WEAK VERSUS PAIRWISE PRIMENESS FOR VIRTUAL LINKS AND THEIR DIAGRAMS

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Abstract. The usual notion of a "prime" virtual link does not correspond to the most common notion ("weakly prime") of a prime link in a thickened surface. We compare these two notions and discuss related issues (split, stabilized, etc.). Our main result is that an alternating virtual diagram V represents a prime (resp. weakly prime, nonsplit) virtual link unless V is "obviously" nonprime (resp. not weakly prime, split). These results generalize classical results due to Menasco. We also introduce the notion of "lassos" for virtual link diagrams and describe how to use lassos to determine by inspection whether or not an alternating virtual diagram is weakly prime, pairwise prime, or nonsplit.

1. Introduction

A Gauss code is a permutation of the tuple $(-n, \ldots, -1, 1, \ldots, n)$, $n \in \mathbb{Z}$. Some Gauss codes, but not all, describe classical knot diagrams. What about the other Gauss codes? Indeed, this line of inquiry led to Kauffman's discovery of *virtual knots* [Ka98]. Moreover, these objects, although defined abstractly, have tangible meaning:[1] by work of Kauffman [Ka98], Kamada–Kamada [KK00], and Carter–Kamada–Saito [CKS02], there is a triple bijective correspondence between (i) virtual links, (ii) abstract links (see below), and (iii) stable equivalence classes of links in thickened surfaces.[2] There is also a correspondence between the associated *diagrams* [Ki22a]. We review these correspondences in §2.1.

A virtual link K is *prime* if, for every diagram V of K and every diagrammatic connect sum decomposition $V = V_1 \# V_2$, either V_1 or V_2 represents the trivial knot. Yet, the corresponding notion for a link L in a thickened surface $\Sigma \times I$ is not the one that typically appears in the literature, much of which addresses the geometry of the link exterior $E = \Sigma \times I \setminus \mathring{\nu} L$ and thus is chiefly interested in whether

[1] There is an obvious historical parallel between the advent of virtual knots and that of complex numbers.

[2] Thus, virtual links pertain more generally to the study of links L in arbitrary 3-manifolds M for which there exists a thickened surface $\Sigma \times I \subset M$ with $L \subset \Sigma \times I$. See [HP20, PT22].

or not E contains an essential torus. Thus, $L \subset \Sigma \times I$ is sometimes considered prime if, for any pairwise connect sum decomposition $(\Sigma \times I, L) = (\Sigma \times I, L_1)\#(S^3, L_2)$, L_2 is the unknot. Following Howie–Purcell, we call such L *weakly prime*. See §2.2 for the definitions of primeness and weak primeness for virtual links and for links in thickened surfaces.[3]

In the classical setting, Menasco proved that a reduced alternating diagram D represents a prime (resp. nonsplit) link if and only if D is diagrammatically prime (resp. connected) [Me84]. Our first main result extends this fact to alternating diagrams on closed orientable surfaces of arbitrary genus:

Theorem 1.1. *Suppose $D \subset \Sigma$ is a fully alternating diagram of a link $L \subset \Sigma \times I$. Then L is (i) nullhomologous over $\mathbb{Z}/2$ and (ii) nonsplit. Moreover, (iii) if (Σ, D) is prime, then (Σ, L) is prime.*

Parts (i) and (ii) are proven by Ozawa in [Oz06] and by Boden-Karimi in [BK20]. We use Menasco's crossing ball technique to give an alternate proof of (ii), as the argument adapts nicely to give a proof of (iii). Adams et al prove a result similar to (iii) in [Aetal19]: if D is weakly prime on Σ, then there is no nontrivial pairwise connect sum decomposition $(\Sigma \times I, L) = (\Sigma \times I, L_1)\#(S^3, L_2)$.

We then turn our attention to virtual link diagrams, where an analogous result follows from the correspondences described above. This result is not immediately practical, however, for the following reason.

Menasco's classical result is sometimes stated as follows: an alternating link is composite (resp. split) if and only if it is *obviously* composite (resp. split) in a given alternating diagram. Theorem 3.2 can be stated in the same manner. Yet, for virtual diagrams, "obvious" feels inaccurate. What does it mean for an alternating virtual diagram V to be "obviously" composite? Certainly, if V decomposes as a diagrammatic connect sum of two nontrivial links, then it is obviously composite; but this is too restrictive (the theorem is untrue with such a strong requirement). To remedy this, i.e. to describe when V is "obviously composite," we introduce *lassos* in §4.

Given a link diagram D on a surface Σ, a lasso is a disk $X \subset \Sigma$ that contains all crossings of D. Similarly, given a virtual diagram $V \subset S^2$, a lasso is a disk $X \subset S^2$ that contains all classical crossings of V and no virtual ones. In both contexts, a lasso X is *acceptable* if the part of the diagram in X is connected and the part of the diagram outside X does not admit an "obvious" simplification. (See §4 for the precise definition.) We show that:

[3]For our purposes, abstract links are useful mainly as an intermediate stage in the correspondence between virtual links and (stable equivalence classes of) links in thickened surfaces.

Proposition 1.2. *An alternating virtual diagram V represents a nonsplit virtual link if and only if there is a diagram V' related to V by virtual (non-classical) Reidemeister moves which admits an acceptable lasso.*

Moreover, still assuming that V is alternating, we describe how to determine from V' and X (by inspection) whether or not the associated virtual link is prime (resp. weakly prime, nonsplit):

Theorem 1.3. *Suppose an alternating virtual diagram V admits an acceptable lasso X. Then (V is connected and) the virtual link represented by V is (nonsplit and):*

(i) pairwise prime if and only if, for every disk $Z \subset S^2$ whose boundary intersects V in two generic points, both in X, all classical crossings of V lie on the same side of ∂Z; and

(ii) weakly prime if and only if, for every disk $Z \subset X$ that intersects V generically and contains at least one crossing, $|\partial Z \cap L| > 2$.

Finally, in §**??**, we introduce *lasso numbers* of virtual links and their diagrams. We establish some basic properties. Computing these invariants seems like a challenging, but approachable, problem.

