The Time-Neutral Exterior Schwarzschild Metric in Polynomial Form

Kevin Loch*

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The time-neutral metric is discussed, and the neutral exterior Schwarzschild metric is converted to polynomial form in r and M. The resulting polynomials are both of degree three, which allows exact solutions using standard cubic solving techniques.

THE TIME-NEUTRAL METRIC

The time-neutral metric *(neutral metric)* is a spacetime metric solution of General Relativity[1] reduced by c^2dt^2 and set equal to one:

$$\frac{d\tau^2}{dt^2} = 1 \tag{1}$$

This represents zero time dialation, as opposed to the maximal time dialation of the *zero metric*: $d\tau^2/dt^2 = 0$. That finite, real, solutions with non-zero radius exist for neutral metrics when $M \neq 0$ may be surprising. It is especially counter-intuitive for the Schwarzschild metric[2] which does not have the opposite sign r_Q^2/r^2 or a^2/r^2 factors of higher order metrics[3–5]. However, these solutions do exist mathematically.

The open-source software tool `knsolver`[6] can be used to generate plots of solutions to the neutral Kerr-Newman metric[5]. It uses computational techniques on the standard form of the Kerr-Newman metric which is flexible but slow, especially for precision results. Converting a neutral metric to polynomial form may allow for fast exact solutions, depending on the degree of the polynomial.

POLYNOMIAL FORM

As shown in the proof below, the neutral exterior Schwarzschild metric, with test particle velocities converted to v^2/c^2, as a polynomial in r is given by:

$$\left(\frac{v_r^2}{c^2} + \frac{v_\Omega^2}{c^2}\right) r_\mathrm{s} r^3 + \left(1 - \frac{v_\Omega^2}{c^2}\right) r_\mathrm{s}^2 r^2 - r_\mathrm{s}^3 r = 0, \qquad (r \geq R)\,, \tag{2}$$

and in M:

$$\frac{8G^3 r}{c^6} M^3 - \left(1 - \frac{v_\Omega^2}{c^2}\right) \frac{4G^2 r^2}{c^4} M^2 - \left(\frac{v_r^2}{c^2} + \frac{v_\Omega^2}{c^2}\right) \frac{2Gr^3}{c^2} M = 0, \qquad (r \geq R)\,. \tag{3}$$

As both polynomials are of degree three, standard cubic solving techniques can be used to efficiently find exact solutions. This also agrees with the number of roots implied by plots generated from `knsolver` when electric charge Q and angular momentum J are set to zero.

PROOF

Deriving the polynomials involves basic algebraic manipulation but special care must be taken to not lose the degrees of freedom represented by the inverse sum in the dr^2 term. This makes the process slightly more complicated, but nowhere near as tedious as with the Reissner–Nordström metric[3]. This proof uses some substitutions that are not strictly necessary here but can be used to help manage the explosive proliferation of length factors in higher order metrics[3–5].

* kevin@loch.me

We begin with the exterior ($r \geq R$) Schwarzschild metric[2] in spherical coordinates (t, r, θ, φ) and metric signature $[+, -, -, -]$:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{r_{\mathrm{s}}}{r}\right) c^2 dt^2 - \left(1 - \frac{r_{\mathrm{s}}}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \quad (r \geq R)\,, \tag{4}$$

$$r_{\mathrm{s}} = \frac{2GM}{c^2}, \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2. \tag{5}$$

Reduce the metric by $c^2 dt^2$ and set equal to one to obtain the time-neutral metric, and also convert test particle velocities to v^2/c^2 . This is the equation we will convert to polynomial form:

$$\frac{d\tau^2}{dt^2} = \left(1 - \frac{r_{\mathrm{s}}}{r}\right) - \frac{v_r^2}{c^2}\left(1 - \frac{r_{\mathrm{s}}}{r}\right)^{-1} - \frac{v_\Omega^2}{c^2} = 1, \quad (r \geq R)\,, \tag{6}$$

$$\frac{v_r^2}{c^2} = \frac{dr^2}{c^2 dt^2}, \quad \frac{v_\Omega^2}{c^2} = \frac{r^2 d\Omega^2}{c^2 dt^2}. \tag{7}$$

Multiply the dt^2 and $d\Omega^2$ terms by $\left(1 - \frac{r_{\mathrm{s}}}{r}\right) / \left(1 - \frac{r_{\mathrm{s}}}{r}\right)$, then simplify the denominator by dividing both sides by r:

$$\frac{\left(1 - \frac{r_{\mathrm{s}}}{r}\right)\left(1 - \frac{r_{\mathrm{s}}}{r}\right) - \frac{v_r^2}{c^2} - \frac{v_\Omega^2}{c^2}\left(1 - \frac{r_{\mathrm{s}}}{r}\right)}{1 - \frac{r_{\mathrm{s}}}{r}} = 1, \tag{8}$$

$$\frac{\left(1 - \frac{r_{\mathrm{s}}}{r}\right)\left(1 - \frac{r_{\mathrm{s}}}{r}\right) - \frac{v_r^2}{c^2} - \frac{v_\Omega^2}{c^2}\left(1 - \frac{r_{\mathrm{s}}}{r}\right)}{r - r_{\mathrm{s}}} = \frac{1}{r}. \tag{9}$$

