ the flype move corresponding to V, B' and W' the checkerboard surfaces from D', and B'' the surface obtained*

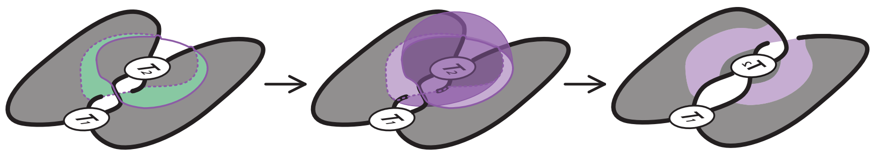

FIGURE 9. A flype move corresponds to an isotopy of one checkerboard surface (here, W) and a re-plumbing of the other.

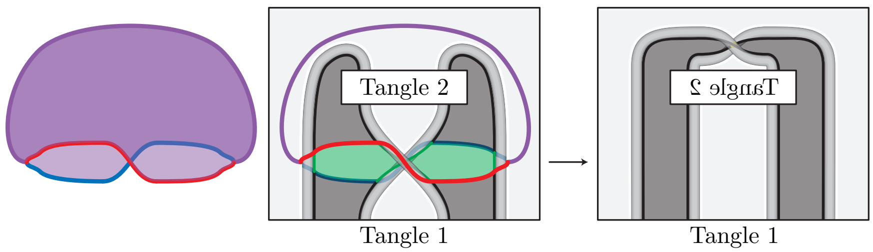

FIGURE 10. A flyping cap and the associated flype move

by re-plumbing B along V. Then B' and B'' are isotopic, as are W' and W. Hence, D' is equivalent to the diagram determined by B'', W via Theorem 2.35.[33]

Proof. As in [Ki23], Figure 9 demonstrates the isotopies. □

Conversely, if γ is a flyping circle for (Σ, D), then there is an flyping cap V for B (or W) with $V \cap W \subset \nu\gamma$ (resp. $V \cap B \subset \nu\gamma$).

3. THE FLYPING THEOREM IN THICKENED SURFACES

The arguments in §§3-5 and 7-8 of [Ki23] have been revised so that they apply directly in the context of this paper (with the obvious replacements of S^3 with $\Sigma \times I$, S^2 with Σ, essential with end-essential, and prime with weakly prime): B, W are the checkerboard surfaces from a weakly prime, fully alternating diagram $D \subset \Sigma$ of a link $L \subset \Sigma \times I$, F is an end-essential positive-definite surface spanning L, v_F is comprised of the vertical arcs at the crossings where F has crossing bands, and $D_{F,W}$ is the diagram determined via Theorem 2.35 by F, W. One then implements Menasco's crossing ball setup, isotopes F into *fair position*, and performs a sequence of isotopy and re-plumbing moves according to a hierarchy: one only performs each move k if F is in $(k-1)$-good position, meaning that F is in fair position and none of Moves 1 through $k-1$ are possible. See [Ki23] for the notations $C, v, \widehat{W}, S_\pm$ etc. associated with the crossing ball setup and for the precise definitions of fair position and Moves 1-10.

[33] An analogous statement holds for flyping caps for W.

Moves 1-9, all of which are isotopy moves, appear in Figure 11. Move 10 is a re-plumbing move and is more complicated; see [Ki23].

A few details are worth noting. First, one must be more careful with push-through moves (see Definition 3.10 of [Ki23]) in thickened surfaces than in S^3. The definition is the same (because it was written with this paper in mind!), but in addition to the three pictures shown top in Figure 19 of [Ki23], three more pictures are possible. See Figure 12. In any case, if we wish to perform (or observe the possibility of) a push-through move along an arc α whose endpoints lie on a circle γ, we must now check that α is parallel in S_+ into γ; in [Ki23], this was free. Importantly, however, this is always the case.[34]

Second, whereas in [Ki23] every circle of $F \cap S_\pm$ was inessential in $S_\pm \approx S^2$, this property holds here only because the assumption that F is end-incompressible allows us to require that $S_+ \cup S_-$ cuts F into disks (c.f. Definition 3.2 (h) and Lemma 3.3 of [Ki23]).

Third, Sublemma 5.2 of [Ki23] implies there that the circles of $F \cap S_+$ are mutually nested, but this is less clear here. The proof of Lemma 5.3 of [Ki23] is thus written with this paper in mind, and is slightly more complicated as a result.