2. Background

2.1. **Correspondences.** A *virtual link diagram* is the image of an immersion $\bigsqcup S^1 \to S^2$ in which all self-intersections are transverse double-points, some of which are labeled with over-under information. These labeled points are called *classical crossings*, and the other double-points are called *virtual crossings*. Traditionally, virtual crossings are marked with a circle, as in Figure 1. A *virtual link* is an equivalence class of such diagrams under generalized Reidemeister moves, as shown in Figure 1. There are seven types of such moves, the three *classical* moves and four *virtual* moves.[4]

Notation 2.1. Given a virtual link diagram $V \subset S^2$, let $[V]$ denote the set of all virtual diagrams related to V by planar isotopy and virtual Reidemeister moves.

An **abstract link diagram** (S, G) consists of a 4-valent graph G embedded in a compact orientable surface S, such that G has over-under information at each vertex, and G is a deformation retract of S. An **abstract link** is an equivalence class of such diagrams under the following equivalence relation $\sim$: $(S_1, G_1) \sim (S_2, G_2)$ if there are embeddings $\phi_i : S_i \to S$, $i = 1, 2$, into a surface S, such that $\phi_1(G_1)$ and $\phi_2(G_2)$ are related by classical Reidemeister moves on S.

[4]The move involving two virtual crossings and one classical crossing is sometimes called a *mixed* move, but we include it as a virtual (non-classical) move.

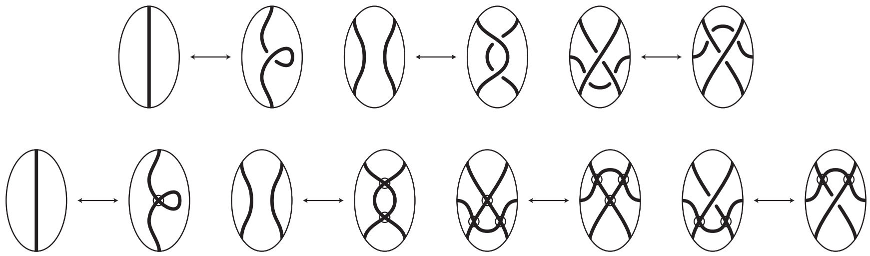

Figure 1. Classical (top) and virtual (bottom) Reidemeister moves

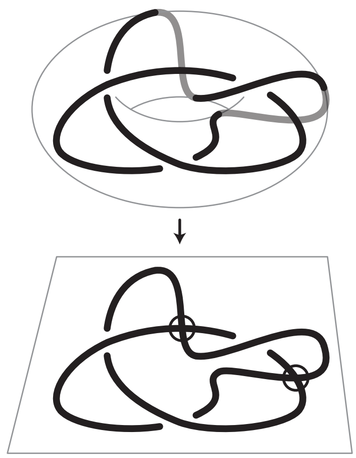

Figure 2. A link diagram on the torus and a corresponding virtual diagram

Notation 2.2. Throughout, Σ is a closed orientable surface, not necessarily connected or of positive genus. We denote the intervals $[-1,1]$ and $[0,1]$ by I and I_+, respectively. In $\Sigma \times I$, we identify Σ with $\Sigma \times \{0\}$, and we denote $\Sigma \times \{\pm 1\} = \Sigma_\pm$. We reserve the notations L and D for links and diagrams as follows: for a pair (Σ, L), L is a link in $\Sigma \times I$ which intersects each component of $\Sigma \times I$, and for a pair (Σ, D), D is a link diagram on Σ which intersects each component of $\Sigma \times I$.

The diagrammatic correspondence works as follows (see [Ki22a] for a proof):

Correspondence 2.3. *There is a triple bijective correspondence between (i) equivalence classes $[V]$ of virtual link diagrams, (ii) abstract link diagrams, and (iii) cellularly embedded link diagrams on closed surfaces. Namely:*

(i) → (ii) *Given an equivalence class $[V]$ of virtual link diagrams, choose a representative diagram $V \subset S^2$, and construct an abstract*

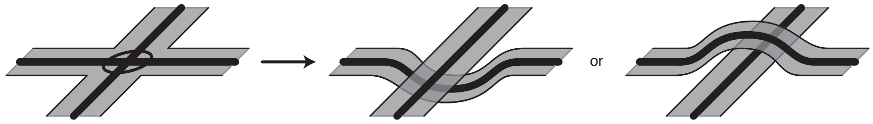

FIGURE 3. Converting the neighborhood of a virtual link diagram to an abstract link diagram

link diagram as follows. First, take a regular neighborhood νV of V in S^2. Second, near each virtual crossing of V, modify νV as shown in Figure 3. Third, view the resulting pair abstractly, forgetting the embedding in S^3.

(ii) → (i) *Given an abstract link diagram (S, G), view $S^3 = (S^2 \times \mathbb{R}) \cup \{\pm\infty\}$, denote $\widehat{S^3} = S^3 \setminus \{\pm\infty\}$ with projection $\pi : \widehat{S^3} \to S^2$, and choose any embedding $\phi : S \to \widehat{S^3}$ such that $\pi|_{\phi(S)}$ has no critical points and all self-intersections in $\pi \circ \phi(G)$ are transverse double-points with neighborhoods as suggested in Figure 3. Now take the 4-valent graph $\pi \circ \phi(G) \subset S^2$, and, for each crossing point c of G, label the double-point $\pi \circ \phi(c)$ with the matching over-under information. (Thus, the double points of V coming from the crossings of D comprise the classical crossings of V, and the remaining double-points comprise the virtual crossings.) See Figure 2.*

(ii) → (iii) *Given an abstract link diagram (S, G), cap off each component of ∂S with a disk to obtain a cellularly embedded link diagram on a closed surface.*

(iii) → (ii) *Given any pair (Σ, D), the pair $(\nu D, D)$ is the associated abstract link diagram.*

This gives a new diagrammatic perspective on a well-known correspondence [Ka98, KK00, **?**]:

Correspondence 2.4. *There is a triple bijective correspondence between (i) virtual links, (ii) abstract links, and (iii) stable equivalence classes of links in thickened surfaces. Namely, choose any representative diagram and apply the diagrammatic Correspondence 2.3.*

There is an important caveat in Correspondences 2.3 and 2.4 which is worth noting explicitly. Namely, recall the requirement in part (ii) → (i) of the proof of Correspondence 2.3 that $\pi \circ \phi|_S$ must have no critical points. Therefore, $\phi(S)$ has no "back side." If one wishes to construct a virtual link diagram directly from diagram D on a closed surface Σ, however, this is too much to require, any embedding of Σ will have a front and a back: given an embedding $\phi : \Sigma \to \widehat{S^3}$, a regular point of $\pi|_{\phi(\Sigma)}$ lies on the *front* or *back* of $\phi(\Sigma)$ depending on whether an *even* or *odd* number of points of $\phi(\Sigma)$ lie directly above it. The salient point is that one must choose an embedding ϕ of Σ under which all *crossings* of D lie on the front of $\phi(\Sigma)$:

Construction 2.5. Given a link diagram D on a surface Σ, one may construct a corresponding virtual link diagram V directly as follows. Choose any embedding $\phi : \Sigma \to \widehat{S^3}$ such that (i) for each crossing point $c \in D$, $\phi(c)$ lies on the front of Σ and (ii) all self-intersections in $\pi \circ \phi(G)$ are transverse double-points. Then let $V = \pi \circ \phi(D)$, with over-under information matching D.