Multiply out the dt^2 term product, then simplify the numerator by multiplying both sides by r^2, then group by terms of r being especially careful with signs of combined factors:

$$\frac{1 - \frac{2r_{\mathrm{s}}}{r} + \frac{r_{\mathrm{s}}^2}{r^2} - \frac{v_r^2}{c^2} - \frac{v_\Omega^2}{c^2}\left(1 - \frac{r_{\mathrm{s}}}{r}\right)}{r - r_{\mathrm{s}}} = \frac{1}{r}, \tag{10}$$

$$\frac{r^2 - 2r_{\mathrm{s}}r + r_{\mathrm{s}}^2 - \frac{v_r^2}{c^2}r^2 - \frac{v_\Omega^2}{c^2}\left(r^2 - r_{\mathrm{s}}r\right)}{r - r_{\mathrm{s}}} = r, \tag{11}$$

$$\frac{\left(1 - \frac{v_r^2}{c^2} - \frac{v_\Omega^2}{c^2}\right) r^2 - \left(2 - \frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}}r + r_{\mathrm{s}}^2}{r - r_{\mathrm{s}}} = r. \tag{12}$$

At this point we introduce temporary variables X and Y, to help capture the extra degrees of freedom in the inverse sum. Let X be the numerator on the left side of equation 12:

$$\frac{X}{r - r_{\mathrm{s}}} = r, \tag{13}$$

$$X = \left(1 - \frac{v_r^2}{c^2} - \frac{v_\Omega^2}{c^2}\right) r^2 - \left(2 - \frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}}r + r_{\mathrm{s}}^2. \tag{14}$$

The denominator sum is now split between numerators X and Y:

$$\frac{X}{r - r_{\mathrm{s}}} = \frac{X}{r} + \frac{Y}{r_{\mathrm{s}}}, \tag{15}$$

$$\frac{X}{r - r_{\mathrm{s}}} = \frac{X r_{\mathrm{s}} + Y r}{r_{\mathrm{s}} r}. \tag{16}$$

Now it is safe to multiply both sides by $(r - r_{\mathrm{s}})\, r_{\mathrm{s}} r$, then we group by terms of X and Y:

$$X r_{\mathrm{s}} r = (X r_{\mathrm{s}} + Y r)\,(r - r_{\mathrm{s}})\,, \tag{17}$$

$$X r_{\mathrm{s}} r = X r_{\mathrm{s}} r - X r_{\mathrm{s}}^2 + Y r^2 - Y r_{\mathrm{s}} r, \tag{18}$$

$$X r_{\mathrm{s}}^2 - Y\left(r^2 - r_{\mathrm{s}} r\right) = 0. \tag{19}$$

Substitute symbols X_f, Y_f, for length factors associated with X and Y, then solve for Y:

$$X X_f - Y Y_f = 0, \tag{20}$$

$$X_f = r_{\mathrm{s}}^2, \qquad Y_f = r^2 - r_{\mathrm{s}} r, \tag{21}$$

$$Y = \frac{X X_f}{Y_f}. \tag{22}$$

With equations 13 and 16, multiply by $r_{\mathrm{s}} r$ and substitute for Y using equation 22:

$$\frac{X r_{\mathrm{s}} + Y r}{r_{\mathrm{s}} r} = r, \tag{23}$$

$$X r_{\mathrm{s}} + \frac{X X_f r}{Y_f} = r_{\mathrm{s}} r^2. \tag{24}$$

Multiply by Y_f, then group by terms of X:

$$X Y_f r_{\mathrm{s}} + X X_f r = Y_f r_{\mathrm{s}} r^2, \tag{25}$$

$$X\,(Y_f r_{\mathrm{s}} + X_f r) - Y_f r_{\mathrm{s}} r^2 = 0. \tag{26}$$

Substitute symbols $F_1, F_{2,}, F_3$, for the combined length factors:

$$X\,(F_1 + F_2) - F_3 = 0, \tag{27}$$

$$F_1 = Y_f r_{\mathrm{s}}, \qquad F_2 = X_f r, \qquad F_3 = Y_f r_{\mathrm{s}} r^2. \tag{28}$$