Adapting the arguments from §§3-5, 7-8 of [Ki23] thus gives:

Theorem 3.1. *If $D = D_{B,W}$ is a weakly prime, fully alternating diagram of (Σ, L), then any end-essential, positive definite surface F spanning L is plumb-related to B; likewise for end-essential negative-definite surfaces and W.* Υ

Corollary 3.2. *With F and D as in Theorem 3.1, $\beta_1(B) = \beta_1(F)$ and $s(B) = s(F')$.* Υ[35]

Theorem 3.3 (Part of Tait's extended first conjecture [Gr17, Ka87, Mu87, Th87, Tu87])**.** *If $D, D' \subset \Sigma$ are alternating diagrams of a link $L \subset \Sigma \times I$, neither containing removable nugatory crossings, then D and D' have the same number of crossings.*[36]

Theorem 3.4. *If F is in 9-good position, then F contains no saddle disks: $F \cap C = v_F$; hence, every circle γ of $F \cap S_+$ is a flyping circle, and $D_{F,W}$ is related to D by a sequence of flypes that preserve the isotopy class of W.* Υ

Theorem 3.5 (Tait's extended flyping conjecture)**.** *All weakly prime, fully alternating diagrams $D = D_{B,W}$ and $D' = D_{B',W'}$ of the same*

---

[34] In [Ki23], see the definition of Move 2 and the proofs of Lemmas 3.22, 4.1, and 5.3 and of Propositions 8.2 and 8.3.

[35] This is also true if L is non-stabilized and/or weakly prime.

[36] When D and D' are weakly prime and fully alternating, Fact 2.16, Theorems 2.8 and 3.1, and Corollary 3.2 immediately imply this. The general case then follows, as the number of crossings is additive under (de)stabilization, diagrammatic connect sum, and split union.