Remark. The requirement in Construction 2.5 that all crossings lie on the front of $\phi(\Sigma)$ is necessary. Without this requirement, different embeddings may produce distinct virtual links. See Example 2.6.

Example 2.6. Suppose Σ is a 2-sphere and D is a minimal diagram of the RH trefoil. Embed Σ in $\widehat{S^3}$ such that the critical locus of $\pi|_\Sigma$ is a circle and D lies entirely on the front of Σ. The corresponding virtual diagram V is also a minimal diagram of the classical RH trefoil. Now isotope D on Σ so that a crossing passes across the critical circle of $\pi|_\Sigma$, as shown locally in Figure 4, top. The effect on the virtual diagram, shown locally in Figure 4, center, produces a new virtual diagram $D_{v'}$. As shown in Figure 4, bottom, this diagram $D_{v'}$ represents the virtual knot 3.5, which is distinct from the classical RH trefoil [Ka98]. Interestingly, the virtual knot 3.5 has the same Jones polynomial as the RH trefoil, but the two can be distinguished using the involutory quandle, also called the fundamental quandle. Indeed, by Lemma 5 of [Ka98], the virtual knot 3.5 has the same involutory quandle as the unknot, which is distinct from that of the RH trefoil, since the former is trivial and the latter is not [Jo82].

The pair (Σ, L) is **stabilized** if, for some essential circle[5] $\gamma \subset \Sigma$, L can be isotoped to be disjoint from the annulus $\gamma \times I$; one can then *destabilize* the pair (Σ, L) by cutting $\Sigma \times I$ along $\gamma \times I$ and attaching two 3-dimensional 2-handles in the natural way; the reverse operation is called *stabilization*. Equivalently, (Σ, L) is *nonstabilized* if every diagram D of L on Σ is **cellularly embedded**, meaning that D cuts Σ into disks. Kuperberg's Theorem states that the stable equivalence class of (Σ, L) contains a unique nonstabilized representative:

Theorem 1 of [Ku03]**.** *If (Σ, L) and $(\Sigma' \times I, L')$ are stably equivalent and nonstabilized, then there is a pairwise homeomorphism $(\Sigma \times I, L) \to (\Sigma' \times I, L')$.*

Correspondence 2.3 yields a new diagrammatic proof of Kuperberg's theorem (see [Ki22a]):

[5] We use "circle" as shorthand for "simple closed curve."

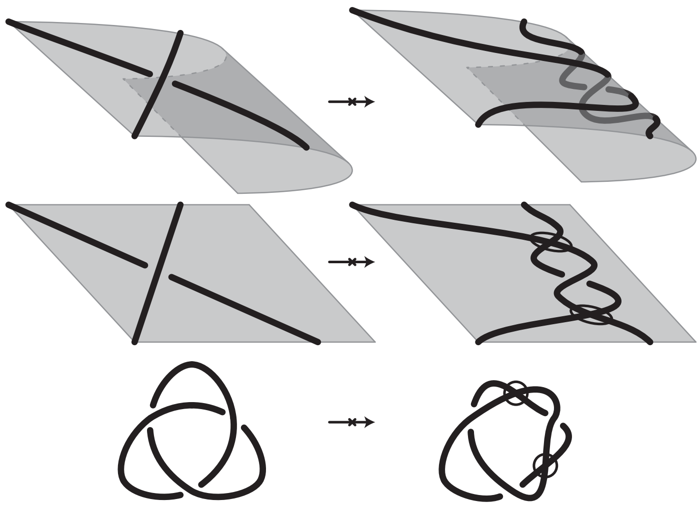

FIGURE 4. Given $D \subset \Sigma$, one obtains $V \subset S^2$ by embedding Σ in S^3 and projecting, but all crossings of D must remain on the front of Σ.

Theorem 2.7 (Theorem 1 of [Ku03])**.** *If (Σ, L) and (Σ', L') are stably equivalent and nonstabilized, then there is a pairwise homeomorphism $(\Sigma \times I, L) \to (\Sigma' \times I, L')$.*

2.2. **Primeness and related notions.** Kuperburg's theorem implies, in particular, that when (Σ, L) is nonsplit, meaning that every diagram of L on Σ is connected, (Σ, L) is nonstabilized if and only if Σ has *minimal genus* in this stable equivalence class. Note that if (Σ, L) is nonstabilized, then it is also nonsplit. The converse is false: if (Σ, L) is split, then so is any stabilization of it, as is any destabilization which fixes $|\Sigma|$.[6]

If L is nonsplit and $g(\Sigma) = 0$, then $(\Sigma \times I) \setminus L$ is irreducible, as $\Sigma \times I$ is always irreducible, since its universal cover is $\mathbb{R}^2 \times \mathbb{R}$.[7] The converse of this, too, is false, [8] due to the next observation, which follows from a standard innermost circle argument:

Observation 2.8. *If $(\Sigma_i \times I) \setminus L_i$ is irreducible for $i = 1, 2$ and $\Sigma = \Sigma_1 \#_\gamma \Sigma_2$ with $L = L_1 \sqcup L_2 \subset \Sigma \times I$, where the annulus $A = \gamma \times I$ separates L_1 from L_2 in $\Sigma \times I$, then $(\Sigma \times I) \setminus L$ is irreducible.*

[6] Throughout, $|X|$ denotes the number of connected components of X.

[7] For more detail, see the proof of Proposition 12 of [BK20], which cites [CSW14].

[8] If $(\Sigma_i \times I, L_i)$ is nonsplit for $i = 1, 2$, then choose disks $X_i \subset \Sigma_i$ with $(X_i \times I) \cap L_i = \varnothing$ and construct the connect sum $\Sigma = (\Sigma_1 \setminus \text{int}(X_1)) \cup (\Sigma_2 \setminus \text{int}(X_2)) = \Sigma_1 \# \Sigma_2$. Let $L = L_1 \sqcup L_2 \subset \Sigma \times I$. Then L is split. Yet, $\Sigma \times I$ is irreducible by Observation 2.8.