With equations 21, and 28, solve $F_1 + F_2$, and with equation 14, $X\,(F_1 + E_2)$:

$$F_1 = r_{\mathrm{s}} r^2 - r_{\mathrm{s}}^2 r, \qquad F_2 = r_{\mathrm{s}}^2 r, \tag{29}$$

$$F_1 + F_2 = r_{\mathrm{s}} r^2, \tag{30}$$

$$X\,(F_1 + F_2) = \left[\left(1 - \frac{v_r^2}{c^2} - \frac{v_\Omega^2}{c^2}\right) r^2 - \left(2 - \frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}} r + r_{\mathrm{s}}^2\right] r_{\mathrm{s}} r^2, \tag{31}$$

$$X\,(F_1 + F_2) = \left(1 - \frac{v_r^2}{c^2} - \frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}} r^4 - \left(2 - \frac{v_\Omega^2}{c^2}\right) r_{\mathrm{s}}^2 r^3 + r_{\mathrm{s}}^3 r^2. \tag{32}$$

Solve F_3 and $X\,(F_1 + F_2) - F_3$:

$$F_3 = r_{\mathrm{s}} r^4 - r_{\mathrm{s}}^2 r^3, \tag{33}$$

$$X\left(F_1+F_2\right)-F_3=\left(1-\frac{v_r^2}{c^2}-\frac{v_\Omega^2}{c^2}\right)r_\mathrm{s}r^4-\left(2-\frac{v_\Omega^2}{c^2}\right)r_\mathrm{s}^2r^3+r_\mathrm{s}^3r^2-r_\mathrm{s}r^4+r_\mathrm{s}^2r^3. \tag{34}$$

With equations 27 and 34, divide by r since the smallest degree of r is 2:

$$\left(1-\frac{v_r^2}{c^2}-\frac{v_\Omega^2}{c^2}\right)r_\mathrm{s}r^4-\left(2-\frac{v_\Omega^2}{c^2}\right)r_\mathrm{s}^2r^3+r_\mathrm{s}^3r^2-r_\mathrm{s}r^4+r_\mathrm{s}^2r^3=0, \tag{35}$$

$$\left(1-\frac{v_r^2}{c^2}-\frac{v_\Omega^2}{c^2}\right)r_\mathrm{s}r^3-\left(2-\frac{v_\Omega^2}{c^2}\right)r_\mathrm{s}^2r^2+r_\mathrm{s}^3r-r_\mathrm{s}r^3+r_\mathrm{s}^2r^2=0. \tag{36}$$

Convert to standard polynomial form in r by consolidating terms by r, then inverting the sign of each term:

$$\left(-\frac{v_r^2}{c^2}-\frac{v_\Omega^2}{c^2}\right)r_\mathrm{s}r^3-\left(1-\frac{v_\Omega^2}{c^2}\right)r_\mathrm{s}^2r^2+r_\mathrm{s}^3r=0, \tag{37}$$

$$\boxed{\left(\frac{v_r^2}{c^2}+\frac{v_\Omega^2}{c^2}\right)r_\mathrm{s}r^3+\left(1-\frac{v_\Omega^2}{c^2}\right)r_\mathrm{s}^2r^2-r_\mathrm{s}^3r=0, \qquad (r\geq R)} \tag{38}$$

Convert to standard polynomial form in M by substituting $r_s=\frac{2GM}{c^2}$, then rearranging terms by M and inverting signs again:

$$\left(\frac{v_r^2}{c^2}+\frac{v_\Omega^2}{c^2}\right)\frac{2GM}{c^2}r^3+\left(1-\frac{v_\Omega^2}{c^2}\right)\frac{4G^2M^2}{c^4}r^2-\frac{8G^3M^3}{c^6}r=0, \tag{39}$$

$$\boxed{\frac{8G^3r}{c^6}M^3-\left(1-\frac{v_\Omega^2}{c^2}\right)\frac{4G^2r^2}{c^4}M^2-\left(\frac{v_r^2}{c^2}+\frac{v_\Omega^2}{c^2}\right)\frac{2Gr^3}{c^2}M=0, \qquad (r\geq R)} \tag{40}$$

[1] A. Einstein. Die Grundlage der allgemeinen Relativitätstheorie. *Annalen der Physik*, 354(7):769–822, January 1916.

[2] Karl Schwarzschild. Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, pages 189–196, January 1916.

[3] H. Reissner. Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie. *Annalen der Physik*, 355(9):106–120, January 1916.

[4] Roy P. Kerr. Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics. *Phys. Rev. Lett.*, 11(5):237–238, September 1963.

[5] E. T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence. Metric of a Rotating, Charged Mass. *Journal of Mathematical Physics*, 6(6):918–919, June 1965.

[6] K. Loch. knsolver. `https://github.com/kevinloch/knsolver`, 2024.