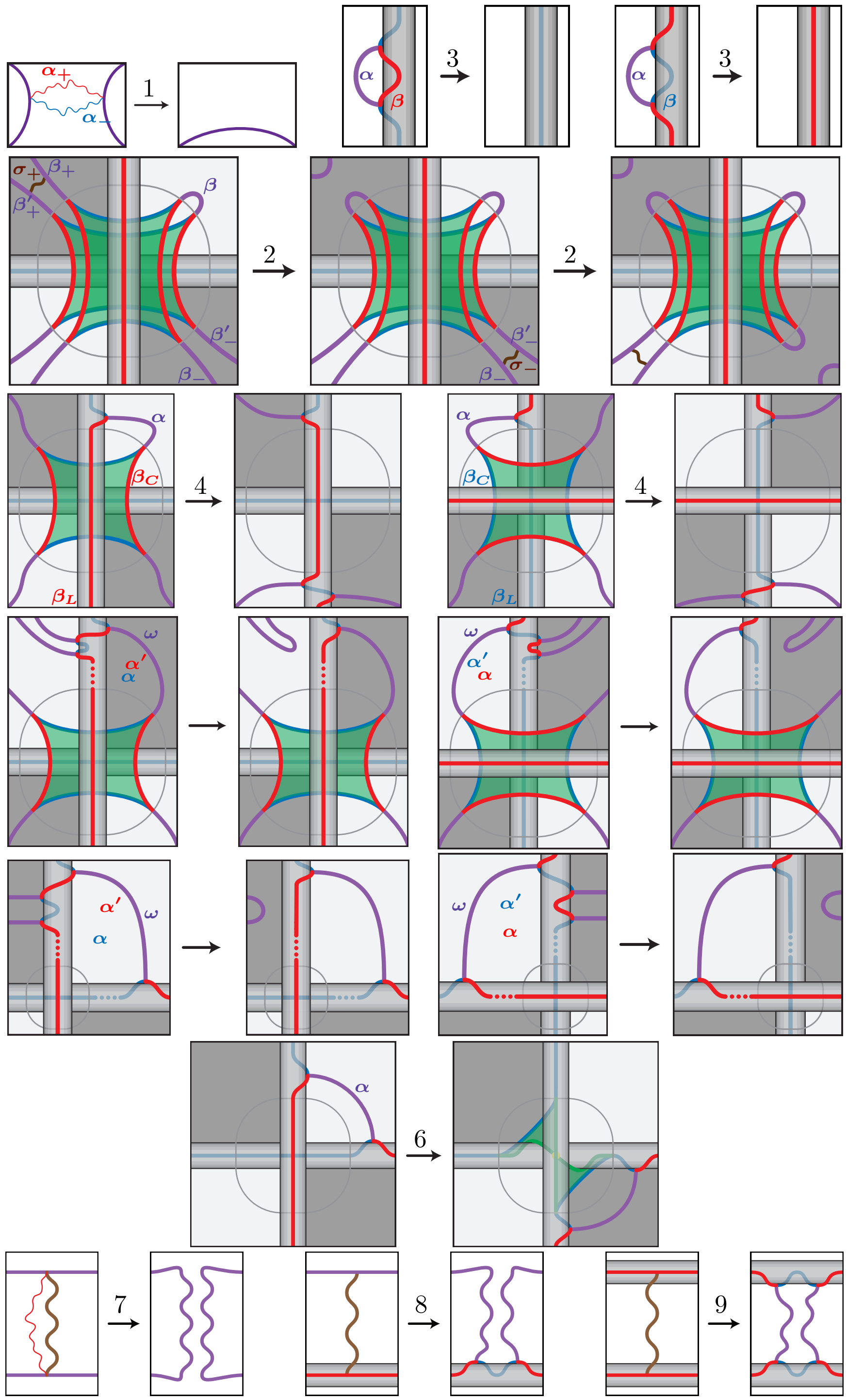

FIGURE 11. Moves 1-9

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FIGURE 12. Push-through moves in $\Sigma \times I$ need not appear as in Figure 19 of [Ki23].

link $L \subset \Sigma \times I$ are related by a sequence of flypes $D \to \cdots \to D'' \to \cdots \to D'$ in which $D \to \cdots \to D''$ preserves the isotopy class of W and $D'' \to \cdots \to D'$ preserves the isotopy class of B'.

Since writhe is invariant under flypes (recall Observation 2.14) and additive under diagrammatic connect sum and disjoint union, we obtain a new geometric proof of Tait's second conjecture:

Theorem 3.6 (Tait's extended second conjecture [BK18, BKS19])**.** *All weakly prime, fully alternating diagrams of a given link $L \subset \Sigma \times I$ have the same writhe.*

Theorem 3.5 implies that, unlike a classical link and a link in $S^2 \times I$, a link in a thickened surface of positive genus is not necessarily isotopic to the link obtained by reflecting horizontally (in the projection surface) and then vertically. More precisely, let $D \subset \Sigma$ be a weakly prime, fully alternating diagram of a link $L \subset \Sigma \times I$; let $\phi : \Sigma \to \Sigma$ be an orientation-reversing involution; let $D' \subset \Sigma$ be the diagram obtained from $\phi(D)$ by reversing all crossing information; and let $L' \subset \Sigma \times I$ be the link represented by D'. Note that L' is the image of L under the map $\Sigma \times I \to \Sigma \times I$ given by $(x, t) \mapsto (\phi(x), -t)$.

Corollary 3.7. *With the setup above, if D is weakly prime and fully alternating, then the links L and L' are isotopic in $\Sigma \times I$ if and only if the diagrams D and D' are flype-related on Σ. In particular, this is always true if $g(\Sigma) = 0$, but not necessarily if $g(\Sigma) > 0$.*

Example 3.8. The diagrams on T^2 shown right in Figure 2 admit no non-trivial flypes and are non-isotopic; thus, by Corollary 3.7, they represent non-isotopic links in $T^2 \times I$.

4. THE FLYPING THEOREM FOR VIRTUAL LINKS

A *virtual link diagram* is the image of an immersion $\bigsqcup S^1 \to S^2$ in which all self-intersections are transverse double-points, some of which are labeled with over-under information. These labeled points are called *classical crossings*, and the other double-points are called *virtual crossings*. Traditionally, virtual crossings are marked with a circle, as in Figure 13. A *virtual link* is an equivalence class of such diagrams under generalized Reidemeister moves, as shown in Figure 13. There are seven types of such moves, the three *classical* moves and four *virtual* moves.[37]

[37] The move involving two virtual crossings and one classical crossing is sometimes called a *mixed* move, but we include it as a virtual (non-classical) move.