A diagram $D \subset \Sigma$ is **fully alternating** on Σ if D is alternating and cellularly embedded [Aetal19]. We will use the following result of Boden–Karimi and the subsequent generalization:

Corollary 3.6 of [BK20]**.** *Corollary 3.6 of* [BK20] *If Σ is connected and (Σ, L) is represented by a fully alternating diagram, then (Σ, L) is nonsplit and nonstabilized.*

Corollary 2.9. *Suppose (Σ, L) is represented by a fully alternating diagram. Then (Σ, L) is nonsplit if and only if Σ is connected. Either way, (Σ, L) is nonstabilized.*

Corollary 2.10. *Suppose (Σ, L) is represented by an alternating diagram D. Then (Σ, L) is nonsplit if and only if D is connected.*

We call (Σ, D) (pairwise) **prime** if Σ is connected and every separating curve intersecting D in two generic points also bounds a disk in Σ which contains no crossings of D, or equivalently if any pairwise connect sum decomposition $(\Sigma, D) = (\Sigma_1, D_1)\#(\Sigma_2, D_2)$ has $(\Sigma_i, D_i) = (S^2, \bigcirc)$ for either $i = 1, 2$. Note that if (Σ, D) is prime, then D is connected, unless $(\Sigma, D) = (S^2, \bigcirc\ \bigcirc)$.

Two other notions of primeness for D on Σ appear in the literature; one is more restrictive than our *pairwise* notion for (Σ, D), the other less restrictive. Removing "separating" in the first version of the definition above gives the (more restrictive) notion of *strong primeness* introduced by Ozawa in [Oz06]. Following Howie–Purcell, we call (Σ, D) **weakly prime** if any pairwise connect sum decomposition $(\Sigma, D) = (\Sigma_1, D_1)\#(S^2, D_2)$ has $D_2 = \bigcirc$ [HP20].[9][10]

We say that (Σ, L) is (pairwise) **prime** if Σ is connected and, whenever $\gamma \subset \Sigma$ is a separating curve and L is isotoped to intersect the annulus $\gamma \times I$ in two points, then γ bounds a disk $X \subset \Sigma$ and L intersects $X \times I$ in a single unknotted arc. Equivalently, (Σ, L) is non-prime if there is a diagram D of L such that $(\Sigma, D) = (\Sigma_1, D_1)\#(\Sigma_2, D_2)$, where neither D_i is a diagram of the unknot on S^2. If there is such a diagram D, we write $(\Sigma, L) = (\Sigma_1, L_1)\#(\Sigma_2, L_2)$, where L_i is the link in $\Sigma_i \times I$ represented by D_i; we may also specify $(\Sigma, L) = (\Sigma_1, L_1)\#_\gamma(\Sigma_2, L_2)$, where $\Sigma \setminus \setminus \gamma = (\Sigma_1 \setminus \setminus (\text{disk})) \sqcup (\Sigma_2 \setminus \setminus (\text{disk}))$. This operation is sometimes called an *annular connect sum.*

If (Σ, L) is (pairwise) prime, then L is prime in $\Sigma \times I$, in the sense of Definition 2 of [Aetal19]: there is no nontrivial pairwise connect sum decomposition $(\Sigma \times I, L) = (\Sigma \times I, L_1)\#(S^3 \times L_2)$. The converse is false.[11] Note that whenever (Σ, L) is prime, it is also nonsplit, hence nonstabilized.

[9] Their definition also allows Σ to be disconnected and for its components to have any genus.

[10] Adams et al use the term *obviously prime* for the same condition [Aetal19].

[11] For example, for $i = 1, 2$, suppose $(\Sigma_i \times I, L_i)$ is prime and Σ_i has positive genus, and take a disk $X_i \subset \Sigma_i$ such that $(X_i \times I)$ intersects L_i in a single

3. Weak and pairwise primeness for links in thickened surfaces

In this section, we establish our main result regarding (pairwise) prime alternating links in thickened surfaces. To prepare, we need:

Proposition 3.1. *Suppose* $(\Sigma, L) = (\Sigma_1, L_1)\#_\gamma(\Sigma_2, L_2)$. *If both* (Σ_i, L_i) *are nonsplit, then* (Σ, L) *is also nonsplit.*

Proof of Proposition 3.1. Note that Σ_1 and Σ_2 are connected, so Σ is too. In the case where, say $\Sigma_2 = S^2$, the proposition follows from a more general fact: if L_1 is a link in a 3-manifold M and $M \setminus L_1$ is irreducible, and if L_2 is nonsplit in S^3, then the complement of $L_1\#L_2$ in $M = M\#S^3$ is also irreducible. The proof is a standard innermost circle argument. We may thus assume that both Σ_i have positive genus, hence that both $(\Sigma_i \times I) \setminus L_i$ are irreducible. A standard innermost circle argument shows that $(\Sigma \times I) \setminus L$ is irreducible.

Suppose that (Σ, L) is split. Then there is a system A of annuli, each with one boundary component on each of $\Sigma_\pm$, such that $(\Sigma \times I) \setminus A$ has two components, both of which intersect L. Among all choices for such A, choose one which lexicographically minimizes $(|A|, |A \pitchfork U|)$,[12] where U is the annulus $\gamma \times I$.

Note that $\Sigma_\pm \setminus \partial A$ each have two components, neither of which is a disk due to the irreducibility of $(\Sigma \times I) \setminus L$, and that A must intersect U, since both $(\Sigma_i \times I, L)$ are nonsplit. The minimality of $|A \pitchfork U|$ further implies that $A \cap U$ consists only of circles, each of which is parallel in $A \subset \Sigma \times I$ to ∂A. The fact that ∂A contains only *essential* circles in $\Sigma_\pm$ thus implies that $A \cap U$ consists only of circles parallel in U to γ, and thus (using the minimality of $|A|$) that A has a single component, which is parallel in $\Sigma \times I$ to U. This contradicts the assumptions that both $(\Sigma_i \times I, L)$ are nonsplit and $|A \pitchfork U|$ is minimized.[13] $\square$

As proven by Menasco in the classical case [Me84], certain diagrammatic conditions constrain an alternating link L as one might wish. This is our first main result:

unknotted arc. Construct the pairwise connect sum $(\Sigma, L) = (\Sigma_1, L_1)\#(\Sigma_2, L_2)$ along these (thickened) disks X_i. Then L is prime in Σ, in the sense of [Aetal19], but (Σ, L) is not (pairwise) prime.

[12] $|A|$ denotes the number of connected components of A. Likewise throughout.

[13] Namely, choose a component A_0 of A whose boundary consists of a circle of $A \cap U$ and a component of ∂A. Then one of the two components U_0 of $U \setminus u$ is disjoint from L. Surger A along U_0 to obtain two annuli, $A_1, A_2 \subset (\Sigma \times I) \setminus L$, where A_1 is parallel in $\Sigma \times I$ to A (and U), and A_2 is parallel in $\Sigma \times I$ to $\Sigma_\pm$. Then $|A_1 \pitchfork U| < |A \pitchfork U|$, so A_1 must not separate L, but this implies that L must intersect the solid torus through which A_2 is parallel to $\Sigma_\pm$, contrary to the assumption that both $(\Sigma_i \times I, L_i)$ are non-split.

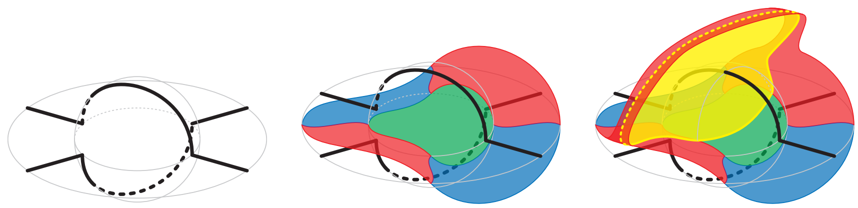

FIGURE 5. A crossing bubble, a saddle, and the disk X in the proof of Theorem 3.2.

Theorem 3.2. *Suppose $D \subset \Sigma$ is a fully alternating diagram of a link $L \subset \Sigma \times I$. Then L is (i) nullhomologous over $\mathbb{Z}/2$ and (ii) nonsplit. Moreover, (iii) if (Σ, D) is prime, then (Σ, L) is prime.*

Proof of Theorem 3.2. Any fully alternating diagram D is checkerboard colorable,[14][15] so L bounds checkerboard surfaces[16] and thus is nullhomologous over $\mathbb{Z}/2$. This confirms (i).

Assume for the rest of the proof that (Σ, D) is prime. Indeed, this is a hypothesis of (iii), and by Proposition 3.1, it will suffice to prove (ii) under this additional assumption.

Implement the crossing ball setup a la Menasco by inserting a tiny ball C_i centered at each crossing point of D and pushing the two strands of D near that crossing point onto opposite hemispheres of that ball's boundary. Denoting $C = \bigcup_i C_i$, this gives an embedding of L in $(\Sigma \setminus \text{int}(C)) \cup \partial C$. See Figure 5, left. Denote the two components of $(\Sigma \times I) \setminus\setminus(\Sigma \cup C)$ by H_+ and H_-.[17]

Assume contrary to (ii) that (Σ, L) is split. Then there is a system $A \subset (\Sigma \times I) \setminus L$ of disjoint, properly embedded annuli, each with one boundary component in Σ_+ and one in Σ_-, which cuts $\Sigma \times I$ into two pieces, both of which intersect L. Of all choices for such A, choose one which lexicographically minimizes $(|A \cap C|, |A \setminus (\Sigma \cup C)|)$, provided $A \pitchfork \partial C$ and $A \pitchfork \Sigma \setminus \text{int}(C)$.

Then, since D is cellularly embedded, each component of A must intersect C, and, by a standard argument, each component of $A \cap C$

[14]This means that one can color the disks of $\Sigma \setminus\setminus D$ black and white so that regions of the same color abut only at crossings.

[15]Since D is fully alternating, it is possible to orient each disk of $\Sigma \setminus D$ so that, under the resulting boundary orientation, over-strands are oriented toward crossings and under-strands away from crossings. Since Σ is orientable, these orientations determine the desired coloring. Interestingly, fully alternating link diagrams on nonorientable surfaces are never checkerboard colorable.

[16]One can use a checkerboard coloring to construct *checkerboard surfaces* B and W for L, where B projects into the black regions, W projects into the white, and B and W intersect in *vertical arcs* which project to the the crossings of D.

[17]$X \setminus\setminus Y$ denotes "X cut along Y," which is the metric closure of $X \setminus Y$. For more detail, see footnote 7 of [Ki20].

FIGURE 6. Part of the circle γ in the proof of Part (ii) of Theorem 3.2. Arrows point inward. The shaded areas may contain more of γ.

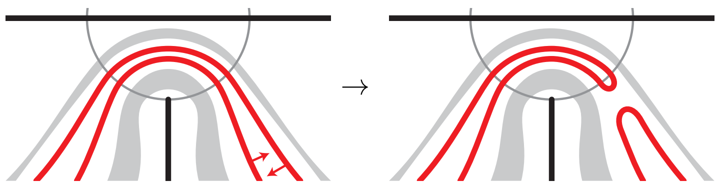

FIGURE 7. The first part of the simplifying move in the proof of Part (ii) of Theorem 3.2

is a saddle, as in Figure 5, center. Also, by minimality, no arc of $A \cap \Sigma \setminus C$ is parallel in $\Sigma \setminus C$ to ∂C. Some component V of $A \setminus\setminus (\Sigma \cup C)$ must be a disk, due to euler characteristic considerations. Assume without loss of generality that $V \subset H_+$. Denote $\gamma = \partial V \subset \partial H_+$. Then γ bounds a disk U in ∂H_+; by passing to an innermost circle in this disk, we may assume that the interior of U is disjoint from A.

We claim that there is an arc δ_0 of $\gamma \cap \partial C$ that is parallel in $\partial C \cap U \setminus\setminus \gamma$ to an overpass. To see this, choose any arc δ of $\gamma \cap \partial C$ that is parallel in $\partial C \cap \partial H_+ \setminus \gamma$ to an overpass. If the overpass lies inside U, take $\delta = \delta_0$. Otherwise, take ε to be either arc of $\gamma \setminus \partial C$ incident to δ, and let δ' be the other arc of $\gamma \cap \partial C$ incident to ε. See Figure 6. If δ' is parallel in $\partial C \cap \partial H_+ \setminus \gamma$ to an overpass, then this overpass lies inside U (because D is alternating), giving $\delta' = \delta_0$. Otherwise, δ' is parallel in $\partial C \cap U$ to an arc δ'' of $\gamma \cap \partial C$, but this allows an isotopy that decreases $|A \pitchfork C|$, contrary to assumption. See Figure 7 (and bigon moves in §**??**).