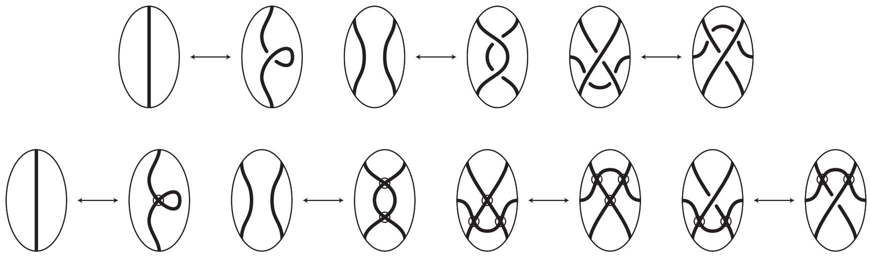

FIGURE 13. Classical (top) and virtual (bottom) Reidemeister moves

FIGURE 14. Converting the neighborhood of a virtual link diagram to an abstract link diagram

Notation 4.1. Given a virtual link diagram $V \subset S^2$, let $[V]$ denote the set of all virtual diagrams related to V by planar isotopy and virtual Reidemeister moves.

4.1. **Correspondences.** By work of Kauffman [Ka98], Kamada–Kamada [KK00], and Carter–Kamada–Saito [CKS02], there is a triple bijective correspondence between (i) virtual links, (ii) abstract links (see below), and (iii) stable equivalence classes of links in thickened surfaces. We review this correspondence now and introduce a new correspondence between the associated *diagrams*.

Definition 4.2. An **abstract link diagram** (S, G) consists of a 4-valent graph G embedded in a compact orientable surface S, such that G has over-under information at each vertex, and G is a deformation retract of S. An **abstract link** is an equivalence class of such diagrams under the following equivalence relation $\sim$: $(S_1, G_1) \sim (S_2, G_2)$ if there are embeddings $\phi_i : S_i \to S$, $i = 1, 2$, into a surface S, such that $\phi_1(G_1)$ and $\phi_2(G_2)$ are related by classical Reidemeister moves on S.

For expository reasons, we introduce our new diagrammatic correspondence before reviewing the related, well-known correspondence. This gives a new diagrammatic proof of the older, well-known correspondence and of Kuperberg's theorem. If this strikes the reader as unexpectedly complicated, there is a good reason for this complication. See Construction 4.6, Remark 4.7, and Example 4.8.

Correspondence 4.3. *There is a triple bijective correspondence between (i) equivalence classes* $[V]$ *of virtual link diagrams, (ii) abstract* *link diagrams, and (iii) cellularly embedded link diagrams on closed surfaces. Namely:*

(i) → (ii) *Given an equivalence class $[V]$ of virtual link diagrams, choose a representative diagram $V \subset S^2$, and construct an abstract link diagram as follows. First, take a regular neighborhood νV of V in S^2. Second, embed S^2 in S^3 and modify νV near each virtual crossing of V as shown in Figure 14. Third, view the resulting pair abstractly, forgetting the embedding in S^3.*

(ii) → (iii) *Given an abstract link diagram (S, G), cap off each component of ∂S with a disk to obtain a cellularly embedded link diagram on a closed surface.*

To confirm the bijectivity of this correspondence, we will use *lassos*, which are introduced and more fully explored in [Ki23b]:

Definition 4.4. A **lasso** for a link diagram D on a closed surface Σ is a disk $X \subset \Sigma$ that intersects D generically and contains all crossings of D. A **lasso** for a virtual link diagram $V \subset S^2$ is a disk $X \subset S^2$ that intersects D generically and contains all classical crossings of D but no virtual crossings; call $V \cap X$ and $V \setminus\setminus X$ respectively the **classical** and **virtual** parts of V (with respect to X).

Proof of Correspondence 4.3. We first describe the reverse constructions (iii) → (ii) and (ii) → (i). The former is easy: given any cellularly embedded link diagram D on a closed surface Σ, the pair $(\nu D, D)$ is the associated abstract link diagram.

For (ii) → (i), consider an abstract link diagram (S, G). View $S^3 = (S^2 \times \mathbb{R}) \cup \{\pm\infty\}$, denote $\widehat{S^3} = S^3 \setminus \{\pm\infty\}$, and denote projection $\pi : \widehat{S^3} \to S^2$. Choose any embedding $\phi : S \to \widehat{S^3}$ such that $\pi|_{\phi(S)}$ has no critical points and all self-intersections in $\pi \circ \phi(G)$ are transverse double-points with neighborhoods as suggested in Figure 14. Now take the 4-valent graph $\pi \circ \phi(G) \subset S^2$, and, for each crossing point c of G, label the double-point $\pi \circ \phi(c)$ with the matching over-under information. (Thus, the double points of V coming from the crossings of D comprise the classical crossings of V, and the remaining double-points comprise the virtual crossings.) See Figure 15.

It remains to justify that this triple correspondence is indeed bijective. For (ii) ↔ (iii), this is immediate.

For (i) ↔ (ii), note that in the construction (i) → (ii), the pairwise homeomorphism type of the resulting abstract link diagram is well-defined and is unchanged by virtual R-moves on V. It thus remains only to show that the choice of embedding $\phi : S \to \widehat{S^3}$ described above for (ii) → (i) does not affect the equivalence class $[V]$ of the resulting virtual diagram V.

Choose a spanning tree T for G, and take a regular neighborhood U of G in S. Now take two embeddings $\phi_i : S \to \widehat{S^3}$, $i = 1, 2$, as

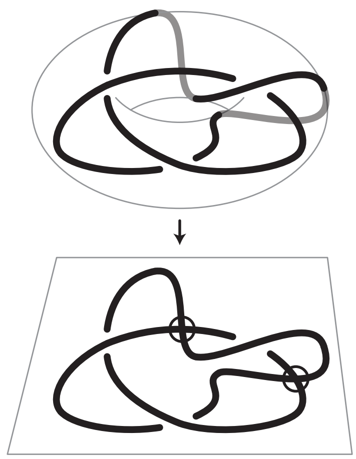

FIGURE 15. A link diagram on the torus and a corresponding virtual diagram

described for (ii) $\to$ (i). Construct ϕ_1 such that $\pi \circ \phi_1(U)$ contains all classical crossings of the virtual diagram $V_1 = \pi \circ \phi_1(G)$ and no classical crossings[38]. Then the disk $\pi \circ \phi_2(U) \subset S^2$ is a lasso for the virtual diagram $V_2 = \pi \circ \phi_2(G)$, and $\pi \circ \phi_2$ restricts to a pairwise homeomorphism of $(U, G \cap U)$. Let ϕ_2 be arbitrary, and denote the virtual diagram $V_2 = \pi \circ \phi_2(G)$. Also denote $\pi \circ \phi_i = f_i$ and $f_i(U) = U_i$ for $i = 1, 2$.

It will suffice to show that V_1 and V_2 are related by virtual R-moves. If $U_2 \cap f_2(G \setminus U) = \varnothing$, then U_2 is a lasso for V_2, and the classical parts of V_2 and V_1 match, as do the combinatorics of the virtual parts, and so V_2 and V_1 are related by virtual R-moves.

Assume instead that $U_2 \cap f_2(G \setminus U) \neq \varnothing$. Consider any (smooth) arc α of $U_2 \cap f_2(G \setminus U)$ in U_2. Note that α contains no classical crossings. Denote one of the disks of $U_2 \setminus\setminus f_2(G \setminus U)$ by U_2'. Then $\partial U_2' = \alpha \cup \beta$ for some arc $\beta \subset \partial U_2 \setminus\setminus f_2(G \setminus U)$. Since α contains no classical crossings, there is a *virtual pass move*, as shown in Figure 16, which takes α through U_2' past β; such a move can be performed via a sequence of *virtual* Reidemeister moves, since α contains no virtual crossings.

Perform such virtual pass moves successively on the arcs of $U_2 \cap f_2(G \setminus U)$ until $U_2 \cap f_2(G \setminus U) = \varnothing$. Then U_2 is a lasso for the resulting diagram, whose classical part matches that of V_1; the combinatorics of the virtual parts of these diagrams also match, and so these diagrams are related by virtual R-moves, as therefore are V_1 and V_2. $\square$

[38]That is, construct ϕ_1 so that $\pi^{-1} \circ \pi \circ \phi_1(U) \cap \phi_1(G) = \phi_1(G \cap U)$.

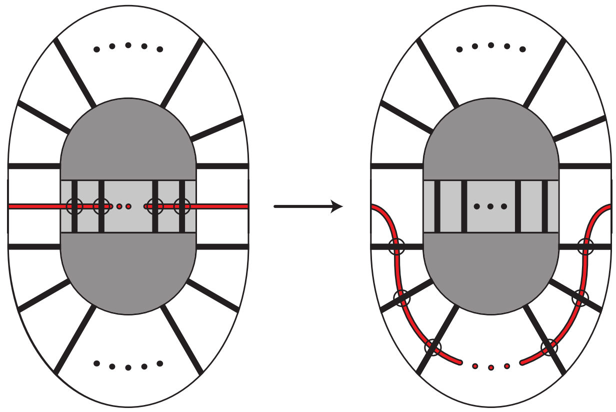

Figure 16. A virtual pass move; the red arc on the left may also contain virtual crossings with itself.