Let S_0 denote the saddle in C_0 incident to δ_0. Then $S_0 \cap \partial S_+$ consists of δ_0 and another arc, δ_1, which must also lie on γ. Choose points $x_i \in \delta_i$, $i = 1, 2$, and construct arcs $\alpha \subset V$ and $\beta \subset S_i$ joining x_1 and x_2. Then, as shown in Figure 5, right, $\alpha \cup \beta$ is a circle in A which bounds a disk $X \subset A$ with $|X \pitchfork L| = 1$, which is impossible.

The proof of (iii) is nearly the same, but taking A to be a single annulus, again with one boundary component in each $\Sigma_\pm$, which

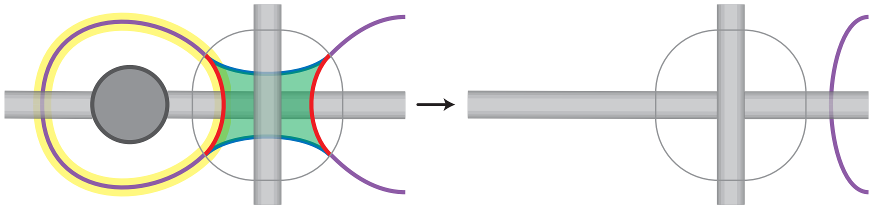

FIGURE 8. The simplifying move in the proof of part (iii) of Theorem 3.2. The highlighted circle is γ.

separates $\Sigma \times I$ and intersects L in exactly two points; of all possibilities for such A, choose one which lexicographically minimizes $(|A \cap C|, |A \setminus (\Sigma \cup C)|)$, provided $A \pitchfork \partial C$ and $A \pitchfork \Sigma \setminus \text{int}(C)$. Proceeding as before, and noting that A must intersect C since (Σ, D) is prime, obtain a circle $\gamma \subset \partial H_+$ which bounds disks $V \subset A \cap H_+$ and $U \subset \partial H_+$, the latter with interior disjoint from A. If there is an arc δ_0 of $\gamma \cap \partial C$ that is parallel in $\partial C \cap U \setminus \gamma$ to an overpass, then we again get the disk $X \subset A$ with $|X \pitchfork L| = 1$, which now contradicts the minimality of $|A \cap C|$.

Assume instead that no such δ_0 exists. Choose any crossing ball C_0 that γ intersects. Then there is an arc δ of $\gamma \cap \partial C_0$ that is parallel in $\partial C_0 \cap \partial H_+ \setminus \gamma$ to the overpass at C_0, which must therefore lie outside U. Denote the arcs of $\gamma \setminus \partial C$ incident to δ by ε_1 and ε_2. Both ε_i must intersect L, by the arguments from (ii). But $|\gamma \cap L| \leq |A \cap L| = 2$, so γ intersects at most two crossing balls and therefore appears as in Figure 8, left. As shown, the fact that D is prime now gives an isotopy which decreases $|A \cap C|$, contrary to assumption. $\square$

4. Lassos

We now turn our attention to virtual links and their diagrams.

4.1. Weak and pairwise primeness for virtual links.

Definition 4.1. Let V be a virtual diagram, and let (Σ, D) be the cellularly embedded diagram corresponding to $[V]$. Say that V is **split** if Σ is connected. Say that V is **prime** (resp. **weakly prime**) if (Σ, D) is pairwise prime (resp. weakly prime).

Remark. This definition of primeness for virtual knot is traditional and is well motivated by Gauss codes [Ka98]. Namely, suppose V comes from a (cyclic) Gauss code G. Then V is nonprime if and only if G has the form $w_1 w_2$, where w_1 and w_2 are nonempty words with no letters in common.

Proposition 4.2. *Whereas connected sum is a well-defined operation for classical knots and for any classical knot with any virtual*

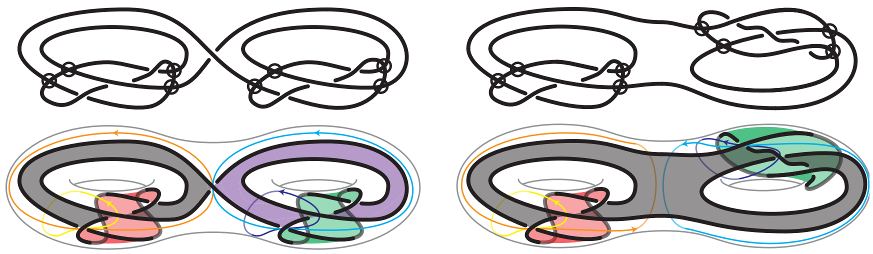

FIGURE 9. Connected sum is not a well-defined operation on virtual knots: the knot on the left is distinct from the knot on the right.

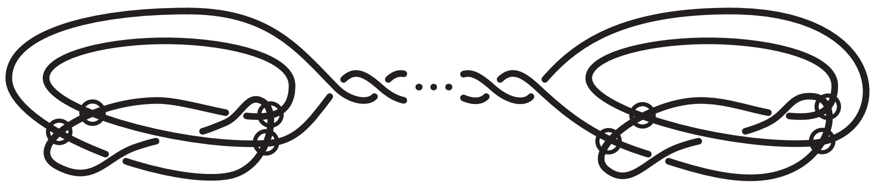

FIGURE 10. There are infinitely many different ways to take the connect sum of any two non-classical alternating knots.

knot, connected sum is not a well-defined operation for virtual knots. In particular, given any two non-classical, weakly prime, alternating virtual links V_1 and V_2, there are infinitely many distinct virtual links that decompose as a connected sum of V_1 and V_2.

This fact follows easily from the flyping theorem for virtual links [Ki22a]. See Figure 10.

We conjecture that the same construction works in the non-alternating case:

Conjecture 4.3. *Given any two non-classical, weakly prime virtual links V_1 and V_2, there are infinitely many distinct virtual links that decompose as a connected sum of V_1 and V_2.*

Definition 4.4. A virtual link $\widetilde{L}$ is **nonsplit** (resp. **weakly prime**, **pairwise prime**) if the unique nonstabilized representative (Σ, L) of the corresponding stable equivalence class is nonsplit (resp. weakly prime, pairwise prime).