This gives a new diagrammatic perspective on a well-known correspondence [Ka98, KK00, CKS02]:

Correspondence 4.5. *There is a triple bijective correspondence between (i) virtual links, (ii) abstract links, and (iii) stable equivalence classes of links in thickened surfaces. Namely, choose any representative diagram and apply the diagrammatic Correspondence 4.3.*

There is an important caveat in Correspondences 4.3 and 4.5 which is worth noting explicitly. Namely, recall the requirement in part (ii) $\to$ (i) of the proof of Correspondence 4.3 that $\pi|_{\phi(S)}$ must have no critical points. Therefore, $\phi(S)$ has no "back side." If one wishes to construct a virtual link diagram directly from diagram D on a closed surface Σ, however, this is too much to require, any embedding of Σ will have a front and a back: given an embedding $\phi : \Sigma \to \widehat{S^3}$, a regular point of $\pi|_{\phi(\Sigma)}$ lies on the *front* or *back* of $\phi(\Sigma)$ depending on whether an *even* or *odd* number of points of $\phi(\Sigma)$ lie directly above it. The salient point is that one must choose an embedding ϕ of Σ under which all *crossings* of D lie on the front of $\phi(\Sigma)$:

Construction 4.6. Given a link diagram D on a surface Σ, one may construct a corresponding virtual link diagram V directly as follows. Choose any embedding $\phi : \Sigma \to \widehat{S^3}$ such that (i) for each crossing point $c \in D$, $\phi(c)$ lies on the front of Σ and (ii) all self-intersections in $\pi \circ \phi(G)$ are transverse double-points. Then let $V = \pi \circ \phi(D)$, with over-under information matching D.

Remark 4.7. The requirement in Construction 4.6 that all crossings lie on the front of $\phi(\Sigma)$ is necessary; otherwise, different embeddings $\Sigma \to \widehat{S^3}$ may yield distinct virtual links. See Example 4.8.

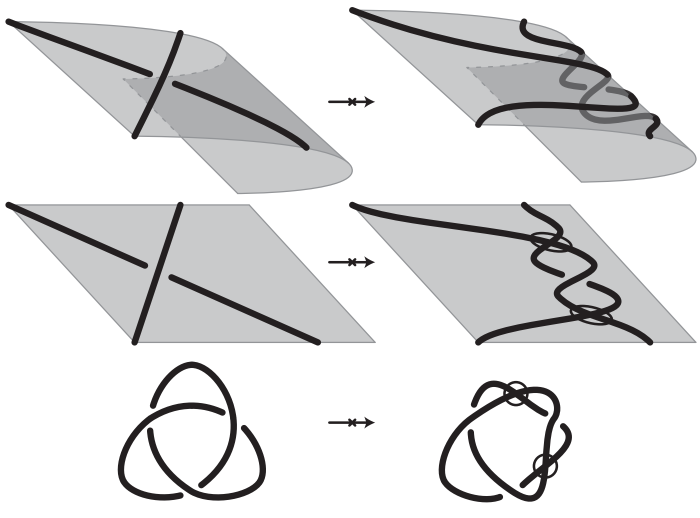

FIGURE 17. Given $D \subset \Sigma$, one obtains $V \subset S^2$ by embedding Σ in S^3 and projecting, but all crossings of D must remain on the front of Σ.

Example 4.8. Let D be a minimal diagram of the RH trefoil on a 2-sphere Σ, and embed Σ in $\widehat{S^3}$ such that the critical locus of $\pi|_\Sigma$ is a circle and D lies entirely on the front of Σ. The corresponding virtual diagram V is also a minimal diagram of the classical RH trefoil. Now isotope D on Σ as shown in Figure 17, so that a crossing passes across the critical circle of $\pi|_\Sigma$. Denote the resulting diagram on Σ by D'. The virtual diagram $V' = \pi \circ \phi(D')$ represents the virtual knot 3.5, which is distinct from the classical RH trefoil [Ka98], even though D and D' are isotopic on Σ.