Theorem 3.2 and Theorem 2 of [Aetal19] give the following generalization of Menasco's classical results that a link if split or non-prime if and only if obviously so in a given alternating diagram [Me84]:

Theorem 4.5. *Suppose V is an alternating diagram of a virtual link $\widetilde{L}$. Then $\widetilde{L}$ is nonsplit if and only if V is. Likewise for weakly prime and pairwise prime.*

4.2. **Lassos of link diagrams on closed surfaces.** One would like to be able to tell whether or not an alternating virtual diagram V is split, weakly prime, or pairwise prime by inspection, as one can for (Σ, D). To enable this, we define:

Definition 4.6. A **lasso** for (Σ, D) is a disk $X \subset \Sigma$ that intersects D generically and contains all crossings of D; X is **acceptable** if:

- $D \cap X$ is connected and
- no crossingless arc of $D \setminus\setminus X$ is parallel in $\Sigma \setminus\setminus X$ to ∂X.

Proposition 4.7. *If D is connected, then (Σ, D) admits an acceptable lasso.*

Proof. Let G be the underlying graph of D. Take a spanning tree T of G. Then the disk $\nu T \subset \Sigma$ is a lasso for (Σ, D). If any crossingless arc of $D \setminus\setminus X$ is parallel in $\Sigma \setminus\setminus X$ to ∂X, isotope ∂X past that arc; note that $D \cap X$ remains connected. Repeat until X is acceptable. $\square$

Proposition 4.7 and Corollary 2.10 imply:

Corollary 4.8. *An alternating diagram D on a surface Σ is connected, and thus represents a nonsplit link, if and only if (Σ, D) admits an acceptable lasso.*

4.3. **Lassos of virtual link diagrams.**

Definition 4.9. A **lasso** for a virtual link diagram $V \subset S^2$ is a disk $X \subset S^2$ that intersects D generically and contains all classical crossings of D but no virtual crossings. Then $V \cap X$ and $V \setminus\setminus X$ are called the **classical** and **virtual** parts of V (with respect to X), respectively. The lasso X is **acceptable** if no crossingless arc of $V \setminus\setminus \partial X$ is parallel in $S^2 \setminus\setminus (V \cup \partial X)$ to ∂X and no pair of arcs in the virtual part of V intersect more than once.

Construction 4.10. Suppose that $[V]$ corresponds to (Σ, D) under Correspondence 2.3. If X is an acceptable lasso for (Σ, D), then one can construct an acceptable lasso for a virtual diagram $V' \in [V]$ as follows:

- Choose an embedding $\varphi : \Sigma \to \widehat{S^3}$ such that $\varphi(X)$ lies entirely on the front of Σ and $(\pi \circ \varphi(D \cap X)) \cap (\pi \circ \varphi(D \setminus X)) = \varnothing$.
- Perturb φ so that all self-intersections in $\pi \circ \varphi(D \setminus X)$ are transverse double-points.
- Then $\pi \circ \varphi(X)$ is an acceptable lasso for the virtual diagram $\pi \circ \varphi(D) = V' \in [V]$ (with over-under information matching D), using Construction 2.5

Conversely, if Y is an acceptable lasso for $V' \in [V]$, then one can construct an acceptable lasso for (Σ, D) as follows:

- Construct (Σ, D) by taking a regular neighborhood of $Y \cup V$, changing the neighborhood of each virtual crossing as in Figure 3, and (abstractly) capping off each resulting boundary component with a disk.
- Then Y embeds naturally in Σ and is an acceptable lasso for D.

Proposition 4.11. *For every virtual diagram V, if every $V' \in [V]$ is connected, then some $V' \in [V]$ admits an acceptable lasso.*

Proof. Let (Σ, D) be the pair corresponding to $[V]$ under Correspondence 2.3. Assume that every $V' \in [V]$ is connected. Then D is also connected. Hence, by Proposition 4.7, (Σ, D) admits an acceptable lasso X. Therefore, Construction 4.10 gives an acceptable lasso for some $V' \in [V]$. $\square$

We note also:

Proposition 4.12. *Every virtual diagram admits a lasso.*

Proof. Let $G \subset S^2$ denote the underlying graph of V, and denote the vertex set of G by P. Construct a planar graph $\Gamma \subset S^2$ as follows. Place one vertex at each classical crossing of V and one in the interior of each component of $S^2 \setminus \setminus G$. Each classical crossing c of V lies on the boundary of four components of $S^2 \setminus \setminus G$; construct an edge from c to the vertex in each of these four components. Each edge e of G is incident to two components of $S^2 \setminus \setminus G$; construct an edge α between the vertices in these components, such that $\alpha \cap G$ consists of a single point in e. Now choose a spanning tree T for Γ, and take a regular neighborhood X' of T in S^2. The disk $X' \subset S^2$ intersects V generically and contains all classical crossings in V but no virtual crossings. Thus, X' is a lasso for V. $\square$

We now turn our attention to alternating virtual diagrams.

Proposition 4.13. *An alternating virtual diagram V represents a nonsplit virtual link if and only if some $V' \in [V]$ admits an acceptable lasso.*

Proof. If V represents a nonsplit virtual link, then each $V' \in [V]$ is connected. Hence, by Proposition 4.11, some $V' \in [V]$ admits an acceptable lasso. Conversely, if some $V' \in [V]$ admits an acceptable lasso X, then so does the corresponding pair (Σ, D), by Construction 4.10. Thus, by Corollary 4.8, D represents a nonsplit link in $\Sigma \times I$, so V represents a nonsplit virtual link. $\square$

Theorem 4.14. *Suppose an alternating virtual diagram V admits an acceptable lasso X. Then (V is connected and) the virtual link represented by V is (nonsplit and):*

(i) pairwise prime if and only if, for every disk $Z \subset S^2$ whose boundary intersects V in two generic points, both in X, all classical crossings of V lie on the same side of ∂Z; and
(ii) weakly prime if and only if, for every disk $Z \subset X$ that intersects V generically and contains at least one crossing, $|\partial Z \cap L| > 2$.

Proof. Parts (i) and (ii) follow immediately from Theorem 3.2 and the correspondence.... For part (iii), the backward direction is immediate. For the converse, suppose otherwise. Then, by the correspondence, there is a disk $Z \subset S^2$ whose boundary intersects V in two generic points, both in X, such that both components of $X \setminus \partial Z$ contain classical crossings, and one of the disks Y of $S^2 \setminus\setminus \partial Z$ contains no virtual crossings. Among such disks Z, choose one which minimizes $|Z \cap \partial X$.