Interestingly, the virtual knot 3.5 has the same Jones polynomial as the RH trefoil, but the two can be distinguished using the involutory quandle, also called the fundamental quandle. Indeed, by Lemma 5 of [Ka98], the virtual knot 3.5 has the same involutory quandle as the unknot, which is distinct from that of the RH trefoil, since the former is trivial and the latter is not [Jo82].

Correspondence 4.3 yields a new diagrammatic proof of Kuperberg's theorem:

Theorem 4.9 (Theorem 1 of [Ku03])**.** *If (Σ, L) and (Σ', L') are stably equivalent and nonstabilized, then there is a pairwise homeomorphism $(\Sigma \times I, L) \to (\Sigma' \times I, L')$.*

Proof. Choose diagrams $D \subset \Sigma$ and $D' \subset \Sigma'$ of L and L'. Then D and D' are cellularly embedded, because L and L' are nonstabilized. Then, using Construction 4.3, D and D' correspond to diagrams V and V' of the same virtual link. There is a sequence of generalized

Reidemeister moves taking V to V'. By Construction 4.3, only the classical R-moves affect the corresponding diagrams on the closed surfaces (all such diagrams are cellularly embedded by assumption), so D is related by Reidemeister moves to a diagram D'' on Σ, for which there is a pairwise homeomorphism $h : (\Sigma, D'') \to (\Sigma', D')$. □

4.2. **Tait's conjectures for virtual links.**

Definition 4.10. Let V be a virtual link diagram, and let (Σ, D) be the cellularly embedded link diagram corresponding to $[V]$. Say that V is **split** if Σ is connected. Say that V is **prime** (resp. **weakly prime**) if (Σ, D) is prime (resp. weakly prime).

Remark 4.11. This definition of primeness for virtual knot is traditional and is well motivated by Gauss codes [Ka98].[39] Namely, suppose V comes from a Gauss code G. Then V is nonprime if and only if, after some cyclic permutation, G has the form $(a_1, \dots, a_k, b_1, \dots, b_\ell)$ where $b_i \neq -a_j$ for all i, j. The distinction between weak and pairwise primeness is at the heart of the companion paper [Ki23b].

Definition 4.12. A virtual link $\widetilde{L}$ is **nonsplit** (resp. **prime**, **weakly prime**) if the unique nonstabilized representative (Σ, L) of the corresponding stable equivalence class is nonsplit (resp. prime, weakly prime).

Theorem 2.6 [Oz06, BK22, Aetal19, Ki23b] gives the following generalization of Menasco's classical results that a link is split or non-prime if and only if obviously so in a given reduced alternating diagram [Me84]:

Theorem 4.13. *Suppose V is an alternating diagram of a virtual link $\widetilde{L}$.*

- *If V is nonsplit, then $\widetilde{L}$ is nonsplit.*
- *If V is weakly prime, then $\widetilde{L}$ is weakly prime.*
- *If V is prime, then $\widetilde{L}$ is prime.*

The trouble is that, unlike with classical diagrams and diagrams on surfaces, it may be challenging to tell by direct inspection whether or not a given virtual diagram is split, weakly prime, or prime. Also, the converses to the second and third statements are false, because of possible nugatory crossings, which we have yet to address. Theorem 4.15 of [Ki23b] uses lassos to rectify all this, giving necessary and sufficient conditions for an alternating virtual diagram to represent a nonsplit, prime, or weakly prime virtual link. See [Ki23b] for details.

[39] A Gauss code is a permutation of the tuple $(-n, \dots, -1, 1, \dots, n)$, $n \in \mathbb{Z}$. Some Gauss codes describe classical knot diagrams, but *all* Gauss codes describe virtual knot diagrams.

Theorem 4.14. *Any two weakly prime, alternating diagrams of a given virtual link $\widetilde{L}$ are related by virtual Reidemeister moves and (classical) flypes.*

Proof. Let V and V' be two such diagrams, and let (Σ, D) and (Σ', D') be the associated pairs under Correspondence 4.3. By Kuperburg's theorem, we may identify $\Sigma \equiv \Sigma'$, and by Theorem 3.5, there is a sequence of flype moves on Σ taking D to D':

$$D = D_0 \to D_1 \to \cdots \to D_n = D'.$$

We will show for each $i = 1, \ldots, n$ that there are virtual diagrams V_{i-1}^2 and V_i^1 which correspond to (Σ, D_{i-1}) and (Σ, D_i) and which are related by a flype. This will produce a sequence of virtual diagrams

$$V = V_0^1 \to V_0^2 \to V_1^1 \to V_1^2 \to \cdots \to\to V_n^1 \to V_n^2 = V'$$

where each $V_i^1 \to V_i^2$ comes from a sequence of virtual R-moves and each $V_{i-1}^2 \to V_i^1$ comes from a flype.

Consider a flype $D_{i-1} \to D_i$; it is supported within a disk $X \subset \Sigma$.[40] Denote the quotient map $q : \Sigma \to \Sigma/X \equiv \Sigma$, and denote the underlying graph of D_{i-1} by G. Choose a spanning tree T for the 4-valent graph $q(G) \subset \Sigma/X$, and take a regular neighborhood νT. Denote $U = q^{-1}(\nu T)$, and observe that U is a disk in Σ that contains X and all crossings of D_{i-1}.[41]

Choose an embedding $\phi : \Sigma \to \widehat{S^3}$ such that $\pi|_{\phi(U)}$ has no critical points and $\pi \circ \phi(U) \cap \pi \circ \phi(D \setminus U) = \varnothing$. Denote $f = \pi \circ \phi$ and $f(D_{i-1}) = V_{i-1}^2$. Observe that $f|_X$ is a homeomorphism onto its image, and so the disk $f(X)$ supports a flype $V_{i-1}^2 \to V_i^1$ where V_i^1 corresponds to (Σ, D_i).

Thus, as needed, each $V_{i-1}^2 \to V_i^1$ comes from a flype. To complete the proof, we note that each $V_i^1 \to V_i^2$ comes from a sequence of virtual R-moves, due to Correspondence 4.3, since both V_i^1 and V_i^2 correspond to the same cellularly embedded diagram D_i on Σ. □

Since crossing number and writhe are invariant under flypes, we can also extend more parts of Tait's conjectures to virtual links:

Theorem 4.15. *All weakly prime, alternating diagrams of a given virtual link have the same crossing number and writhe.*

Finally, as an additional corollary, we observe that, whereas connect sum is a well-defined operation for classical knots and for any classical knot with any virtual knot, connect sum is *not* a well-defined operation for virtual knots:

[40] That is, take X to be the oval-shaped disk shown left in Figure 1.
[41] Thus, U is a lasso for (Σ, D_{i-1}).

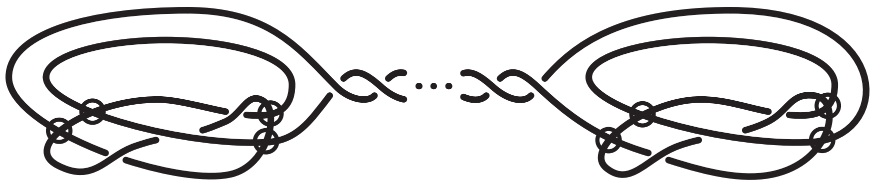

FIGURE 18. There are infinitely many different ways to take the connect sum of any two non-classical alternating knots.

Corollary 4.16. *Given any two non-classical, weakly prime, alternating virtual links V_1 and V_2, there are infinitely many distinct virtual links that decompose as a connect sum of V_1 and V_2.*

This follows immediately from Theorem 4.15, using the construction suggested in Figure 18. We conjecture that the same construction works more generally:

Conjecture 4.17. *Given any two non-classical, weakly prime virtual links V_1 and V_2, there are infinitely many distinct virtual links that decompose as a connect sum of V_1 and V_2.*

APPENDIX A: CROSS-REFERENCING WITH [Ki23]

here	in [Ki23]	here	in [Ki23]
Prop. 2.9	Prop. 2.5	Prop. 2.10	Prop. 2.6
Obs. 2.11	Fact 2.7	Rem. 2.12	Rem. 2.8
Def. 2.13	Def. 2.9	Obs. 2.14	Obs. 2.10
Proc. 2.22	Proc. 2.23	Proc. 2.26	Proc. 2.24
Fact 2.27	Subl. 6.3	Fact 2.28	Subl. 6.4
Fact 2.29	Prop. 6.5	Prop. 2.32	Prop. 6.8
Prop. 2.33	Prop. 6.9	Thm. 2.35	Thm. 2.35
Cor. 2.36	Cor. 2.36	Thm. 3.1	Thm. 4.5
Cor. 3.2	Cor. 4.6	Thm. 3.4	Thm. 5.4

TABLE 1. Cross-listing information with [Ki23]

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