If $Y \cap V \subset X$, then the boundary of the disk $Y \cap X$ intersects V in two generic points, but this disk also contains classical crossings, contrary to assumption. Instead, $Y \setminus X$ must intersect V, but with no virtual crossings. Choose an outermost disk Y_0 of $Y \setminus\setminus \partial X$ whose boundary is disjoint from D. One may push an outermost arc of V in $Y \setminus X$ past ∂X, decreasing $|\partial X \cap V|$ contrary to the fact that X is an efficient lasso for V. □

5. Lasso numbers

Definition 5.1. Given a diagram (Σ, D) of a link $L \subset \Sigma$, denote the set of all lassos for (Σ, D) by $\text{lassos}(\Sigma, D)$. Define the **lasso number** of (Σ, D) to be

$$\text{lasso}(\Sigma, D) = \min_{X \in \text{lassos}(\Sigma, D)} |\partial X \cap D|.$$

If X is a lasso for (Σ, D) and $|\partial X \cap D| = \text{lasso}(\Sigma, D)$, say that X is **efficient** for D. Define the **lasso number** of (Σ, L) to be

$$\text{lasso}(\Sigma, L) = \min_{\text{diagrams } D \text{ of } L} \text{lasso}(\Sigma, D).$$

If X is a lasso for a diagram D of L and $|\partial X \cap D| = \text{lasso}(\Sigma, L)$, say that X is **efficient** for L.

Note that every efficient lasso is acceptable. In general, computing lasso numbers of diagrams, let alone links, seems difficult. In practice, however, computing lasso numbers of diagrams is not so bad, using the following fact:

Proposition 5.2. *Consider a pair (Σ, D), where D has n crossings and Σ has genus g. Then every acceptable lasso X for (Σ, D) satisfies*

$$|\partial X \cap D| = 2g - 1 + |\Sigma \setminus\setminus D| - n,$$

where n is the number of faces of $\Sigma \setminus\setminus D$ contained in X. Hence, X is efficient for D if and only if it contains as many faces of $\Sigma \setminus\setminus D$ as possible.

Proof. Collapse X to a point to obtain the graph $G = D/X$ in the surface $\Sigma/X \equiv \Sigma$. Then G has one vertex and $|D \cap \partial X|$ edges.

The fact that X is acceptable implies that, for each face U of $\Sigma\setminus\setminus D$, $U\setminus\setminus X$ is either empty or connected. Let n denote the number of faces of $\Sigma \setminus\setminus D$ contained in X. Therefore, $|\Sigma\setminus\setminus G| = |\Sigma\setminus\setminus D| - n$. Hence, as claimed:

$$\begin{aligned} 2 - 2g &= 1 - |D \cap \partial X| + |\Sigma \setminus\setminus D| - n \\ |\partial X \cap D| &= 2g - 1 + |\Sigma \setminus\setminus D| - n. \end{aligned}$$

Of the quantities on the right-hand side, only n depends on X. Thus, X is efficient for D if and only if it contains as many faces of $\Sigma \setminus\setminus D$ as possible. □

Thus, the problem of computing the lasso number of (Σ, D) is equivalent to the following. Denote the faces of $\Sigma \setminus\setminus D$ by U_i, $i \in \mathcal{C}$ for some index set $\mathcal{C}$. The problem is to find the largest subset $\mathcal{L} \subset \mathcal{C}$ such that $\bigcup_{i\in\mathcal{L}} U_i$ is simply connected.

One can show that one can always choose a maximal subset $\mathcal{L}$ which contains all bigons (and monogons) U_i, but this is as far as we pursue this matter in this paper. Given, say, a 100-crossing fully alternating diagram D on a surface Σ of genus five (with no symmetry), it seems challenging to compute $\mathrm{lasso}(\Sigma, D)$.

Continue. Say more? Prove that lasso numbers for [virtual diagrams] equal lasso numbers for pairs.

Definition 5.3. Given a virtual diagram V, denote the set of all lassos for V by $\mathrm{lassos}(V)$. Define the **lasso number** of V to be

$$\mathrm{lasso}(V) = \min_{V'\in[V]} \min_{X\in(V')} |\partial X \cap V'|.$$

If X is a lasso for V and $|\partial X \cap V| = \mathrm{lasso}(V)$, say that X is **classically efficient**. If no two arcs of $V \setminus\setminus X$ intersect more than once, say that X is **virtually efficient**. Say that X is **efficient** if it is both classically and virtually efficient.

Definition 5.4. Given a pair (Σ, D) where Σ is connected, the **lasso number** of (Σ, D) is:

$$\mathrm{lasso}(\Sigma, D) = \min_{X\in\mathrm{lasso}(\Sigma,D)} |\partial X \cap D|.$$

Call $X \in \mathrm{lasso}(\Sigma, D)$ **efficient** if $|\partial X \cap D| = \mathrm{lasso}(\Sigma, D)$.

Proposition 5.5. *If the lasso $X \subset \Sigma$ for D in Construction* **??** *is efficient, then the lasso $\pi(X)$ for $V = \pi(D)$ is classically efficient.*

If the lasso X for V in Construction **??** *is classically efficient, then X is an efficient lasso for (Σ, D).*

Proof. Both statements follow from the observation that Constructions **??** and **??** are inverse procedures: given any lasso X for (Σ, D), performing Construction **??** and then Construction **??** returns X and (Σ, D), up to isotopy, and given any lasso X for V, performing Construction **??** and then Construction **??** returns a lasso X'' for some $V'' \in [V]$ such that there is a pairwise homeomorphism between $(X, V \cap X)$ and $(X'', V'' \cap X'')$. □

Corollary 5.6. *Suppose D is a cellularly embedded diagram on a connected surface Σ, and the corresponding virtual class is $[V]$. Then $lasso(\Sigma, D) = lasso(V)$.*

Observation 5.7. *Given a lasso X for a virtual diagram V, there exists $V' \in [V]$ with $V' \cap X = V \cap X$, for which X is an efficient lasso; in particular, if X is classically efficient, for V, then it is efficient for V'. Moreover, if one allows only isotopy and virtual Reidemeister moves that fix $V \cap X$ pointwise, then V' is unique up to isotopy.*

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